Lecture 9

Analytic solutions to the Black-Scholes equation

The next few lectures of the course will deal with solving the heat conduction or diffusion equation and how to adapt these techniques to solve the Black-Scholes equation for some standard option pricing problems. Before doing that we will study the analytic solutions to the valuation problems and a few more key features of options.

The **Black-Scholes formulae** for the price of European call and put options are as follows:

$$C(S,t) = SN(d_1) - Xe^{-r(T-t)}N(d_2) \tag{9.1}$$

$$P(S,t) = Xe^{-r(T-t)}N(-d_2) - SN(-d_1) \tag{9.2}$$

where

$$\begin{aligned} d_1 &= \frac{\log(S/X) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{\log(S/X) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned} \tag{9.3}$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}s^2} ds \tag{9.4}$$

which we recognise as the cumulative distribution function for a Normal distribution. Note that these expressions satisfy the put call parity and so by calculating one it is routine to calculate the other, also note that the boundary conditions at $S = 0$ and $S \to \infty$ are satisfied.

For those students interested in probability it may be worth noting that $N(d_2)$ is the probability that the option will be exercised, i.e. $S > X$ at expiry. $SN(d_1)$ is the current value of a variable that equals S_T at $t = T$ *if* $S_T > X$ and is zero otherwise.

So, what does a graph of underlying asset against option price looks like as time moves backwards from expiry? As one would expect from a PDE which is a close relative of the diffusion equation, the payoff function $\max(S-X,0)$ gradually diffuses out as time moves backwards. The same is also true for a cash or nothing option even though the payoff is in fact discontinuous.

Example 9.1 (Option Pricing Formula)**.** The price of an asset (today) is £5. Find the value of a put and a call option, both with an exercise price of £6, and both with expiration dates in 9 months time. The risk-free interest rate is 3% per annum (fixed) and the volatility (constant) is 10% per $(\text{annum})^{\frac{1}{2}}$.

Solution 9.1.

$r = .03$, $T-t = 0.75$, $\sigma = .1$, $S = 5$, $X = 6$.
Using the formulae.$d_1 = -1.8021$, $d_2 = -1.8888$
Then

$$\begin{aligned} N(d_1) &= N(-1.8021) = N(-1.80) - .21[N(-1.80) - N(-1.81)] \\ &= 0.0359 - .21 \times (0.0359 - 0.0351) \\ &= 0.0357 \end{aligned}$$

Similarly $N(d_2) = .0295$
Leads to $C = .0060$.
Put can be calculated similarly - but best to use put-call parity:

$$P = C - S + Xe^{-r(T-t)}.$$

and this leads to $P = 0.8725$.

9.1 Delta hedging and the other hedge parameters

A tedious, yet straightforward, calculation (see MATH20912) will show that using the known expressions for the values of call and put options, that they have the following Δ's

$$\Delta_C = \frac{\partial C}{\partial S} = N(d_1)$$

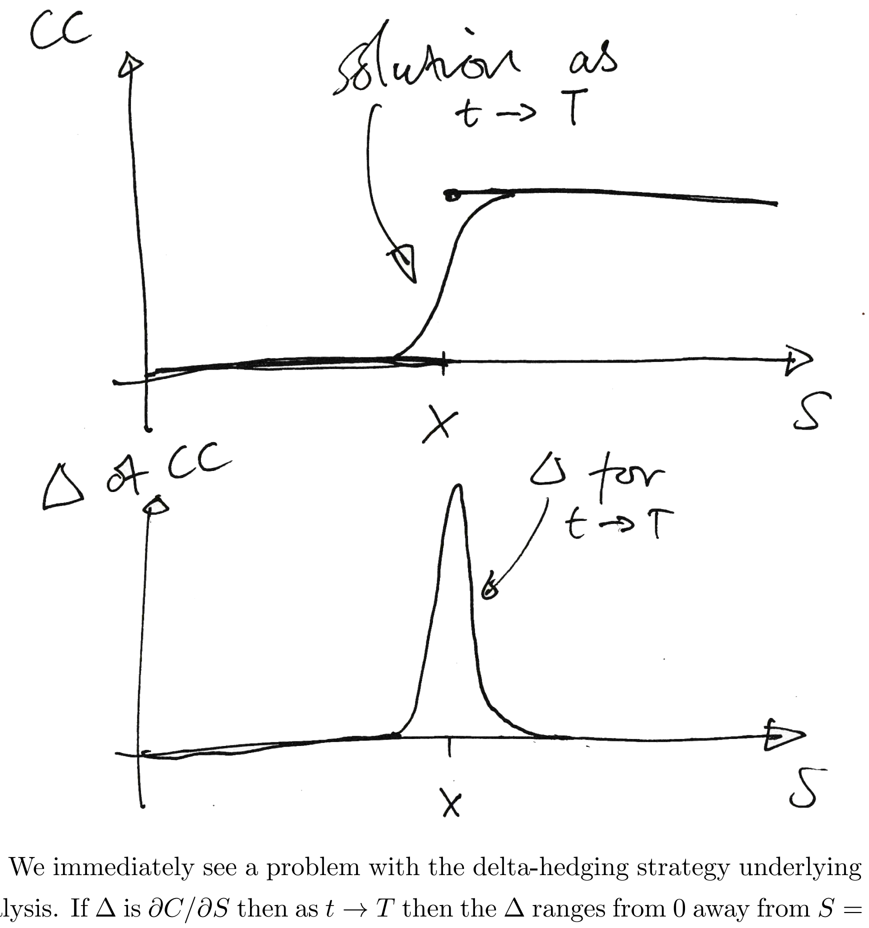
$$\Delta_P = \frac{\partial P}{\partial S} = N(d_1) - 1$$

