Lecture 4

Model of stock price movements

In order to value more complex products than forward and futures contracts we will have to use stochastic processes in an attempt to accurately mirror the real life movements of underlying asset prices. Fortunately, although dealing with stochastic variables it is often possible to transform the problem into a deterministic one. In this case it is achieved through the employment of Itô's lemma, which can be seen to be the analogue of Taylor's theorem for stochastic calculus.

4.1 Efficient markets and Markovian processes

Most of modern option pricing theory is based on the **efficient market** hypothesis. The hypothesis states that all the data available about a particular company or commodity is reflected in the current price. So it is impossible to gain an edge by having studied the historical data or by examining in intricate detail company reports or the financial press. This means that any movements in underlying price will be unpredictable (i.e. random and without memory).

Obviously to come to such a conclusion empirical experiments will have to be made to check that increments in stock prices are random. Unsurprisingly this is the case although there is a lot of academic dispute about the true efficiency of markets.

Example 4.1 (Technical Analysis by Machines)**.** If we employ machine learning techniques to analyse data, can we expect to make a profit?

Solution 4.1.

The idea of a process randomly evolving without memory fits very nicely into the theory of basic random processes. **Markov** processes are processes which have no memory, in that whatever movement or information has occurred before a certain time in the process, has no impact on where its next movement will be.

Example 4.2 (A Random Walk)**.** One of the simplest random processes, which also happens to be Markovian, is the simple symmetric discrete random walk. Write down a process S_t that can move up or down (same distance, with equal probability) at fixed intervals of time.

Solution 4.2.

4.2 Brownian motion and the model of stock price movement

This discrete random walk is one possible way of modelling stock price movements. It is, however, very simplistic and only works for discrete time. The latter of these problems can be easily overcome by using its continuous time analogue: Brownian motion, or as it's sometimes referred to, the Wiener process.

The stochastic model of Brownian motion was, obviously, defined to mirror the movement of tiny particles in water but has applications in more fields than that, one being in option pricing theory. There is a lot of rigorous mathematics surrounding such processes but as regards this course a heuristic overview will be provided.

Definition (Brownian motion) A real valued stochastic process W_t is a Brownian motion (or Wiener process) under a probability measure P if

1. For each $t \geq 0$ and $s > 0$ the random variable $W_{t+s} - W_t$ (often termed dW) is distributed Normally with mean 0 and variance s.

2. For each n and for any times $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$ the random variables $\{W_{t_r} - W_{t_{r-1}}\}$ are independent.

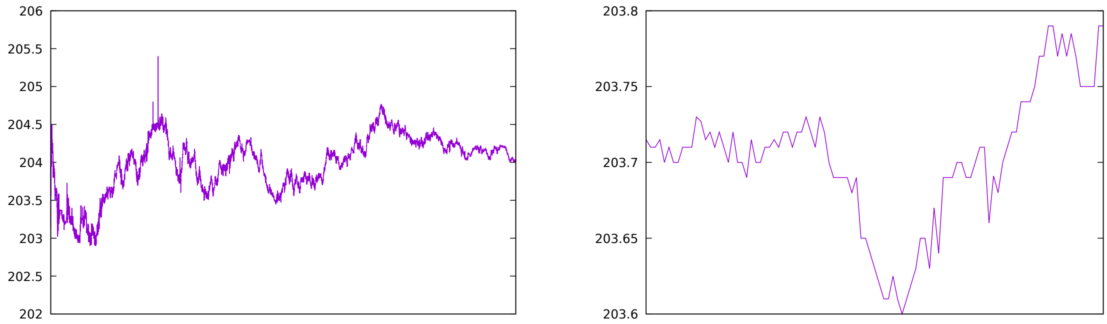

Figure 4.1: The figure on the left shows markets prices from around 8 hours of trading 8am until 4pm. The zoomed in graph on the right shows around 2 minutes of trading around 11:30am.

3. $W_0 = 0$ (this is merely a convention, it can start from any point).
4. W_t is continuous in $t \geq 0$.

