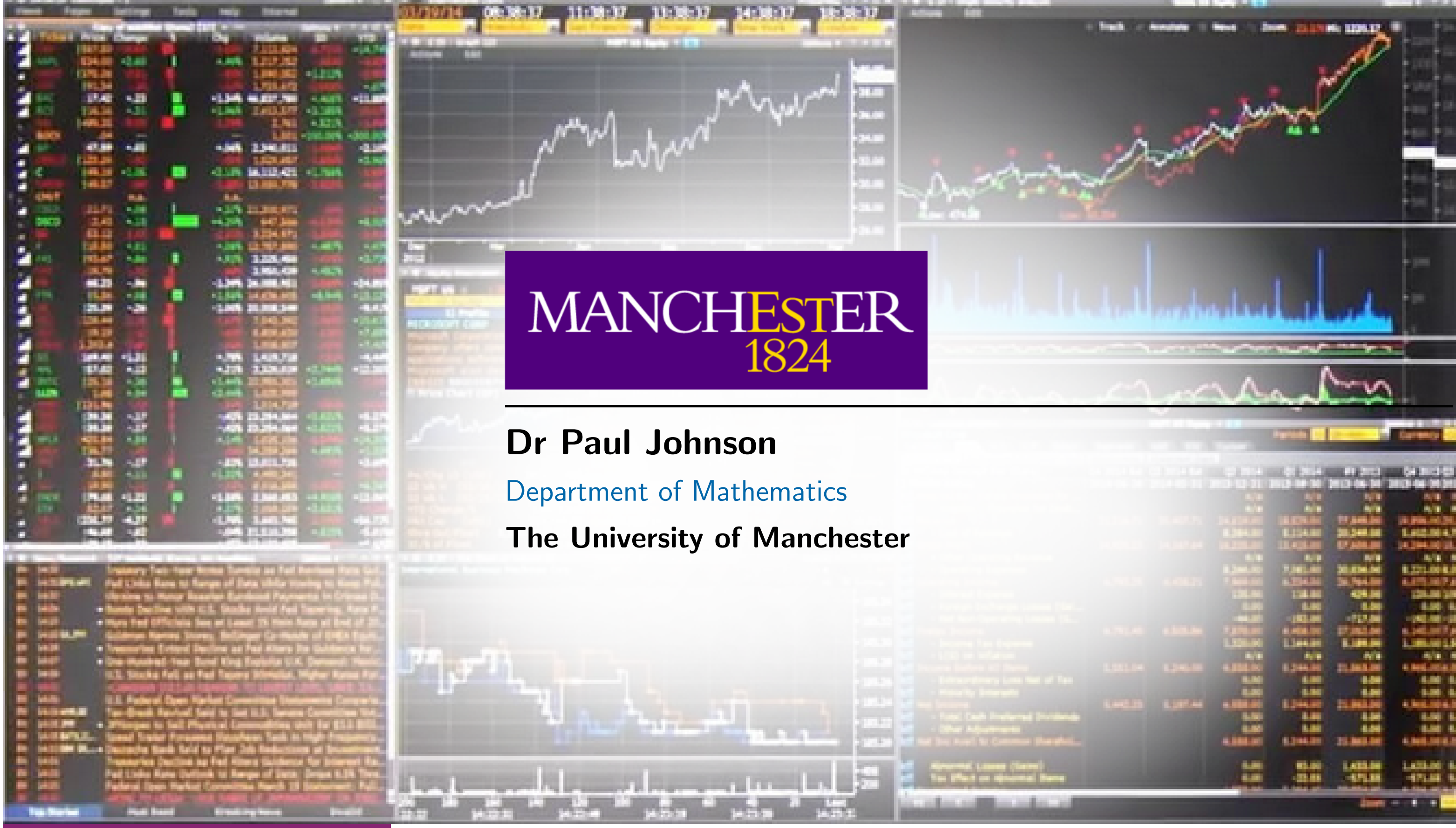

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Dr Paul Johnson

Department of Mathematics

The University of Manchester

MATH39032: Mathematical Modelling of Finance

Semester 2 2020

Course Outline

MATH39032 Mathematical Modelling of Finance

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Getting in Contact

- If you need to see me in person come along during my office hour (Tuesday 11:30-12:30) or catch me in one of the examples classes (there are 2 per week).
- Preferred method of answering queries is to use the Blackboard forum (so that everyone can see the answer and I don't get asked same questions again and again). I am automatically notified every time someone posts and will try answer queries here within one working day.
- When answering emails I will normally post the query and response on the blackboard forum (anonymised) or point you towards a similar question on the forum.

