Lecture 15

American Options

American options are options which can be exercised at any time to receive $S - X$ or $X - S$ for call and put options respectively. Unfortunately this gives rise to a **non-linear** problem and as such it is not possible in general to derive explicit formulae like those for European options.

15.1 American put options

The first problem is to decide at which values of S and t it is optimal to exercise. To consider the problem, treat the American put option as a European put option with the extra early exercise feature. At expiry the early exercise condition has no effect, as the value of the American put, $P(S, t)$, is given by

$$P(S, T) = \max(X - S, 0).$$

Moving back from expiry there will, however, be certain values of S for which

$$X - S > P_{BS}(S, t)$$

where $P_{BS}(S, t)$ is the value of the European put option derived from the Black-Scholes PDE. In this case the holder of the option would exercise their right and receive $X - S$. The major problem is to locate the value of S at which it becomes optimal to exercise the option, if we call this value $S_f(t)$ then we have

$$P(S, t) = \begin{cases} X - S & \text{for} \quad S \leq S_f(t) \\ P_{BS}(S, t) & \text{for} \quad S > S_f(t). \end{cases}$$

This is known as a free boundary problem and they are very difficult to solve. More formally when pricing American options the Black-Scholes equation becomes an inequality, which is an equality when it is optimal to hold the option:

$$S_f(t) < S < \infty : \qquad P > X - S, \quad \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0,$$

and an inequality when it is optimal to exercise

$$0 \leq S < S_f(t) : \qquad P = X - S, \quad \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP < 0.$$

Example 15.1 (Variational Inequality)**.** A problem of this type is known as a variational inequality. Explain using no arbitrage arguments why the inequality on the Black-Scholes PDE holds.

Solution 15.1.

The boundary conditions are as follows:

$$\begin{aligned} P(S,T) &= \max(X - S, 0), \\ P(S_f(t),t) &= X - S_f(t), \\ P(S,t) \to 0 \quad &\text{as} \quad S \to \infty. \end{aligned}$$

where the first and third are as for a European put option but the second is one of the conditions on the free boundary, $S_f(t)$. There is another, less obvious condition at $S = S_f(t)$, known as the smooth pasting condition which ensures that the Δ $(= \partial P/\partial S)$ is smooth across the early exercise boundary.

Example 15.2. Derive and justify the smooth pasting condition

$$\frac{\partial P}{\partial S}(S_f(t), t) = -1 \tag{15.1}$$

for an American put option using no arbitrage arguments.

Solution 15.2.

In general, numerical methods must be used to price American put options. There is one exception though and that is the perpetual case, which we will cover in Lecture 16.

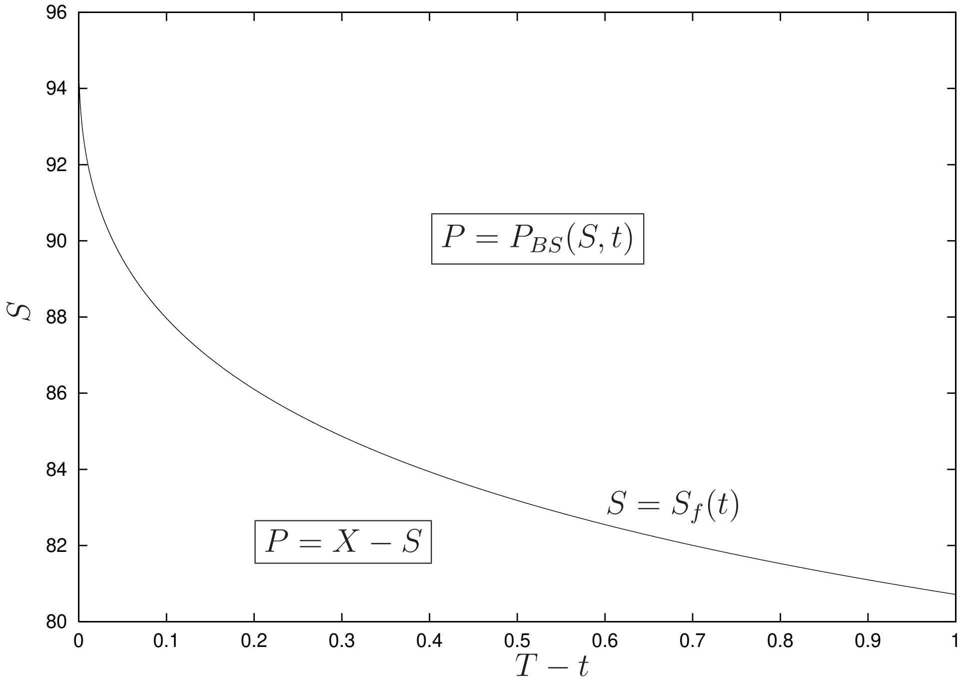

Figure 15.1: The position of $S_f(t)$ and the valuation regions for an American put option.