What does this mean? During the lifetime of the option Δ varies between 0 for out of the money calls (puts) and 1 (-1) for in the money calls (puts) and very close to T there is in fact a step function between these two extremes. The Δ simply approximates the rate of change of the option price wrt the underlying asset and so any slight movement in the option price value will be offset by a roughly equivalent movement in Δ of the underlying. Clearly the portfolio will have to be rebalanced as regularly as possible to have a perfect hedge. In practise the number of times a portfolio can be hedged will be limited by transaction costs.

Example 9.2. Consider the value of a cash-or-nothing call option, plot the payoff of this option.

What problems might arise in a delta hedging strategy for this option.

Solution 9.2.



We immediately see a problem with the delta-hedging strategy underlying the Black-Scholes analysis. If Δ is $\partial C/\partial S$ then as $t \to T$ then the Δ ranges from 0 away from $S = X$ to approaching

∞ close to $S = X$. Thus as the underlying asset price moves, huge amounts of the underlying will have to be bought and sold to keep the portfolio properly hedged.

There are ways of hedging away other risks, not just those to do with the movement of the asset price. There are hedge parameters (also known as, somewhat loosely, as *The Greeks*) for each of the principle parameters in the Black-Scholes model, namely:

- The sensitivity to the decay of time of any option V is known as the **theta** and is defined as
$$\Theta = \frac{\partial V}{\partial t}$$
- The sensitivity to the volatility is known as the **vega** and is defined as
$$\mathcal{V} = \frac{\partial V}{\partial \sigma}$$
- The sensitivity to interest rates is known as **rho** and, unsurprisingly to be
$$\rho = \frac{\partial V}{\partial r}$$
- Finally, the sensitivity of the Δ to the underlying asset is known as **gamma** and is defined as follows
$$\Gamma = \frac{\partial^2 V}{\partial S^2}$$

Often these hedge parameters are used to see what would happen if there was a small change in one of the parameters, this is important as both r and σ are not fixed or even time dependent in practice.

In order to investigate how these risks factors contribute to the prices of options we observe in the market, we must consider the position of the bank. If you are on a trading desk and have been asked to hedge a particular option that has been sold by one of your colleagues at the price V, you will need to assess the profit and loss in trading (normally called P&L) over a set period of time, say δt. In practice continuous rebalancing of a hedge (as prescribed by Black-Scholes model) is not feasible as we can only trade at finite time intervals. Assume that we set up a hedging portfolio Φ with Δ lots of the underlying asset S

$$\Phi = -V + \Delta S.$$

For the following analysis we consider what would happen in the real world, so we denote δS to be the real observed change in stock price over δt. We do not need to prescribe a model for S to carry out this analysis and in the real world it is not necessarily log normal. The P&L, which will be $d\Phi$, will comprise of two parts, the change in value of the option V that has been sold and the hedge S that has been bought, plus the interest rate earned on V minus the borrowing cost to buy ΔS, which gives

$$d\Phi = -[V(S+\delta S, t+\delta t) - V(S,t)] + \Delta\delta S + rV\delta t - r\Delta S\delta t.$$

Example 9.3 (How to Choose Δ Model Free?)**.** Assume that V is a smooth function in S and t, expand with a Taylor series and choose Δ to minimise P&L. Show the resulting P&L is of the form

$$d\Phi = -A(S,t)\delta t - B(S,t)\left(\frac{\delta S}{S}\right)^2. \tag{9.5}$$

Solution 9.3.

Expanding with a Taylor series

$$V(S+\delta S, t+\delta t) - V(S,t) = \frac{\partial V}{\partial t}\delta t + \frac{\partial V}{\partial S}\delta S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\delta S)^2 + \dots$$

and we have

$$d\Phi = -\left[\frac{\partial V}{\partial t}\delta t + \frac{\partial V}{\partial S}\delta S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\delta S)^2 + \dots\right] + \Delta\delta S + rV\delta t - r\Delta S\delta t.$$

Since the δS terms are likely the largest terms in the equation we should choose $\Delta = \frac{\partial V}{\partial S}$ to eliminate them. This gives (after some rearranging)

$$d\Phi = -\left[\frac{\partial V}{\partial t} + rV - rS\frac{\partial V}{\partial S}\right]\delta t - \frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2}\left(\frac{\delta S}{S}\right)^2 + \dots.$$

We can show that $A = \frac{\partial V}{\partial t} + rV - rS\frac{\partial V}{\partial S}$ and $B = \frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2}$. Note that under Black-Scholes analysis we assume a model for S meaning that we can quantify $\left(\frac{\delta S}{S}\right)^2$ exactly cancelling out both terms and returning a zero P&L (in the limit $\delta t \to 0$). In practice $\left(\frac{\delta S}{S}\right)^2$ is likely to retain an element of uncertainty meaning that non-zero P&L is inevitable.