Assessment

- This module is entirely assessed by a final examination (2 hours). The past three years exam papers will provide you with a good overview of the standard of questions and also the topics which will be covered. These may be obtained closer to exam time on the course website.
- The exam rubric will be identical to last year (2019 exam): "*Answer all 6 questions.*".

Lectures

- Mondays 1pm:- Rutherford Theatre, Schuster Building
  Located on the ground floor of the Schuster Building
- Wednesday 11am:- Theatre A, University Place
  Located on the third floor of University Place

Support

There will be **8 examples sheets** to accompany the course. The first feedback/examples class will be in Week 2 (on Tuesday 6th February). I would strongly encourage you to attend these classes - they will be a useful forum for feedback. I would also encourage you to attempt the questions on the sheets prior to the classes.

Syllabus

1. Introduction to options, futures, no arbitrage principle [3]
2. Models for stock prices, basics of stochastic calculus and Itô's lemma. [3]
3. Deriving the the pricing partial differential equation, and the assumptions behind it. Formulating the mathematical problem. Analytic solutions and Implied volatility. [3]
4. Connection with the heat conduction equation, solution of the heat conduction equation - similarity solutions and the Dirac delta function. Derivation of the price of European options. [3]
5. Extension to consider options on assets paying dividends and multi-factor models. [4]
6. American options and free boundary problems. [2]
7. Interest-rate models and bonds. [2]
8. Barrier options. [2]

Learning outcomes

Upon completion of this unit, students should be able to:

I recognise the role that financial derivatives play in reducing risk

II construct payoff diagrams for standard options (and portfolios of options)

III construct a PDE, using the concepts of stochastic calculus and hedging

IV solve analytically the standard Black-Scholes equation

V use the Black-Scholes formulae to evaluate fair prices for European options

VI extend the basic European option model (to include dividends and/or early exercise) and where possible to solve the resulting models analytically

Recommended Texts

Text books:

- The best book for this course is still:

  Wilmott, P., Howison, S., Dewynne, J., 1995: The Mathematics of Financial Derivatives, Cambridge U.P. ISBN: 0521497892

- Alternatively, as an introductory text to the area:

  Wilmott, P., 2001: Paul Wilmott Introduces Quantitative Finance, 2nd Edition, Wiley. ISBN: 0471498629.

- For a very detailed (and expensive) look at mathematical finance:

  Wilmott, P., 2000: Paul Wilmott on Quantitative Finance, Wiley. ISBN: 0471874388

- There are more probabilistic ways of approaching the area (as considered in other modules) and for those seeking to obtain a full knowledge of the area, including more on stochastic processes and Martingale theory, these courses are highly recommended. Some introductory books for stochastic calculus as applied to finance are:

  Etheridge, A., 2002: A Course in Financial Calculus, Cambridge U.P. ISBN: 0521890772

  Neftci, S. N., 2000: An Introduction to the Mathematics of Financial Derivatives, 2nd Ed., Academic Press. ISBN: 0125153929

- For a more financial look at options and derivatives the following is excellent and is the course text for finance students (usually MBA or PhD) studying derivatives:

  Hull, J. C., 2002: Options, Futures and other Derivatives, 5th edition, Prentice Hall. ISBN: 0130465925.

- For a readable book on Stochastic Finance:

  Higham, D.J. 2004: An introduction to financial option valuation. Cambridge University Press. ISBN 0521 54757 1 for paperback and ISBN 0521 83884 3 for hardback.

