Lecture 6

More General Stochastic Processes

The Ornstein-Uhlenbeck process is a diffusion process that was first introduced to help model the velocity of a particle undergoing Brownian motion. More recently, it has been used in finance to model stochastic volatility of an asset, or to model an underlying asset which naturally exhibits mean-reversion, such as interest rates and commodities. As a model of underlying assets this appears to contradict the efficient markets hypothesis, as if the values are above the long term average we know almost certainly that the next movement in the price will be to move down. However this contradiction can exist if there are difficulties (such as storage costs or assets that can't be traded) that stop agents in the market exploiting what looks in the data like an arbitrage opportunity.

Example 6.1 (Commodities)**.** What are storage effects, and how do they affect price?

Solution 6.1.

In the next section we will introduce the model for an OU process using the notation of assets, before later on in the course talking about how this may be used to model hidden variables (such as volatility) in our model.

6.1 OU Process

Definition 6.1 (Ornstein-Uhlenbeck process). Suppose that the price of a stock S_t satisfies the following Stochastic Differential Equation

$$dS = \kappa(\theta - S)dt + \sigma dW \tag{6.1}$$

where W is a Wiener process. Then S is said to follow an Ornstein-Uhlenbeck (OU) process.

If we are given a known value of the process, say S_0 (so we know the stock price today), then we can solve (6.1) to find that the distribution of S_t is given by

$$S_t \sim N\left(S_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right), \frac{\sigma^2}{\kappa}\left(1 - e^{-2\kappa t}\right)\right). \tag{6.2}$$

Therefore S_t is normally distributed with mean and variance that are functions of time.

Example 6.2 (Plotting an OU process)**.** Sketch some sample paths for an OU process.

Solution 6.2.

When we are looking at a stochastic model for some quantity in finance, one important consideration is the range of values that this process can take. For instance, there are many situations where one requires asset values that should be bounded below by zero and therefore never go negative, such as stock prices and interest rates. Although, recent history suggests the restriction on the latter should not necessarily be considered. There are other situations as well. If we wish to model coefficients in our model in a stochastic way, they may also have restrictions on them.

Example 6.3 (Bounding coefficients.)**.** A trader wishes to model two asset prices by setting the correlation between them as a stochastic process ρ_t. How will the resulting process need to be bounded?

Solution 6.3.

Example 6.4 (What are the chances?)**.** Calculate the probability that $S_t < 0$ when $t = 5$ if S follows an OU process with $\kappa = 0.25$, $\theta = 1$ and $\sigma = 0.2$.

Solution 6.4.

6.2 Square Root Process

If we were to try and model the volatility σ of an asset price as a stochastic process itself, then we would require that $\sigma_t > 0$ for all t. It turns out we can modify to the standard OU process to guarantee positivity of the process. This can been done by adjusting the function in the SDE multiplying the Wiener process (we denoted this as $b(S, t)$ in Lecture 5), so that we have for instance

$$dS = \kappa(\theta - S)dt + \sigma\sqrt{S}dW$$

or

$$dS = \kappa(\theta - S)dt + \sigma S dW.$$

It is possible to show that the former case can guarantee positivity subject to a constraint on parameter values, whilst the latter case will guarantee positivity under all scenarios. In fact, it can be shown that if the function b takes the form

$$b(S, t) = \sigma S^{\alpha}$$

then the process will never reach zero if $\alpha > \frac{1}{2}$.

Definition 6.2 (Square Root Ornstein-Uhlenbeck process). Suppose that the price of a stock S_t satisfies the following Stochastic Differential Equation

$$dS = \kappa(\theta - S)dt + \sigma\sqrt{S}dW \tag{6.3}$$

where W is a Wiener process. Then S is said to follow an Sqaure Root Ornstein-Uhlenbeck (SROU) process.

Example 6.5 (Plotting a SROU process)**.** Sketch some sample paths for an SROU process.

Solution 6.5.

6.3 Electricity Prices

Electricity prices are quite an unusual commodity to work with when pricing derivative contracts due to the expensive cost associated with storage. This means that the price at any time of the day is highly dependent on the demand for electricity, which is largely predictable. In fact, demand shows a repeating pattern every day, with low prices overnight and high prices during early evening. However there are still some variations in this caused by small variations in demand and uncertain supply (e.g. wind). A common way to model these variations on top of a predictable process is to write down a stochastic differential equation for the *difference* between the observed price and the predicted values.