9.2 Implied volatility and Modelling in Finance

One of the most important parameters, and the only one which is very difficult to know for definite is the volatility, σ. There are several conventions for calculating the volatility of an underlying asset. One would perhaps assume that the best way is to look at the volatility of past returns and use this as a decent guess as to what would happen in the future. However, another way is to assume that the Black-Scholes analysis is correct and use the market prices for options to back-out the volatility, using a suitable iterative procedure such as Newton-Raphson, the only unknown being σ itself.

If one attempts this they will see a problem with the volatility. Depending on how far in or out of the money the option is the volatility may well not be constant for a given r, S, and t. So, not only is it dependent on time but also on the exercise and asset prices. Such a result is often termed the *volatility smile* although many other shapes can be observed depending on the market conditions such as a *frown*, *wry smile* etc. This is another example of the faults in the Black-Scholes model.

Lecture 10

Solving the heat conduction and Black-Scholes equations

The PDE which defines the price of a derivative is now known to be a second-order **parabolic** equation, in the majority of cases this equation is also a linear one. The next few lectures are concerned with the nature of these equations, focusing attention on the heat conduction equation and then extending to the Black-Scholes equation itself.

10.1 Properties of the Heat conduction equation

The heat conduction equation takes the form

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

where τ is the time and x is the spatial variable, it normally models the flow of heat or its diffusion and has been extensively studied over the years. Its fundamental properties are as follows

- It is a second order linear PDE, as such if u_1 and u_2 are solutions then so is $a_1u_1 + a_2u_2$ for any constants a_1 and a_2.
- It is a parabolic equation and its characteristics are simply along the lines $\tau = c$ (where c is a constant) which means that this is where information propagates along. So any change in the boundary conditions is felt along these lines.
- The heat conduction equation generally has analytic solutions in x, technically in that for $\tau > 0$, $u(x, \tau)$ has a convergent power series of $(x - x_0)$ for $x_0 \neq x$.

Crucially, the heat conduction (diffusion) equation is a smoothing out process, and as such discontinuities in the boundary or initial (final) conditions can be catered for. Recall that in the Black-Scholes equation the final conditions are often discontinuous.

By way of demonstration consider the following *initial value problem.*

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

for $\tau > 0$ and $-\infty < x < \infty$ where $u(x, 0) = u_0(x)$ and $u \to 0$ as $x \to \pm\infty$. $u(x, \tau)$ is analytic for $\tau > 0$.

Example 10.1. Show that the following function is an analytic solution to the initial value problem:

$$u(x, \tau) = u_\delta(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} e^{-x^2/4\tau} \tag{10.1}$$

for $-\infty < x < \infty$ and $\tau > 0$.

Solution 10.1.

Now we verify that this indeed satisfies the PDE.

$$\frac{\partial u}{\partial x} = \frac{-x}{4\tau^{3/2}\sqrt{\pi}} e^{-x^2/4\tau}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-1}{4\tau^{3/2}\sqrt{\pi}} e^{-x^2/4\tau} + \frac{x^2}{8\tau^{5/2}\sqrt{\pi}} e^{-x^2/4\tau}$$

$$\frac{\partial u}{\partial \tau} = \frac{-1}{4\tau^{3/2}\sqrt{\pi}} e^{-x^2/4\tau} + \frac{x^2}{8\tau^{5/2}\sqrt{\pi}} e^{-x^2/4\tau}.$$

So, this is a solution which is well behaved except at one instance, the initial point in time $\tau = 0$. At this point when $x \neq 0$ then $u_\delta(x, 0) = 0$ but at $x = 0$ it has infinite value. This clearly has discontinuous initial conditions yet gives rise to a, reasonably, well behaved solution.

What more can we say about this special solution to the heat conduction equation? Well,

$$\int_{-\infty}^{\infty} u_\delta(x, \tau) dx = 1, \quad \forall\tau.$$

This function has all of the *heat* initially ($\tau = 0$) concentrated at $x = 0$ and then this immediately dissipates out as for *any* $\tau > 0$, $u_\delta(x, \tau) > 0$ for *all* values of x, figure 10.1.

Finally note the close similarity between the probability density function for the Normal distribution ($\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$) and the value of $u_\delta(x, \tau)$. Clearly it is the same only with a mean(μ) of zero and a variance (σ^2) of 2τ. As such it is possible to interpret this particular solution as the probability density function of the future position of a particle following a Brownian motion ($\sqrt{2}dW$) along the x-axis, with the particle starting at the origin.

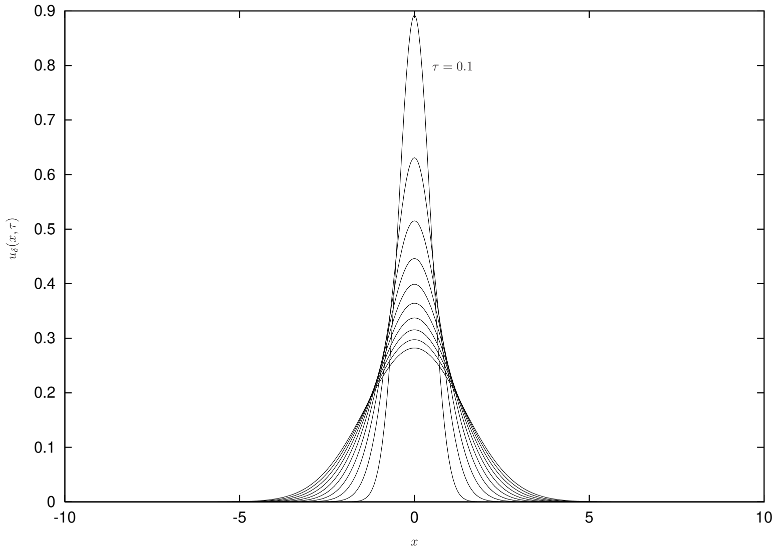


Figure 10.1: A graphical representation of $u_\delta(x, \tau)$ for $\tau = 0.1, 0.2, 0.3, \ldots, 1$.