15.2 American call options

If the underlying asset pays no dividends then pricing an American call option is remarkably simple.

Example 15.3. Calculate the price of an American call option with no dividends.

Solution 15.3.

The solution however is not so easy when the underlying asset is paying continuous dividends, as one can observe from the option profiles in figures 15.2 and 15.3.

In the continuous dividend case the problem becomes similar to that for the American put, with analogous boundary conditions.

$$0 < S < S_f(t): \qquad C > S - X, \quad \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D)S\frac{\partial C}{\partial S} - rC = 0,$$

and with the BSE being an inequality when it is optimal to exercise

$$S_f(t) < S < \infty: \qquad C = S - X, \quad \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D)S\frac{\partial C}{\partial S} - rC < 0.$$

Example 15.4. Derive and justify the boundary conditions for an American call option.

Solution 15.4.

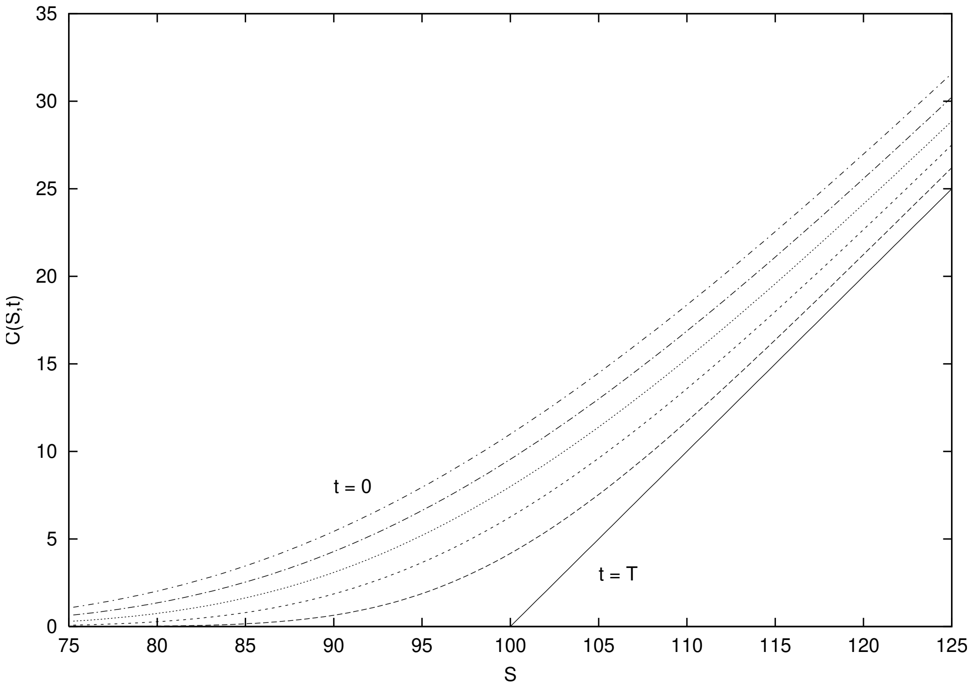

Figure 15.2: The value of $C(S,t)$ at $t = 0, \ldots$ and $t = T$ on a non-dividend paying asset - note how the value of $C(S,t)$ does not drop below $S - X$.

