Lecture 17

Interest rate models and bonds

So far we have assumed that interest rates are constant or at best known functions of time; this is clearly not the case in reality. Although the effects of interest-rate changes on option prices are generally small (because of their short lifetime), many other securities with much longer durations can be very susceptible to interest rate changes.

17.1 Bonds

A bond is a contract, paid for up-front, that yields a known amount on a known date in the future, the **maturity date**, $t = T$. The bond may also pay a known cash dividend (the **coupon**) at fixed times during the life of the contract. If there are no coupons, the bond is known as a **zero-coupon bond**. Bonds may be issued by both governments and companies to raise capital, and the up-front premium can be regarded as a loan.

A typical question related to this is: *how much should I pay now to get a guaranteed \$1 in 10 years' time?*

In the simple case of a zero-coupon bond $V(t)$ which pays Z at $t = T$ we may equate the return to that of a bank deposit, i.e.

$$dV = r(t)V dt,$$

with $V(T) = Z$. If the interest rate is deterministic, then

$$V(r, t; T) = Ze^{-\int_t^T r(\tau)d\tau}.$$

Example 17.1. What is the price of a the bond that pays a single coupon amount Z_1 at $t = T_1 < T$, and the face value Z at maturity?

Solution 17.1.

Example 17.2. If a the bond pays K coupons of fixed amount Z_K at times $T_i = i\Delta T$ for $1 \leq i \leq K$ and $\Delta T = T/K$, and it has a face value of Z at maturity, what is the price at $t = 0$? If we let $Z_k = z\Delta T$ where z is the constant continuous coupon yield, and $K \to \infty$, what is the price of the bond?

Solution 17.2.

17.2 Stochastic interest rates

In the same way we developed a model for the asset price as a lognormal walk, suppose that the interest rate r is governed by a stochastic differential equation

$$dr = w(r,t)dX + u(r,t)dt. \tag{17.1}$$

The functional form of $w(r,t)$ and $u(r,t)$ determines the behaviour of the **spot rate** r.

Deriving the bond-pricing equation

Pricing a bond is trickier than pricing an option, since there is no underlying asset with which to hedge: we cannot go out and buy an interest rate of 5%. Instead, we hedge with bonds of different maturity dates. In this case we denote $V_1(r,t)$ as the value of a zero-coupon bond with a maturity of T_1, and $V_2(r,t)$ as the value of a zero-coupon bond with a maturity of T_2.

Example 17.3. Consider a portfolio comprising long one V_1 bond and short Δ of V_2 bonds. Using Itô's Lemma, show that the choice

$$\Delta = \frac{\partial V_1}{\partial r} \bigg/ \frac{\partial V_2}{\partial r}$$

eliminates the random component of the portfolio.

Solution 17.3.

We then have

$$
\begin{aligned}
d\Pi &= \left(\frac{\partial V_1}{\partial t} + \tfrac{1}{2} w^2 \frac{\partial^2 V_1}{\partial r^2} - \frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r} \left(\frac{\partial V_2}{\partial t} + \tfrac{1}{2} w^2 \frac{\partial^2 V_2}{\partial r^2} \right) \right) dt \\
&= r\Pi dt \\
&= r(V_1 - V_2 \frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r}) dt,
\end{aligned}
$$

and here we have used arbitrage arguments to set the return on the portfolio to equal the risk-free (spot) rate.

Gathering all the V_1 terms on the left-hand-side and all the V_2 terms on the right-hand-side yields

$$\left(\frac{\partial V_1}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - rV_1\right)/\frac{\partial V_1}{\partial r} = \left(\frac{\partial V_2}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2} - rV_2\right)/\frac{\partial V_2}{\partial r}$$

So what can we do with this equation? We have one equation and two unknowns V_1 and V_2, so there is no explicit answer. Instead we look for the relationship between the two contracts which must always hold due to no arbitrage.

Example 17.4. Show that the value of both V_1 and V_2 must satisfy the equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} - a\frac{\partial V}{\partial r} - rV = 0, \tag{17.2}$$

where $\lambda(r, t)$ is an arbitrary function that does not depend on either T_1 or T_2.

Solution 17.4.