10.2 The Dirac delta function

The function $u_\delta(x, \tau)$ when $\tau = 0$ is one representation of the **(Dirac) delta function** which is not a function in the normal sense but is known as a **generalised** function. It's definition is as a linear map representing the limit of a function whose effect is confined to a smaller and smaller interval but remains finite.

An informal definition is to consider a function

$$f(x) = \begin{cases} 1/2\epsilon, & |x| \leq \epsilon \\ 0, & |x| > \epsilon \end{cases}$$

and as $\epsilon \to 0$ the graph becomes taller and narrower but at all points

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

regardless of the value of ϵ although for all $x \neq 0$, $f(x) \to 0$ as $\epsilon \to 0$. In general the delta function $\delta(x)$ is the limit as $\epsilon \to 0$ of any one-parameter family of functions δ_ϵ with the following properties

- for each ϵ, $\delta_\epsilon(x)$ is piecewise smooth;
- $\int_{-\infty}^{\infty} \delta_\epsilon(x)dx = 1$;
- for each $x \neq 0$, $\lim_{\epsilon\to 0} \delta_\epsilon(x) = 0$.

Note that the specific solution to the heat conduction equation u_δ satisfies the above constraints with τ replaced by ϵ. The best way to look at the delta function is to only consider its integral which we know to be 1 and which smooths out the function's bad behaviour, especially when $x = 0$ and $\epsilon \to 0$ (of $\tau \to 0$).

Example 10.2. When concentrating on the integral form, show that the delta function can be seen as a **test function**.

Solution 10.2.

We see that

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(x)\phi(x)dx &= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \delta_\epsilon(x)\phi(x)dx \\ &= \lim_{\epsilon\to 0}\left\{\int_{-\infty}^{-\epsilon} \delta_\epsilon(x)\phi(x)dx + \int_{-\epsilon}^{\epsilon} \delta_\epsilon(x)\phi(x)dx + \int_{\epsilon}^{\infty} \delta_\epsilon(x)\phi(x)dx\right\} \\ &= \lim_{\epsilon\to 0}\left\{\phi(0)\int_{-\epsilon}^{\epsilon} \delta_\epsilon(x)dx\right\} \\ &= \phi(0)\end{aligned}$$

In fact, for any $a, b > 0$

$$\int_{-a}^{b} \delta(x)\phi(x)dx = \phi(0)$$

and, as importantly, for any x_0

$$\int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)dx = \phi(x_0)$$

and so integrating picks out the value of ϕ at x_0, the reason why $\delta(x)$ is also known as a test function.

Other properties concern its links with the Heaviside function as

$$\int_{-\infty}^{x} \delta(s)ds = \mathcal{H}(x)$$

and conversely,

$$\mathcal{H}'(x) = \delta(x)$$

where, as before

$$\mathcal{H}(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

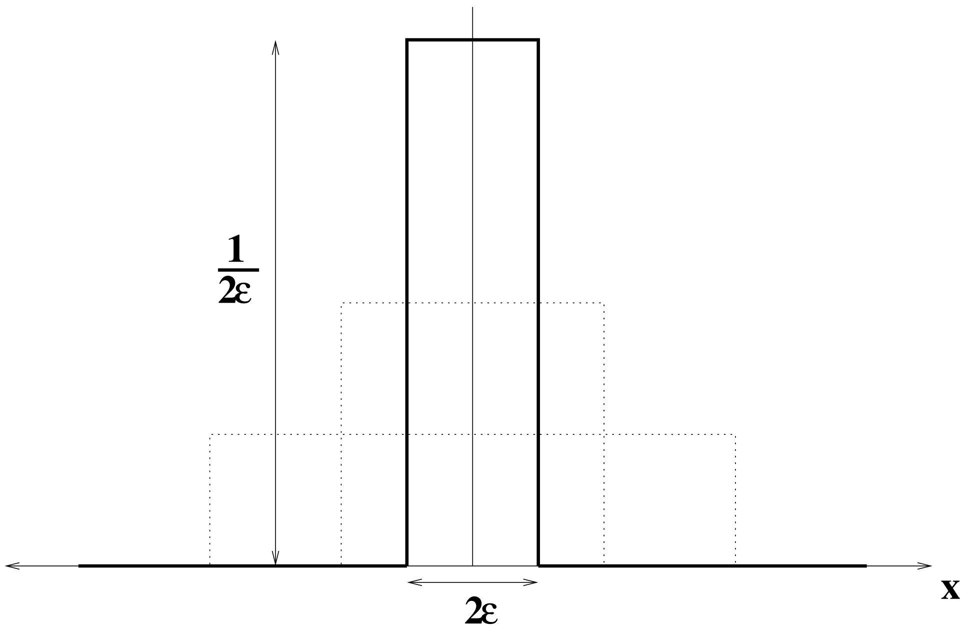


Figure 10.2: The epsilon representation of $\delta(x)$ which is the limit as $\epsilon \to 0$.

