Lecture 13

Options on assets paying dividends

13.1 Introduction

The majority of companies who have issued shares pay out dividends of some form another, fortunately it is relatively easy to incorporate dividend payments into the option pricing methodology. Of even greater use is that the methods used for pricing options on dividend paying stocks can be routinely extended to deal with other, analogous, problems such as options on foreign currency where the dividend becomes the foreign risk-free interest rate and options on commodities where the dividend becomes minus the cost of carry.

There are two main ways of modelling dividend payments: as continuous and as discrete.

Example 13.1 (Forward Prices)**.** What is the link between forward prices and dividends?

Solution 13.1.

13.2 Continuous dividend yield

This is the simplest payment structure, assume that over a period of time dt the underlying asset pays out a dividend $D(S,t)Sdt$ in that $D(S,t)$ is the proportion of the value of the asset paid out over this period of time. $D(S,t)$ is often considered to be constant and independent of t so that analytic results are available although even in this case the size of the dividend will obviously depend on S which is dependent on t.

How does this affect our model? By using arbitrage arguments (see Examples 1) a payment of dividends results in the underlying asset price dropping by the value of the dividend. Hence with a continuous dividend the stochastic process is given by

$$dS = (\mu - D(S,t))Sdt + \sigma SdW.$$

Here μ is defined as the return on investment including dividend payments that are reinvested by buying new shares.

Example 13.2 (Risk Neutral Process Prices)**.** Show that the Risk-Neutral process following by an asset paying dividends is

$$dS = (r - D(S,t))Sdt + \sigma SdW. \tag{13.1}$$

Solution 13.2.

Example 13.3 (Put-Call Parity with dividends)**.** Show that the Put-Call parity with a continuous constant dividend yield is given by

$$C(S,t) - P(S,t) = S_t e^{-D(T-t)} - Xe^{-r(T-t)} \tag{13.2}$$

Solution 13.3.

Now let us derive the governing PDE. This is a similar process to that to derive the standard Black-Scholes, so the portfolio starts off with the same format

$$\Pi = V - \Delta S.$$

However, in the case of dividends the change in value of the portfolio is different, given by

$$d\Pi = dV - \Delta(dS + D(S,t)Sdt)$$

as the holder of the portfolio receives the dividend as well. Proceeding as for the non-dividend case

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt,$$

and so

$$d\Pi = \frac{\partial V}{\partial t}dt + (\frac{\partial V}{\partial S} - \Delta)\,[(\mu - D)Sdt + \sigma S dW] - \Delta DSdt + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt.$$

Setting

$$\Delta = \frac{\partial V}{\partial S}$$

leads to a deterministic result, to which we can apply the usual no-arbitrage argument, i.e.

$$\begin{aligned} d\Pi &= \frac{\partial V}{\partial t}dt + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt - DS\frac{\partial V}{\partial S}dt \\ &= r\Pi dt \\ &= r(V - S\frac{\partial V}{\partial S})dt \end{aligned}$$

which gives the following PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0. \tag{13.3}$$

The standard Black-Scholes equation derived earlier in the course is just a special case of this equation for the case when $D = 0$. Now if the dividend rate is constant, valuing European call and put options is reasonably straightforward, the main difference being that r is replaced by $r - D$ but only in the coefficient of the $\partial C/\partial S$. To account for this slight difference introduce

$$V(S,t) = e^{-D(T-t)} V_1(S,t)$$

so that we now have

$$\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (r - D)S\frac{\partial V_1}{\partial S} - (r - D)V_1 = 0$$

which is the Black-Scholes equation only with r replaced by $r - D$ and with the same final conditions. As such

$$C(S,t) = e^{-D(T-t)} SN(d_{10}) - Xe^{-r(T-t)} N(d_{20})$$

where

$$d_{10} = \frac{\log(S/X) + (r - D + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_{20} = \frac{\log(S/X) + (r - D - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Example 13.4 (Non-constant dividends)**.** Consider an asset where the dividend yield is

$$D(S,t) = r + \kappa(\ln(S) - \theta)$$

Show that the modified Black-Scholes equation will be

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \kappa(\theta - \ln(S))S\frac{\partial V}{\partial S} - rV = 0. \tag{13.4}$$

Solution 13.4.

Example 13.5 (Forward Price)**.** Consider again the asset where the dividend yield is

$$D(S,t) = r + \kappa(\ln(S) - \theta).$$

Assume that the risk neutral process S evolves according to the equation

$$dS = \kappa(\theta - \ln(S))Sdt + \sigma SdW,$$

show that

$$E[\ln S_T] = e^{-\kappa T}\ln S_0 + (1 - e^{-\kappa T})\left(\theta - \frac{\sigma^2}{2\kappa}\right)$$

taking expectations under the risk neutral measure.

Solution 13.5.

Lecture 14

Discrete dividend payments

When considering options where the underlying is a stock then a more realistic model is to treat dividends as being paid at discrete points in time. This is because most companies pay out their dividends periodically, every quarter, every six months, every year etc.