This is basically just an extension of the discrete simple random walk to continuous time. The change $W_{t+s} - W_t$ over a very small period of time dt is often denoted by dW and obviously is distributed accordingly(mean of zero and variance of dt). Brownian motions obviously have very strange paths and, in fact, the expected length of path followed by W in a any time interval is infinite, this will make calculus difficult on Brownian motions (see Lecture 5).

4.2.1 Intraday Data

One way of understanding dW is to see it as $\epsilon\sqrt{dt}$ where ϵ is distributed normally with a mean of 0 and variance of 1. The standard Brownian motion W_t will model stock prices quite well on very short timescales.

In figure 4.1 we see some real trading data over quite short time periods of around 8 hours and 2 minutes. For the 2 minute data, we can see that the stock goes up and down frequently like our discrete random walk model in Example 4.2. A statistical analysis would show this data to be distributed approximately normal.

Example 4.3 (Stock Model)**.** What are the problems associated with such a model over longer timescales?

Solution 4.3.

4.2.2 Generalised Brownian motion

On top of the random increments generating by the Brownian motion term it is possible to add in deterministic terms. When dealing with stock prices there is a general upward drift, call this μ and so if the stock price is denoted by S_t then we have the following **stochastic differential equation**

$$S_{t+dt} - S_t = dS = \mu dt + \sigma dW \tag{4.1}$$

where σ is just scaling the effect of Brownian motion. So, in this case over a period of time dt the stock price increases from S_t by an amount μdt plus an unknown amount σdW where dW is a Brownian motion.

Example 4.4. What is the distribution of the stock price increases, dS?

Solution 4.4.

This is an improvement on Brownian motion but still has the problem that there is nothing to prevent S_t from dropping below zero.

4.2.3 Geometric Brownian motion

To over come this adapt the above process ever so slightly to

$$dS = \mu S dt + \sigma S dW. \tag{4.2}$$

This is saying that both the deterministic and random terms are scaled depending on the size of S at time t. The larger S is the bigger, on average, its movements are. This makes sense as a share with price \$3 is more likely to move by a cent than one worth 2c. More importantly, $S \geq 0$ as as soon as $S = 0$ then the process remains there as $dS \equiv 0$

This process does not give rise to increments which are distributed Normally but rather ones which are distributed **lognormally** a point seen on Examples 2.

Example 4.5 (Fundamental Value)**.** Stocks are said to have a fundamental value that can be inferred from the available financial reporting data (dividends, leverage ratio, income to earnings ratio etc), how does our random walk model fit in with this?

Solution 4.5.

Lecture 5

Basics of Stochastic calculus and Itô's lemma

The usual way of approximating derivatives is to use a Taylor expansion. Consider a function of the stock price $f(S)$ and look at the change in value of f over a small change in S, δS

$$f(S+\delta S) = f(S) + \delta S \frac{df}{dS} + \frac{1}{2}(\delta S)^2 \frac{d^2 f}{dS^2} + O((\delta S)^3). \tag{5.1}$$

Usually, as $\delta S \to 0$ then the $(\delta S)^2$ term disappears enabling the usual representation of $\frac{df}{dS}$ as

$$\lim_{\delta S \to 0} \frac{f(S+\delta S) - f(S)}{\delta S}.$$

However in this case we have

$$dS^2 = \mu^2 S^2 dt^2 + 2\mu\sigma S^2 dt dW + \sigma^2 S^2 dW^2$$

but as dW is a random variable then it clearly has some variance hence $E[dW^2] \geq 0$, in fact $E[dW^2] = dt$ and so this term will not disappear as $dt \to 0$. This means that it is not possible to perform calculus on stochastic variables in the same way as it is for deterministic variables. In order to overcome this we need to refer to the work of Japanese mathematician Kiyoshi Itô.

There is a huge amount of theory behind Itô calculus but we shall refer only to the main results and most of the explanation will, hence, be heuristic. For a better treatment see the books by Neftci or, better, Etheridge, alternatively attend other (probability) modules.