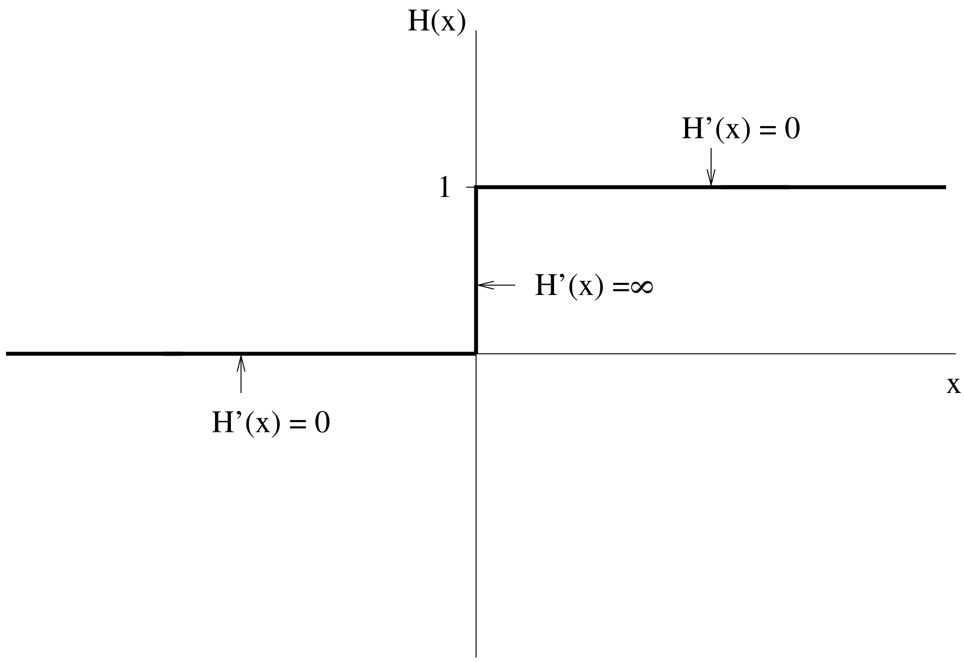


Figure 10.3: Demonstration that $H'(x) = \delta(x)$.

10.3 Transforming the Black-Scholes equation

Consider again the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Example 10.3. By making the following three substitutions

$$\begin{aligned} S &= Xe^x (\text{or} \quad x = \log\frac{S}{X}) \\ t &= T - \frac{\tau}{\frac{1}{2}\sigma^2} (\text{or} \quad \tau = \frac{\sigma^2}{2}(T-t)) \\ V &= Xv(x,\tau), \end{aligned} \tag{10.2}$$

show that the Black-Scholes equation can be reduced to the form

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv$$

where

$$k = \frac{r}{\frac{1}{2}\sigma^2}$$

Solution 10.3.

We have

$$\begin{aligned} \frac{\partial V}{\partial t} &= X\frac{\partial v}{\partial \tau}\frac{d\tau}{dt} = X\frac{\partial v}{\partial \tau}. - \frac{\sigma^2}{2} = -\frac{X\sigma^2}{2}\frac{\partial v}{\partial \tau} \\ \frac{\partial V}{\partial S} &= X\frac{\partial v}{\partial x}\frac{dx}{dS} = X\frac{\partial v}{\partial x}\frac{1}{S} = e^{-x}\frac{\partial v}{\partial x} \\ \frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{e^{-x}}{X}\frac{\partial}{\partial x}\left(e^{-x}\frac{\partial v}{\partial x}\right) = \frac{e^{-x}}{X}\left(e^{-x}\frac{\partial^2 v}{\partial x^2} - e^{-x}\frac{\partial v}{\partial x}\right) = \frac{e^{-2x}}{X}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) \end{aligned}$$

which leads to

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv$$

where

$$k = \frac{r}{\frac{1}{2}\sigma^2}$$

Now attempt to remove the $\frac{\partial v}{\partial x}$ and v terms by introducing the substitution

$$v(x,\tau) = e^{\alpha x + \beta \tau} u(x,\tau)$$

where α and β are constants to be determined.

Example 10.4. Show that this substitution results in the following expression for V:

$$V(S,t) = Xe^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau}u(x,\tau) \qquad (10.3)$$

where

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad \begin{array}{l} -\infty < x < \infty \\ \tau > 0 \end{array}$$

Solution 10.4.