Assume, as a starting point, that just one dividend payment is made during the lifetime of the option. Assume that this is paid at time t_d and can be expressed as a percentage of the level of the underlying, i.e. as $d_y S$ where $0 \leq d_y < 1$. Thus the holder of the asset receives a payment of $d_y S$ at t_d where S is the asset price **prior** to the dividend payment. How does this affect the asset price? By the usual arbitrage arguments if t_d^- is the time immediately before the dividend is paid and t_d^+ is the time immediately after we have

$$\begin{aligned} S(t_d^+) &= S(t_d^-) - d_y S(t_d^-) \\ &= (1 - d_y) S(t_d^-) \end{aligned}$$

where $S(t)$ is the value of the underlying asset at time t. There is a **jump** in the value of S, in that the value of the underlying asset is discontinuous across the dividend date. What effect will this have on the option price? Again in order to eliminate any possible arbitrage opportunities, the value of the option must be **continuous** as a function of time across the dividend date. In which case the value of the option immediately before the dividend payment must be the same as the value immediately after (recall that the owner of the option does *not* receive the dividend) thus

$$V(S(t_d^-), t_d^-) = V(S(t_d^+), t_d^+).$$

This brings to light something interesting in the relationship between S and t. In the Black-Scholes methodology S and t are considered to be independent variables although S is clearly dependent on t, this is possible as we consider every possible value of S at a particular point in time, rather than just one. This is because given the random movement of stock prices, S can take any value.

Example 14.1. Derive the relationship between $V(t_d^-)$ and $V(t_d^+)$in terms of $S(t_d^-)$.

Solution 14.1.

14.1 Example: pricing a European call option when there is one dividend payment

As usual we work back from the known conditions at expiry to derive the option value at a previous time. Moving backwards from expiry to just after the dividend payment time, namely t_d^+. At the dividend payment date we implement the jump condition

$$C(S, t_d^-) = C(S(1 - d_y), t_d^+).$$

then value the option back to any desired time t using these option values as new final conditions. Essentially you have to solve the Black Scholes equation twice

- Once for $T > t > t_d$ with $C(S, T) = \max(S - X, 0)$.
- Once for $t_d > t > 0$ with $C(S, t_d) = C(S(1 - d_y), t_d^+)$

We can simplify the methodology slightly by the following procedure:

Let $C(S,t)$ be the standard European call option and $C_d(S,t)$ be an option on an underlying asset paying discrete payments. If there is just one payment at t_d then from above we have

$$\begin{aligned} C_d(S,t) &= C(S,t;X), \quad t_d^+ \le t < T \\ C_d(S,t_d^-) &= C_d(S(1-d_y),t_d^+) \\ &= C(S(1-d_y),t_d^+;X). \end{aligned}$$

For $t < t_d^-$ there is a shortcut to using the BSE. Prior to the dividend payment the value of the call option is just subject to a scaling in S, i.e $S \mapsto S(1-d_y)$ as such $C(S(1-d_y),t;X)$ still satisfies the Black-Scholes equation. As this is equal to the value of $C_d(S,t)$ at t_d then the two are also equivalent for $t < t_d$. Thus if we can find the value of $C(S(1-d_y),t;X)$ then we'll know the value of C_d for $t < t_d$ and hence for all t.

At expiry,

$$\begin{aligned} C(S(1-d_y),T;X) &= \max(S(1-d_y)-X,0) \\ &= (1-d_y)\max(S-\frac{X}{1-d_y},0) \end{aligned}$$

which is the same as $(1-d_y)$ calls with an exercise price of $X/(1-d_y)$, hence we now know the value of the call option for $0 \le t < t_d$, which is

$$C_d(S,t) = (1-d_y)C(S,t;\frac{X}{1-d_y}).$$

In conclusion

$$C_d(S,t) = \begin{cases} (1-d_y)C(S,t;\frac{X}{1-d_y}) & \text{for} \quad 0 \le t < t_d \\ C(S,t;X) \quad \text{for} \quad t_d \le t < T \end{cases}.$$

which can be valued using the standard option pricing formulae.

Remark: Note that if the underlying asset pays a dividend then this *decreases* the value of the call option, since the holder of the the option does not receive the dividend yet a dividend payment reduces the value of the underlying asset. Correspondingly the value of a put option *increases* when dividends are paid.

Example 14.2. Calculate the value of a call on an asset (value S) that pays out *two* dividends $(S(t_{d_1}^-)d_1, S(t_{d_2}^-)d_2)$ at times (t_{d_1}, t_{d_2}) during the life of the option.

Solution 14.2.

Example 14.3. Consider a call on an asset (value S) that pays out kT dividends

$$S(t_{d_1}^-)d_1, S(t_{d_2}^-)d_2, \ldots, S(t_{d_{kT}}^-)d_{kT}$$

at times

$$t_{d_1}, t_{d_2}, \ldots, t_{d_{kT}}$$

during the life of the option. Now if each of the dividends are fixed and $d_i = \frac{D}{k}$, calculate the value of the call option.

Solution 14.3.

14.2 Non Proportional Dividends

There are many cases in which the dividend payments will **not** depend on the current share price. In fact dividend payments are usually fixed amounts that are announced some time in advance of the dividend date. So why do we always assume that they are proportional? So that the mathematics is simple and the formulas are easily adjusted. However we may think that assuming proportional dividends is causing inconsistencies in our model, so how do we deal with that?

Consider a dividend that this is paid at time t_d and can be expressed as the total amount paid, i.e. as d_S where $d_S \approx d_y S$. Then the same analysis can be followed and we have

$$S(t_d^+) = S(t_d^-) - d_S$$

and for options priced on this asset we have the relationship

$$V(S, t_d^-) = V(S - d_S, t_d^+).$$

Using this formulation introduces a couple of problems. Firstly, this implies that S can go negative if $S(t_d^-) < d_S$, in fact this would send the company bust. So we could write

$$V(S, t_d^-) = \begin{cases} V(S - d_S, t_d^+) & \text{if } S(t_d^-) > d_S \\ 0 & \text{if } S(t_d^-) \le d_S \end{cases}$$

if options expire worthless when the company ceases to exist.

Secondly we can no longer use the scaling adjustments outlined above, but we still be able to solve the problem. In fact we can use a numerical technique to solve this problem.

Example 14.4. Outline how financial contracts can be priced in the case of non-proportional dividends.

Solution 14.4.