Consider the electricity price S_t can be written as

$$S_t = f(t) + X_t \tag{6.4}$$

where $f(t)$ is the predictable part of the process. The function f can be estimated by assuming that the pattern repeats itself over a 24 hour period, so that if we want $f(t = 6pm)$, we simply take an average of the electricity price at $t = 6pm$ over recent history. We can see an example of this in figure 6.1, which plots in black the average electricity price at each time of the day in the UK energy market.

Now let us model the deviations from the predicted value as an OU process so that

$$dX = -\kappa X + \sigma dW. \tag{6.5}$$

Then it can be shown that the stochastic differential equation followed by the electricity price is given by

$$dS = \kappa \left[\frac{1}{\kappa} f'(t) + f(t) - S \right] dt + \sigma dW. \tag{6.6}$$

The appearance of f' in this equation indicates that we need the function f to be differentiable (at least piecewise). A common trick here is to calculate the predicted values and then fit a smooth function through the points, using say a Fourier series. A example of this is plotted in figure 6.1 given by the red line.

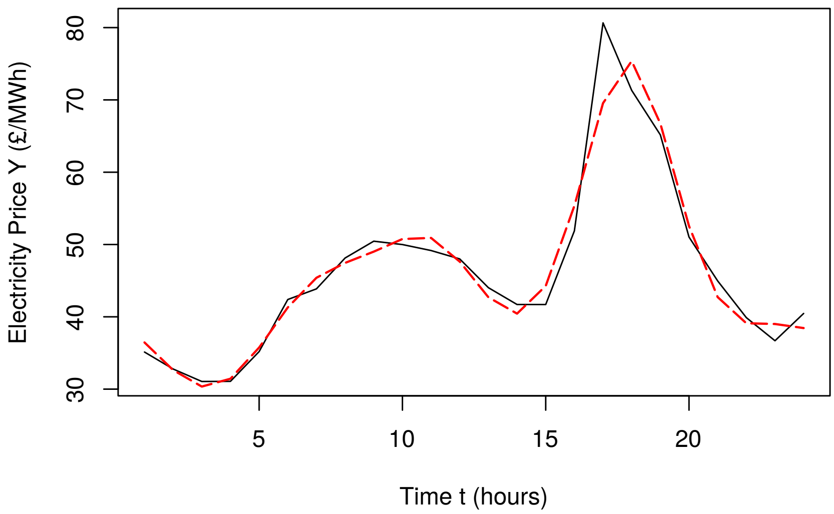

Figure 6.1: The black line show the empirical average electricity price throughout a typical UK winter day. The dashed red line shows a smooth fitted function $f(t)$.

Example 6.6 (Electricity Prices)**.** Calculate the expected value of S_t for large t.

Solution 6.6.

Lecture 7

The Black-Scholes analysis

7.1 Converting a stochastic process to a deterministic one

In the previous section we have defined a particular model for the movement of stock prices. This is by no means the only possible process used for underlying assets but is the one which is used for the Black-Scholes analysis, which still remains the most popular model for practitioners. From here we now proceed to derive the Black-Scholes PDE.

The main problem with the process followed by the function of S, F, is that there is still a random term present which makes constructing a Partial Differential Equation (PDE) somewhat problematic. The solution to this is to create a new function g which is completely deterministic. Consider a function

$$g = f - \Delta S$$

where Δ is an as yet unknown parameter which is constant across a time period dt. In which case the change in the value of g over this period is

$$dg = df - \Delta dS$$

and by substituting in the expressions for df and dS from equations (5.3) and (4.2) we obtain

$$\begin{aligned} dg &= \left[\mu S \tfrac{\partial f}{\partial S} + \tfrac{\partial f}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \tfrac{\partial^2 f}{\partial S^2}\right] dt + \sigma S \tfrac{\partial f}{\partial S} dW - \Delta[\mu S dt + \sigma S dW] \\ &= \sigma S \left[\tfrac{\partial f}{\partial S} - \Delta\right] dW + \left[\mu S\left(\tfrac{\partial f}{\partial S} - \Delta\right) + \tfrac{\partial f}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \tfrac{\partial^2 f}{\partial S^2}\right] dt \end{aligned}$$

Thus, if we choose

$$\Delta = \frac{\partial f}{\partial S}$$

then the equation reduces to one which has only deterministic variables. This is the basis of the technique employed by Black and Scholes to derive their PDE.