Substitution gives

$$\begin{aligned} \frac{\partial v}{\partial \tau} &= \beta e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau} \\ \frac{\partial v}{\partial x} &= \alpha e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} \\ \frac{\partial^2 v}{\partial x^2} &= \alpha^2 e^{\alpha x+\beta\tau}u + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2} \end{aligned}$$

which gives

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku$$

to remove the $\frac{\partial u}{\partial x}$ and u terms we require

$$\begin{aligned} \alpha &= -\frac{1}{2}(k-1) \\ \beta &= -\frac{1}{4}(k+1)^2. \end{aligned}$$

Thus,

$$V(S,t) = Xe^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau}u(x,\tau)$$

and

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad \begin{array}{l} -\infty < x < \infty \\ \tau > 0 \end{array}$$

To transform the final conditions, or the payoff from the option we have for a **call** option

$$V(S,T) = \max(S-X,0)$$

so, from the definition of x, τ and $v(x,\tau)$ in (10.2)

$$Xv(x,0) = \max(Xe^x - X, 0)$$

or

$$v(x,0) = \max(e^x - 1, 0)$$

and so, from (10.3)

$$u(x,0) = u_0(x) = \max\left[e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0\right] \tag{10.4}$$

and similarly for a put option

$$u(x,0) = u_0(x) = \max\left[e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0\right] \tag{10.5}$$

As such the Black-Scholes equation has been converted to the heat conduction equation for $-\infty < x < \infty$ and, for European call and put options, initial condition $u_0(x)$ from (10.4) and (10.5) above. If we can determine a procedure for valuing the initial value problem for the heat conduction equation we'll be able to determine the correct values for call and put options.

Lecture 11

Similarity solutions to the Heat conduction equation

In order to solve the Heat equation using analytic methods we will need to introduce the method of similarity solutions. The method takes a high order partial differential equation and can reduce the order hopefully down to ODEs that can be solved easily. We can normally spot possible transformations as the solution appears self-*similar* at different scales. For instance, the normal distribution type solutions of u_δ in figure 10.1 could be laid over the top of one another if the x and y axes were changed for each of the different values of τ. Here we attempt to show how the method works by way of two examples.

Example 11.1. Suppose that $u(x,\tau)$ satisfies the heat conduction equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad x, \tau > 0$$

with the following boundary conditions

$$u(x, \tau = 0) = 0 \qquad (11.1)$$

$$u(x = 0, \tau) = 1 \qquad (11.2)$$

$$u(x, \tau) \to 0 \quad \text{as} \quad x \to \infty \qquad (11.3)$$

i.e. the bar initially has heat zero and then immediately the heat at one end is raised to 1 and kept there.

Seek a solution of the form $u(x,\tau) = U(\xi)$ where $\xi = x/\sqrt{\tau}$.

Solution 11.1.

Now on substitution

$$\frac{\partial u}{\partial \tau} = \frac{dU}{d\xi}\frac{\partial \xi}{\partial \tau} = -\frac{1}{2}x\tau^{-3/2}\frac{dU}{d\xi}$$

$$\frac{\partial u}{\partial x} = \frac{dU}{d\xi}\frac{\partial \xi}{\partial x} = \tau^{-1/2}\frac{dU}{d\xi}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \tau^{-1/2}\frac{d}{d\xi}\left(\tau^{-1/2}\frac{dU}{d\xi}\right) = \tau^{-1}\frac{d^2U}{d\xi^2}$$

and so, replacing $x/\sqrt{\tau}$ by ξ and multiplying by τ gives the ODE

$$\frac{d^2U}{d\xi^2} + \frac{1}{2}\xi\frac{dU}{d\xi} = 0$$

the boundary conditions become

$$U(0) = 1$$

and

$$U(\infty) = 0$$

with this second condition catering for both the initial condition and $u(x,\tau) \to 0$ as $x \to \infty$. Integrating the ODE once gives

$$\frac{dU}{d\xi} = Ce^{-\xi^2/4}$$

(C constant) and on solving gives

$$U(\xi) = C\int_0^{\xi} e^{-s^2/4}ds + D$$

(D constant). Upon substituting the boundary conditions, first $U(0) = 1$ gives

$$1 = D$$

and then $U(\infty) = 0$ gives

$$0 = C\int_0^{\infty} e^{-s^2/4}ds + 1$$

but we know that

$$\int_0^{\infty} e^{-s^2/4}ds = \sqrt{\pi}$$

thus

$$-1 = C\sqrt{\pi}$$

Thus,

$$U(\xi) = -\frac{1}{\sqrt{\pi}}\int_0^{\xi} e^{-s^2/4}ds + 1$$

but

$$\int_0^{\xi} = \int_0^{\infty} - \int_{\xi}^{\infty}$$

hence

$$U(\xi) = -\frac{1}{\sqrt{\pi}}\left(\int_0^\infty e^{-s^2/4}ds - \int_\xi^\infty e^{-s^2/4}ds\right) + 1$$

or

$$U(\xi) = -\frac{1}{\sqrt{\pi}}\left(-\int_\xi^\infty e^{-s^2/4}ds\right) - 1 + 1$$

so

$$U(\xi) = \frac{1}{\sqrt{\pi}}\int_\xi^\infty e^{-s^2/4}ds$$

and on replacing ξ by its definition we get

$$u(x,\tau) = \frac{1}{\sqrt{\pi}}\int_{x/\sqrt{\tau}}^\infty e^{-s^2/4}ds$$

The key trick being that to solve the equation we replace two variables (x and τ) by just one (ξ) and then the problem reduces to an ODE. Even more useful is the next example, for $-\infty < x < \infty$.