Now it is convenient to write

$$a(r, t) = w(r, t)\lambda(r, t) - u(r, t)$$

for given $w(r,t)$ and $u(r,t)$, but $\lambda(r,t)$ is left unspecified. We will talk more about λ in the next lecture. Hence the zero-coupon bond pricing equation is therefore

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0,$$

subject to the final condition $V(r,T;T) = Z$, and generally $V(r \to \infty, t) \to 0$; the boundary condition on $r = 0$ is generally dependent on λ, u and w. We will go through particular cases of these in the next lecture.

Example 17.5. Derive the equation for a bond with a continuous constant coupon yield z.

Solution 17.5.

Lecture 18

Specifying the Bond Model

18.1 The market price of risk

Consider now in more detail the unknown function $\lambda(r,t)$. In a timestep dt the bond V changes in value by

$$dV = w\frac{\partial V}{\partial r}dX + (\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + u\frac{\partial V}{\partial r})dt. \tag{18.1}$$

Example 18.1. Using (18.1) and the PDE (17.2) calculate the excess return on holding a bond as

$$w\frac{\partial V}{\partial r}(dX + \lambda dt). \tag{18.2}$$

Solution 18.1.

The function λ is called the **market price of risk**. The expression in (18.2) is the excess return above the risk-free rate for accepting a certain level of risk. In return for taking the extra risk the portfolio profits by an extra λdt per unit of extra risk dX.

Example 18.2. How do we calculate the market price of risk?

Solution 18.2.

18.2 The Vasicek model

Now we introduce two of the most common pricing models for dealing with interest rates. The first, the 'Vasicek' model, takes the form

$$dr = (\eta - \gamma r)dt + \beta^{\frac{1}{2}} dX.$$

Here η, γ and β are assumed to be constants. This is clearly modelling the interest rates as an OU process, where $\frac{\eta}{\gamma}$ is the long term mean (of the real interest rates). Remember that the OU process can be solved through stochastic calculus, so analytic solutions will exist for a variety of financial contracts. These include zero coupon bonds, and options on bonds.

The pricing equation for the vasicek model can be shown to be

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\beta \frac{\partial^2 V}{\partial r^2} + (\eta - \lambda\beta^{1/2} - \gamma r)\frac{\partial V}{\partial r} - rV = 0. \tag{18.3}$$

Normally the market price of risk λ is also assumed to be constant.

Example 18.3. Assume that for a zero-coupon bond with $V(r, T; T) = Z$, the analytic value takes the form

$$Ze^{A(t;T) - rB(t;T)}.$$

Derive and solve the equations for A and B to find the solution to the bond equation.

Solution 18.3.

This model is mean reverting which is good, and this is supported by evidence in the markets. The model contains three parameters that could be estimated from historical interest rates (η, γ and β), but the market price of risk can only come from option prices. It might be that models fit better when all parameters are derived from otion prices. There is however one drawback to this model, because as we have shown earlier (Example **??**) it allows for interest rates to go negative. This is of course not supposed to happen, unless a crisis of unforeseen proportions takes hold (like for instance a worldwide flu pandemic breaks out).

18.3 Cox, Ingersoll, Ross Model

The CIR model was adapted from the earlier Vasicek model to supposedly take care of the deficiency of negative interest rates. It does this by using a square root OU process, which takes the form

$$dr = (\eta - \gamma r)dt + \sqrt{\alpha r}dX.$$

Here the spot rate is again mean reverting to a long term mean value of $\frac{\eta}{\gamma}$, but this time remains positive if $\eta > \alpha/2$. Again in the standard model, η, γ, α are all assumed constant and the market price of risk is taken as $\lambda(r,t) = \lambda_c\sqrt{r}$ with λ_c constant.

Example 18.4. Show that the bond pricing equation under a CIR process is given by

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\alpha r\frac{\partial^2 V}{\partial r^2} + (\eta - (\lambda_c\sqrt{\alpha} + \gamma)r)\frac{\partial V}{\partial r} - rV = 0. \tag{18.4}$$

Given the zero-coupon bond with $V(r,T;T) = Z$, value is again of the form

$$A(t;T)e^{-rB(t;T)},$$

derive an analytic solution for V.

Solution 18.4.