7.2 The Black-Scholes PDE

Notation:

- S is the current value of the underlying asset, can also be denoted by S_t especially in SDEs but the t is usually dropped.
- t is the time elapsed since the option was created and the option expires at time T.
- $V(S,t)$ is the value of generic financial contract.
- $C(S,t)$ is the value of a call option.
- $P(S,t)$ is the value of a put option.
- X is the exercise price of the option.
- σ is the volatility of the underlying asset or a measure of the uncertainty of its movements. For example, a telecommunications startup company's shares will have a higher volatility than Tesco's shares.
- μ is the drift of the underlying asset.
- r is the risk-free interest rate, the return that you would receive from a risk-free investment such as a government bond.

Example 7.1 (Black-Scholes assumptions)**.** List the main assumptions for Black Scholes analysis.

Solution 7.1.

Consider the option price $V(S,t)$ depends on the underlying asset, S, which follows geometric Brownian motion

$$dS = \mu S dt + \sigma S dW \tag{7.1}$$

and by Itô's lemma we have

$$dV = \left[\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt + \sigma S \frac{\partial V}{\partial S} dW. \tag{7.2}$$

Now construct a portfolio which consists of an option $V(S,t)$ and short in Δ of the underlying. Π is defined to be the value of the portfolio where

$$\Pi = V - \Delta S. \tag{7.3}$$

Example 7.2. Derive the famous Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{7.4}$$

Solution 7.2.

Remarks:

- This equation defines the price of **any** derivative claim on an underlying asset which follows geometric Brownian motion. The boundary conditions will determine which type of derivative we are evaluating.
- This is a backwards parabolic partial differential equation, a class of equations about which a lot more will be said below.
- Notice that by setting up the portfolio Π using what is known as the *Delta Hedge* the Black Scholes equation does not depend on the drift term μ in any way. The only parameter which needs to be empirically estimated is σ.
- The Delta (Δ) which is the rate of change of the derivative with respect to the underlying asset is a very important value.
- The linear operator
$$\mathcal{L}_{BS} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} - r$$
is a measure of the difference between the return on the hedged portfolio (Π) which are the first two terms ($d\Pi$) and the return on a bank deposit which are the last two terms. For a European option these will be the same, though they are not necessarily for an American option.
- For many types of options it is not possible to obtain closed-form analytic values but more often than not numerical procedures must be employed. In this lecture course, though, emphasis will remain on analytic solutions.

Example 7.3 (Market Price of Risk)**.** What do we do if the dW terms don't cancel out?

Solution 7.3.

Lecture 8

Formulating the mathematical problem

8.1 Classifying the PDE

For there to be no arbitrage, the option value obtained from the Black-Scholes PDE must provide a unique option price. Later it will be shown that, given suitable boundary conditions, this is indeed the case. First, in order to determine the type of boundary conditions required it is necessary to find out some general information about the PDE itself.

We know that in general a PDE with solution $u(x, t)$ of the form

$$au_{xx} + bu_{xt} + cu_{tt} + du_x + eu_t + fu = g \tag{8.1}$$

is classified depending on the sign of $b^2 - 4ac$ as follows:

- If $b^2 - 4ac < 0$ then the equation is **elliptic**.
- If $b^2 - 4ac = 0$ then the equation is **parabolic**.
- If $b^2 - 4ac > 0$ then the equation is **hyperbolic**.

The most commonly seen parabolic equation is the diffusion or heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

which typically models the evolution of heat along a bar. As they are second order in x and only first order in t parabolic equations usually require two boundary conditions in x (or S in the Black-Scholes case) and just the one in t.

Example 8.1. The heat equation requires an initial condition, where as we know Black Scholes requires a final condition. Why is that?

Solution 8.1.

It is essential to always solve parabolic equations *'in the correct direction'*.

8.2 Characteristics

The classification of PDEs in the above section is closely related to the notion of **characteristics**. Characteristics are families of curves along which information moves or across which discontinuities may occur. The trick is to attempt to write the derivative terms in the PDE in terms of directional derivatives reducing the equation to one which behaves like an ODE along these characteristic curves.