Example 11.2. Consider the following equation for $u(x,\tau)$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \qquad \begin{array}{l} -\infty < x < \infty \\ \tau > 0 \end{array}$$

where

$$\int_{-\infty}^\infty u(x,\tau)dx = k, \forall \tau \quad \text{where } k \text{ is a constant.}$$

Choosing the normalised case where $k = 1$ we search for a solution of the form $u(x,\tau) = \tau^{-1/2}U(\xi)$ where $\xi = x/\sqrt{\tau}$. The other boundary condition is a somewhat odd one but is that as $|\xi| \to \infty$ then

$$U(\xi) = o(1/\xi)$$

which says that the solution must decay faster than $1/\xi$ as ξ gets very big (or alternatively $u(x,\tau) = o(1/x)$ as $|x| \to \infty$).

Solution 11.2.

On transforming the derivatives we get

$$\frac{\partial u}{\partial \tau} = -\frac{1}{2}\tau^{-3/2}U + \tau^{-1/2}\frac{dU}{d\xi}.-\frac{1}{2}x\tau^{-3/2} = -\frac{1}{2}\tau^{-3/2}U - \frac{1}{2}\xi\tau^{-3/2}\frac{dU}{d\xi}$$

$$\frac{\partial u}{\partial x} = \tau^{-1/2}\frac{dU}{d\xi}\frac{\partial \xi}{\partial x} = \tau^{-1}\frac{dU}{d\xi}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \tau^{-1/2}\frac{d}{d\xi}\left(\tau^{-1}\frac{dU}{d\xi}\right) = \tau^{-3/2}\frac{d^2U}{d\xi^2}$$

which gives

$$\frac{d^2U}{d\xi^2} + \frac{1}{2}\xi\frac{dU}{d\xi} + \frac{1}{2}U = 0$$

or

$$\frac{d^2U}{d\xi^2} + \frac{d}{d\xi}\left(\frac{1}{2}\xi U\right) = 0.$$

Integrating both sides wrt ξ gives

$$\frac{dU}{d\xi} + \frac{1}{2}\xi U = C$$

where C is a constant. Now as $\xi \to \infty$, $U = o(1/\xi)$ so the LHS is $o(1)$ thus this constant $C = 0$. So then on solving the ODE

$$U(\xi) = Ae^{-\xi^2/4},$$

where A is a constant. Putting in the condition we have

$$A\int_{-\infty}^{\infty} \tau^{-1/2}e^{-x^2/4\tau}dx = 1$$

however, set $x' = x/\sqrt{\tau}$ and we get $dx = \sqrt{\tau}dx'$ and the equation becomes

$$A\int_{-\infty}^{\infty} e^{-x'^2/4}dx' = 1$$

and so using the usual result

$$2A\sqrt{\pi} = 1$$

thus

$$A = \frac{1}{2\sqrt{\pi}}$$

and so,

$$u(x,\tau) = \tau^{-1/2}\left(\frac{1}{2\sqrt{\pi}}e^{-x^2/4\tau}\right)$$

or

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}e^{-x^2/4\tau}$$

which is precisely the special solution u_δ from Example 10.1. [**Note:** The derivation in Wilmott where he states that $U(\xi) = Ce^{-\xi^2/4} + D$ is *wrong*.]

How similarity solutions work

The reason why the above similarity solution worked was because the governing equations and the boundary conditions do not change under the scalings $x \to \lambda x$ and $\tau \to \lambda^2 \tau$, where $\lambda \in \mathbf{R}$. In particular consider new variables $x^* = \lambda x$ and $\tau^* = \lambda^2 \tau$, these clearly satisfy the heat-conduction equation and in Example 11.1 the boundary conditions become $u(x^*, 0) = 0$ and $u(0, \tau^*) = 1$ for *any* λ.

Combining these two results to get a variable which is independent of λ the only possible combination is $x/\sqrt{\tau} = x^*/\sqrt{\tau^*}$. Hence the solution to the problem must be a function of $x/\sqrt{\tau}$ only.

Similarity solutions only work in special cases where all the boundary and initial conditions are invariant under the scaling transformation. It is also possible to multiply $U(\xi)$ by a function of τ as in Example 11.2 because as the heat-conduction equation is linear it is invariant under the scaling $u \to \mu u$.

In general with similarity solutions a good practical test to see if they'll work is to search for a solution of the form $u = \tau^\alpha U(x\tau^\beta)$ in the hope that the PDE will reduce to an ODE in $\xi = x\tau^\beta$ **and** the boundary conditions will be satisfied. For the heat conduction equation then in all cases $\beta = -1/2$ but the value of α will be dependent on the specific boundary conditions. For Example 11.1 $\alpha = 0$ because of the condition at $x = 0$ and, in Example 11.2, $\alpha = -1/2$ to remove τ from the integral condition.

Lecture 12

General solution to the Heat-Conduction equation initial value problem

Searching for a solution to the initial value problem in which we have to solve

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \qquad \begin{array}{l} -\infty < x < \infty \\ \tau > 0 \end{array}$$

with initial data $u(x,0) = u_0(x)$ and there are suitable growth conditions at $|x| \to \infty$ (usually $\lim_{|x|\to\infty} u(x,\tau)e^{-ax^2} = 0$ for $a > 0$ and $\tau > 0$).