5.1 Itô's Lemma

Definition 5.1 (Itô's Lemma:). If we have the standard stochastic differential equation

$$dS = a(S,t)dt + b(S,t)dW$$

and $F = f(S,t)$ then if f is twice continuously differentiable on $[0,\infty) \times \mathbf{R}$ then F is also a stochastic process given by

$$dF = \left[a(S,t)\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}b^2(S,t)\frac{\partial^2 f}{\partial S^2} \right] dt + b(S,t)\frac{\partial f}{\partial S} dW \tag{5.2}$$

(Note: the process for S can also be described in its integral form

$$S = S_0 + \int_0^t a(S,s)ds + \int_0^t b(S,s)dW$$

where again the problem in evaluation comes with the random dW term, a problem which can be overcome by defining the Itô integral which is, obviously, closely linked to Itô's lemma.)

Example 5.1 (Itô's Lemma)**.** For a continuous n-times differentiable function $f(S,t)$, perform a Taylor series expansion in two dimensions and show that if $dt \to 0$,

$$dW^2 \to dt \quad \text{and} \quad dtdW = o(dt)$$

then we can derive Itô's lemma.

Solution 5.1.

Example 5.2 (Itô's Lemma)**.** If $dS = adt + bdW$ where a and b are constants then what process is followed by $G = S^2$?

Solution 5.2.

In our particular case of stock price movements we have the particular case where $a(S,t) = \mu S$ and $b(S,t) = \sigma S$ and so the process followed by any function (satisfying the Itô conditions), $F = f(S,t)$ will be as follows:

$$dF = \left[\mu S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right] dt + \sigma S \frac{\partial f}{\partial S} dW \tag{5.3}$$

What is the importance of this result? Well it is clear that the option price, or any derivative price, is a function of the underlying asset and time. This above notation has enabled us to define the process followed by any function of these variables (within very broad constraints). This is a crucial building block for the derivation of the Black-Scholes partial differential equation.

5.2 Multi-Factor Models

There are many circumstances where single factor models are not flexible enough to capture all features of the market, or possibly we have financial contracts that depend on multiple assets. Multi-factor models are common in option pricing where stochastic volatility models have been introduced to better fit the market data for option prices, which will **not** in turn necessarily improve the model for stock prices. To deal with these more complex models, we will need to know how Itô's lemma behaves when we are faced with more than one stochastic differential equation.

Definition 5.2 (Itô's Lemma Extended:). Let $W^1, W^2, \dots, W^n$ be n correlated Weiner processes and $\boldsymbol{X} = (X^1, X^2, \dots, X^n)$ be a vector of stochastic processes such that each X^i satisfies the stochastic differential equation

$$dX^i = a^i(X,t)dt + b^i(X,t)dW^i.$$

Suppose $F = f(\boldsymbol{X})$, and $f : \mathbf{R}^n \to \mathbf{R}$ is twice continuously differentiable then F is also a stochastic process given by

$$dF = \sum_i^n \frac{\partial f}{\partial X^i} dX^i + \frac{1}{2} \sum_i^n \sum_j^n \frac{\partial^2 f}{\partial X^i \partial X^j} b^i b^j \rho_{ij} dt \tag{5.4}$$

where ρ_{ij} describes the linear correlation between the Weiner processes.

Note here that if we set $X^1 = S$, $a^1 = \mu S$ and $b^1 = \sigma S$ for the first process, and then $X^2 = t$ with $a^2 = 1$ and $b^2 = 0$, we will obtain the original Itô's lemma back. In fact if any deterministic process can be included in such a way by setting $b^i \equiv 0$.

Example 5.3 (Multi Asset Processes)**.** Assume that there are n assets satisfying the following SDE:

$$dS^i = \mu^i S^i dt + \sigma^i S^i dW^i.$$

The dW^i are the usual Brownian motions and as such

$$E[dW^i] = 0, \qquad E[(dW^i)^2] = dt$$

with the additional rule that

$$E[dW^i dW^j] = \rho_{ij} dt$$

where $-1 \leq \rho_{ij} = \rho_{ji} \leq 1$. For a continuous, differentiable function $f(S^1, \dots, S^n)$, determine the process followed by $F = f(S^1, \dots, S^n)$.

Solution 5.3.