Definition 8.1 (Characteristic curve). A curve Γ is a characteristic for a general second order PDE if, for a general PDE in x and t,

$$\frac{\partial t}{\partial x} - \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = 0$$

along Γ.

Example 8.2 (Characteristic curve)**.** Determine the characteristic curve for the heat conduction equation.

Solution 8.2.

8.3 Boundary conditions for the Black-Scholes equation

Returning to the Black-Scholes equation, for each particular type of option we will require the following boundary conditions:

$$\begin{aligned} V(S,t) &= V_a(t) \quad \text{on} \quad S = a \\ V(S,t) &= V_b(t) \quad \text{on} \quad S = b \\ V(S,t) &= V_T(S) \quad \text{on} \quad t = T \end{aligned}$$

where $V_a(t)$ and $V_b(t)$ are known functions of time and $V_T(S)$ is, correspondingly, a known function of the underlying asset price. To demonstrate how to do this for different types of options we'll consider three cases: the standard European call and put options and a cash-or-nothing call option.

European call option, $C(S,t)$**:**

The most straightforward of the conditions to determine is the final condition $C(S,t=T)$ as this is the known payoff for the call option, $(\max(S-X,0))$, hence

$$C(S,T) = \max(S-X,0). \tag{8.2}$$

The conditions for specific values of S are also reasonably straightforward. Note that from the process followed by S, namely

$$dS = \mu S dt + \sigma S dW$$

if $S=0$ then $dS=0$ and , hence, the underlying asset remains at 0 from then on. Hence for a call option, however small the strike price X is, this scenario will always result in the option being worthless, hence

$$C(0,t) = 0 \tag{8.3}$$

For large S the situation is not as clear and there are three standard conventions (of which two are provided here for brevity). As $S\to\infty$ then clearly the call option is more and more likely to be exercised and in comparison to the size of S, X will be small and so one can simply use

$$C(S,t) \to S \quad \text{as} \quad S\to\infty.$$

However, the S boundary conditions are more important when dealing with numerical procedures where a large, but finite, limit is put on S ($S_{\max}$ say). In which case, more accurate conditions are required.

Example 8.3 (Boundary Conditions)**.** Determine a more accurate boundary condition for the call option.

Solution 8.3.

European put option, $P(S,t)$**:**

The case for a put option is far more straightforward. Again determining the final condition is trivial as a result of the discussion in Chapter 1, so we have

$$P(S,T) = \max(X - S, 0). \tag{8.4}$$

The conditions for particular values of S are extensions of the above arguments for calls, only more routine. When $S = 0$ at a particular time then by the nature of the underlying process then it will stay at 0 until expiry. Hence the put option will definitely be exercised and thus worth $X - 0 = X$ at expiry. A guaranteed amount of money, in this case X, to be received at time T is worth $Xe^{-r(T-t)}$ at time t and hence

$$P(0,t) = Xe^{-r(T-t)} \tag{8.5}$$

As S becomes very large then the put options will certainly not be exercised as S will be much larger than the exercise price X and so

$$P(S,t) \to 0 \quad \text{as} \quad S \to \infty. \tag{8.6}$$

As before the most important conditions are the final ones, but the other conditions are essential for numerical schemes as well as giving us more information about the option prices.

8.4 Formulating the Problem for Arbitrary Options

Cash-or-nothing/binary options:

Cash-or-nothing call (put) options (denoted $CC(S,t)$ or $CP(S,t)$) are options where, at expiry, if the underlying asset price is above (below) a certain strike price, X, then the holder receives a pre-designated cash amount A, whereas if it is below (above) this amount the holder receives nothing. Hence at expiry, $t = T$, the final condition for a cash-or-nothing call is

$$CC(S,T) = A\mathcal{H}(S - X)$$

where $\mathcal{H}(.)$ is known as the **Heaviside function**. The Heaviside function is defined as follows

$$\mathcal{H}(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

and will be important when solving PDEs later in the course. Cash-or-nothing options are a special type of option in that their payoff is completely discontinuous yet it is still possible to find an option value for them.

Example 8.4 (More Boundary Conditions)**.** Determine the boundary conditions for the cash-or-nothing call option.

Solution 8.4.