The key to the formulation is the delta function, $\delta(x)$ as we can write the initial conditions as

$$u_0(x) = \int_{-\infty}^{\infty} u_0(\xi)\delta(\xi - x)d\xi.$$

Now recall that the fundamental solution to the initial value problem from Example 10.1 is

$$u_\delta(s,\tau) = \frac{1}{2\sqrt{\pi\tau}} e^{-s^2/4\tau}$$

and has initial value $u_\delta(s,0) = \delta(s)$. Noting that because $u_\delta(s-x,\tau) = u_\delta(x-s,\tau)$ we have

$$u_\delta(s-x,\tau) = \frac{1}{2\sqrt{\pi\tau}} e^{-(s-x)^2/4\tau}$$

which is still a solution to the heat conduction equation with either s or x as the spatial independent variable and it has initial value

$$u_\delta(s-x,0) = \delta(s-x).$$

Example 12.1. Show that the general solution to the heat conduction equation is

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-(x-s)^2/4\tau} ds$$

with initial data $u(x, 0) = u_0(x)$.

Solution 12.1.

Now comes the important bit, hence, for each s the function

$$u_0(s) u_\delta(s - x, \tau)$$

as a function of x and τ with s held fixed, satisfies the heat conduction equation as $u_0(s)$ is simply a constant.

Now using the fact that the diffusion equation is linear we can add together linear combinations of these solutions for any s all the way from $-\infty$ to ∞ and obtain another solution to the heat conduction equation, namely

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-(x-s)^2/4\tau} ds$$

and the initial data is

$$u(x,0) = \int_{-\infty}^{\infty} u_0(s)\delta(s - x) ds = u_0(x).$$

What does all this mean? Well, this solution satisfies the heat conduction equation for all x and for $\tau > 0$ and is also satisfies the initial conditions for *all* initial conditions $u_0(x)$. It is also possible to show that this solution is unique (see Examples 4). Hence we have found the general solution.

12.1 Pricing European call and put options

Example 12.2 (Analytic Solutions)**.** Derive the analytic solution for the European call option

$$C(S,t) = V(S,t) = SN(d_1) - Xe^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\log(S/X) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$
$$d_2 = \frac{\log(S/X) + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}.$$

Solution 12.2.

We now know the general solution to the initial value problem for the heat conduction equation, where $u(x,0) = u_0(x)$ for $\tau > 0$ and $-\infty < x < \infty$, namely

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-(x-s)^2/4\tau} ds.$$

We start by valuing a European call option but the procedure is similar for a put option. In section 5.3 we transformed the European call option pricing problem to the following system

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \qquad \begin{array}{l} -\infty < x < \infty \\ \tau > 0 \end{array}$$

where

$$u(x,0) = u_0(x) = \max\left[e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0\right].$$

By using the known general solution to this problem we have

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} \left\{ \max[e^{\frac{1}{2}(k+1)s} - e^{\frac{1}{2}(k-1)s}, 0] e^{-(x-s)^2/4\tau} \right\} ds$$

but $u_0(x) = 0$ for $x < 0$ hence

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{0}^{\infty} \left\{ [e^{\frac{1}{2}(k+1)s} - e^{\frac{1}{2}(k-1)s}] e^{-(x-s)^2/4\tau} \right\} ds.$$

We make another change of variable, define

$$x' = \frac{s-x}{\sqrt{2\tau}}.$$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \{ \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k+1)(x'\sqrt{2\tau}+x) - \frac{1}{2}x'^2} dx' - \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k-1)(x'\sqrt{2\tau}+x) - \frac{1}{2}x'^2} dx' \}.$$

Completing the square and removing the terms not dependent on x' yields

$$\begin{aligned} u(x,\tau) &= \frac{e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}(x' - \frac{1}{2}(k+1)\sqrt{2\tau})^2} dx' \\ &\quad - \frac{e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}(x' - \frac{1}{2}(k-1)\sqrt{2\tau})^2} dx' \\ &= I_1 - I_2 \end{aligned} \tag{12.1}$$

Noting that the expression for the cumulative Normal distribution is as follows

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}s^2} ds$$

we transform the dependent variable, x', once again to

$$x_1 = x' - \frac{1}{2}(k+1)\sqrt{2\tau}$$

and

$$x_2 = x' - \frac{1}{2}(k-1)\sqrt{2\tau}$$

in I_1 and I_2 respectively and then

$$u(x,\tau) = e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}N(d_1) - e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k-1)^2\tau}N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau} \\ d_2 &= \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}. \end{aligned}$$

Transforming the variables back using the usual definitions

$$\begin{aligned} V(S,t) &= Xe^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau}u(x,\tau) \\ x &= \log\left(\frac{S}{X}\right) \\ \tau &= \frac{\sigma^2}{2}(T-t) \\ k &= \frac{2r}{\sigma^2} \end{aligned}$$

gives the following expression for the value of the European call option

$$C(S,t) = V(S,t) = SN(d_1) - Xe^{-r(T-t)}N(d_2),$$

where

$$\begin{aligned} d_1 &= \frac{\log(S/X) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{\log(S/X) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

The European put can be valued in a similar manner or, more easily, by use of the put-call parity, equation. Either approach yields the following expression for its value, $P(S,t)$

$$P(S,t) = Xe^{-r(T-t)}N(-d_2) - SN(-d_1).$$

(To use put-call parity note that $N(x) + N(-x) = 1$).