Lecture 19

Barrier options

Barrier options are path dependent options - they have a payoff that depends on the realised asset price via its level; certain aspects of the contract are triggered if the asset price becomes too high or too low.

Example 19.1 (Barrier Options). Describe the different types of barrier options:-

- up and out
- down and out
- up and in
- down and in

Solution 19.1.

Barrier options are useful for a number of reasons, including

(i) The purchaser has precise views about the direction of the market.

(ii) The purchaser wants the payoff from an option, but does not want to pay for the upside potential, believing that the movement of the underlying will be limited prior to expiry.

(iii) These options are cheaper than their corresponding vanilla 'cousins'.

19.1 Pricing Barrier Options with PDEs

Although barrier options are path dependent, this dependency can be quite readily incorporated into the PDE methodology - we only need to know whether or not the barrier has been triggered; we do not need any other information about the path. This is in contrast to other more exotic types of option, such as Asian options (where, for example, the payoff may depend on the average value of the underlying during the lifetime of the option contact).

Consider the value of a barrier contract before the barrier has been triggered. The value still satisfies the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{19.1}$$

The barrier feature is to be incorporated in the boundary conditions of the problem.

Out Barriers

If the underlying reaches the barrier in an 'out' barrier option, then the contract becomes worthless. This leads to the boundary condition

$$V(S_u, t) = 0 \quad \text{for} \quad t < T,$$

for an up-and-out barrier option with the barrier level at $S = S_u$. We must solve the Black-Scholes equation for $0 \leq S \leq S_u$ with the above condition on $S = S_u$ and the usual payoff condition if the barrier is not triggered.

If we have a down-and-out option with a barrier at S_d, then we solve for $S_d \leq S < \infty$ with

$$V(S_d, t) = 0,$$

and the relevant final condition at expiry.

Example 19.2. Write down the domain and boundary conditions for a down-and-out European put option $P(S,t)$, with barrier at $S = S_d$ and strike price X.

Solution 19.2.

In Barriers

An 'in' barrier option only has a payoff if the barrier is triggered. If the barrier is not triggered, then the option expires worthless. The value in the option is the potential to hit the barrier. If the option is an up-and-in contract then on the upper barrier the contract must have the same value as a vanilla contract (say $V_v(S,t)$). We then have

$$V(S_u, t) = V_v(S_u, t) \quad \text{for} \quad t < T.$$

A similar boundary condition holds for a down-and-in option.

The contract we receive when the barrier is triggered is a derivative itself, and therefore the 'in' option is a second-order contract. We must therefore solve for the vanilla option first, before solving for the value of the barrier option.

Example 19.3. Write down the domain and boundary conditions for a up-and-in European call option $C(S,t)$, with barrier at $S = S_u$ and strike price X.

Solution 19.3.

Rebates

Sometimes these barrier options will include a rebate to compensate the holder if the option is 'knocked out' (or doesn't get knocked in). This slightly raises the value of the option and may make it more attractive to buyers.

Example 19.4. If the rebate takes the form of a fixed fee at expiry, say R_b, write down the domain and boundary conditions for a down-and-out European call option $C(S,t)$, with barrier at $S = S_d$ and strike price X.

Solution 19.4.

Lecture 20

Analytic Solutions for Barrier Options

In this final lecture we demonstrate how to use the techniques covered in this course to derive a solution for the Barrier option. It uses similarity solutions, transformations and substitutions and in particular the similarity solution technique to extend to solution outside the domain to match a boundary condition (see Examples Sheet 5, qu 1, and Worksheet 6, qu 1).

20.1 Down-and-out call options

Consider that we wish to value the down-and-out call option $C_{DO}(S,t)$ with barrier level S_d below the strike price X. This option must satisfies the Black-Scholes equation, that for an arbitrary option $V(S,t)$ (in the usual notation), is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{20.1}$$

where the volatility σ and interest rate r are both constants. The boundary conditions are

$$C_{DO}(S,T) = \max(S - X, 0),$$

$$C_{DO} \to S - Xe^{-r(T-t)} \text{ as } S \to \infty.$$

$$C_{DO}(S_d, t) = 0 \text{ for } t < T.$$

Example 20.1. By using the substitution $V(S,t) = S^\alpha V_1(S,t)$ into (20.1), where $\alpha = 1-2r/\sigma^2$, show that the equation satisfied by $V_1(S,t)$ is

$$\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (\sigma^2 - r)S\frac{\partial V_1}{\partial S} - rV_1 = 0 \tag{20.2}$$

Solution 20.1.

Example 20.2. By further substituting $V_1(S,t) = V_2(\xi,t)$ into (20.2), where $\xi = S_d^2/S$ and S_d is constant, show that $V_2(\xi,t)$ satisfies (20.1).

Solution 20.2.

From this we can infer the value of a down-and-out call option takes the form

$$C_{DO}(S,t) = AV(S,t) - BS^{1-2r/\sigma^2}V(S_d^2/S,t) \tag{20.3}$$

with A and B constants, and where V is the value of the corresponding European call option. Now C_{DO} is a linear combination of two solutions of (20.1), which is a linear PDE, and hence C_{DO} is also a solution of (20.1).

Example 20.3. Find A and B so that all the boundary conditions for a down-and-out call option are satisfied.

Solution 20.3.

20.2 Down-and-in call options

Consider a down-and-out call option C_{DO}, and a down-and-in call option C_{DI} are both trading in the market with the same strike price X and barrier S_d.

Example 20.4. What is the payoff at expiry for a trader that purchases both options?

Solution 20.4.

So the relationship between an 'in' barrier option and an 'out' barrier option (with the same payoff and barrier level) is actually quite simple

$$\text{in} + \text{out} = \text{vanilla}.$$

If the 'in' barrier is triggered, then so is the 'out' barrier, so whether or not the barrier is triggered, we still obtain the vanilla payoff at expiry. Thus the value of a down-and-in call option is

$$C_{DI}(S,t) = \left(\frac{S}{S_d}\right)^{1-2r/\sigma^2} V(S_d^2/S, t),$$

where V is the value of a European call option with strike price X.