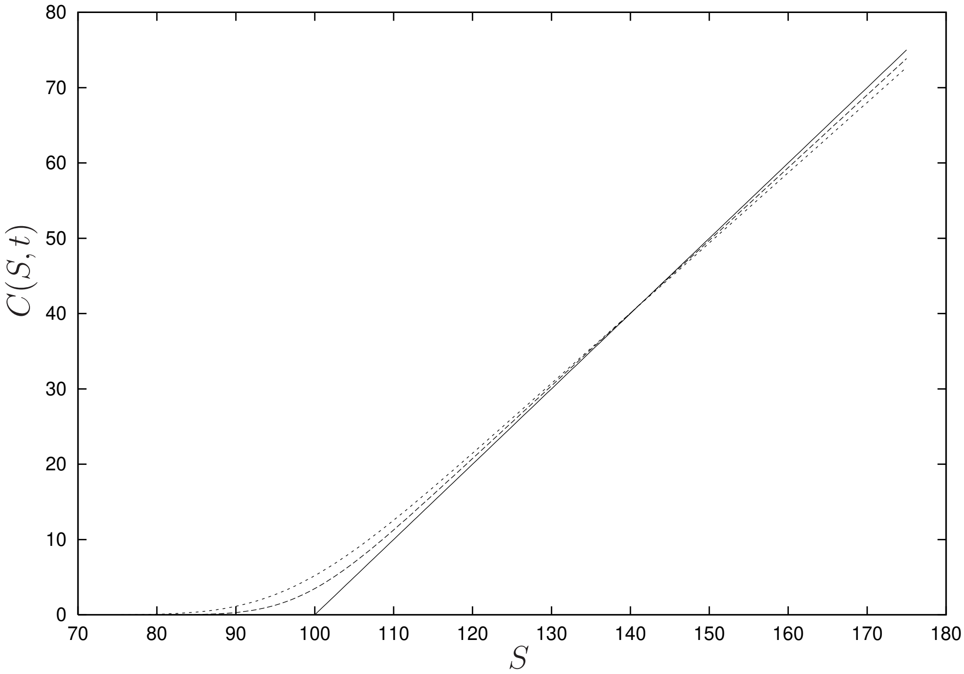

Figure 15.3: The value of $C(S,t)$ at $t = 0$, … and $t = T$ on a dividend paying asset - note how the value of $C(S,t)$ can drop below $S - X$.

Lecture 16

Small Time to Expiry and Perpetual options

Although it appears that we have already written down all of the required conditions for the American put option problem, there is in fact one final condition that is still missing. The final condition for the free boundary must be provided in order for a solution to be found. This can only be discovered by considering an asymptotic analysis of the problem in the limit as we approach maturity.

Asymptotic Analysis:- American Call option near expiry

In order to use the asymptotic analysis we must convert the BSM PDE to it's non-dimensional form.

Example 16.1. Making the following substitutions

$$S = Xe^x, \quad t = T - \tau/\frac{1}{2}\sigma^2, \quad C(S,t) = S - X + Xe^{-\rho\tau}c(x,\tau),$$

with

$$\rho = \frac{r}{\frac{1}{2}\sigma^2} \qquad \nu = \frac{D}{\frac{1}{2}\sigma^2}$$

into (13.3) results in the following equation

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (\rho - \nu - 1)\frac{\partial c}{\partial x} + e^{\rho\tau}(\rho - \nu e^x), \tag{16.1}$$

for $x < x_f(\tau)$ where $S_f(t) = Ee^{x_f(\tau)}$.

Solution 16.1.

Example 16.2. Show that the boundary conditions may be expressed as:

$$c = \frac{\partial c}{\partial x} = 0 \quad \text{at} \quad x = x_f(\tau), \tag{16.2}$$

$$c \sim e^{\rho\tau}(1 - e^x) \quad \text{as} \quad x \to -\infty, \tag{16.3}$$

$$c = \max(1 - e^x, 0) \quad \text{at} \quad \tau = 0, \tag{16.4}$$

Solution 16.2.

From equations (16.1) to (16.4) we may confirm the limit of x_0, the free boundary as time tends to expiry. At expiry, we have that $\partial^2 c/\partial x^2 = \partial c/\partial x = 0$ for $x > 0$. Then (16.1) becomes

$$\frac{\partial c}{\partial \tau} = \rho - \nu e^x \quad \text{for} \quad x > 0. \tag{16.5}$$

Now in order to satisfy the constraint $c \geq \max(1 - e^x, 0)$, we require that $\partial c/\partial \tau > 0$.

Example 16.3. Use the approximate solution to the problem with small time (16.5), along with no arbitrage arguments to derive an expression for x_0 and hence $S_f(T)$.

Solution 16.3.

For the American put, we similarly find the terminal conditions of the free boundary value to be

$$S_f(T) = \min\left[X, \frac{r}{D}X\right], \tag{16.6}$$

where D is the continuous dividend yield. Hence for the American put with no dividend, the final condition is simply $S_f(T) = X$.

16.1 Perpetual Options

These are options with an infinite life, corresponding to $T \to \infty$. In this case we look for solutions (for American puts) of the form $P(S)$ only. The Black-Scholes equation then becomes the following ODE:

$$\tfrac{1}{2}\sigma^2 S^2 \frac{d^2 P}{dS^2} + rS\frac{dP}{dS} - rP = 0. \tag{16.7}$$

This is a form of Euler's equation, and hence has solutions of the form $P = AS^{\alpha}$.

Example 16.4 (Perpetual American Put Option)**.** Substitute $P = AS^{\alpha}$ into (16.7) to obtain a solution to the Perpetual American put option problem.

Solution 16.4.