General interest books:

- One description (not the best, as the best one is, sadly, out of print) of how the Nobel prize winning academics (whose work underpins this course) *tried* to make money from their theories is:

  Lowenstein, F., 2002: When Genius Failed: The Rise and Fall of Long Term Capital Management, Fourth Estate. ISBN: 1841155047.

- For a very readable discussion about investment banks using and abusing derivatives:

  Partnoy, F., 1998: F.I.A.S.C.O: Guns, Booze and Bloodlust: the Truth About High Finance, Profile Books. ISBN: 1861970773.

  Partnoy, F., 2004: Infectious Greed, Profile books. ISBN: 1861974736.

- The original description of what it's really like working and making money on Wall Street was the following:

  Lewis, M., 1999: Liar's Poker, Coronet. ISBN: 0340767006.

- For those of you who are interested in the history of modern finance theory and the major players, I thoroughly recommend:

  Bernstein, P., 1995: Capital Ideas: The Improbable Origins of Wall Street, The Free Press. ISBN: 0029030129.

  Finally, Peter Bernstein has also written an excellent book on risk and its origins:

  Bernstein, P., 1998, Against the Gods: The Remarkable Story of Risk, Wiley. ISBN: 0471295639.

Lecture 1

Introduction

Mathematics has myriad applications in the world of finance and as such the title of this module may be a little broader than what is actually studied. The main focus of this course is on the financial instruments known as **options** and, most importantly, how to calculate their value.

This proved to be a remarkably interesting problem both mathematically and financially and one which took centuries to satisfactorily solve. The earliest known use of options was by the Greek philosopher Thales in 600 B.C. who used them to make money from his predictions about the harvest, in this example the price of olives is the **underlying asset**. Since then options have been traded the world over although rarely in a regulated manner and, until 1973, their values were primarily calculated by guesswork.

In 1973, Fischer Black and Myron Scholes, together with help from Bob Merton derived the **Black-Scholes partial differential equation** which describes the value of an option, V (which is dependent on the time since the option had been sold, t, the value of the underlying asset, S, the interest rate, r, and the volatility of the underlying asset, σ) together with an appropriate set of boundary conditions, as follows

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{1.1}$$

This equation changed the face of option pricing, not only did it earn Nobel prizes for Merton and Scholes in 1997 (Black having died in 1995) but it paved the way for an explosion in the trading of options and other **derivative** products (an option is a type of financial derivative and you'll see more about these shortly). The first organised options exchange also opened in Chicago in 1973 and the volume of trade in options has increased from 5.7m contracts in 1974, to 673m in 2000, 3,899,068,670 in 2010, 4,111,275,659 in 2013, 4,265,368,807 in 2014. During the month of August last year (2019) over 35 million contracts were traded on the Chicago Board Of Exchange alone (just one of hundreds of exchanges around the world). This boom in the, occasionally mathematically complex, derivatives markets has also led to many investment banks actively recruiting skilled mathematicians and physicists to help value such products.

1.1 Terminology

Definition 1.1 (Derivatives). In a financial sense a derivative is a product whose value is *derived* from the price or value of another product.

This is normally an underlying asset, such as a stock or share (Marks and Spencer shares, Parmalat shares etc.), a commodity (oil, gold, tin etc.), an exchange rate (Euro to Sterling etc.). The most common types of derivative products are forwards, futures and options.

Underlying assets

Throughout this course we will be considering options on underlying assets, the value of which is denoted by S in the Black-Scholes equation (equation (1.1) above). This underlying asset is usually assumed to be a share price, but can also be a commodity price or an exchange rate. Underlying assets have an associated **drift** (μ) and **volatility** (σ), where the drift is the expected percentage increase over a certain period of time and the volatility is the measure of uncertainty of this return. For example, one would expect a fledgling technology share to have a higher volatility than a blue-chip company like AT&T.

Interest rates and the time value of money

The famous mantra from courses on economics and finance is 'a dollar today is worth more than a dollar tomorrow', as it is possible to invest your dollar in a risk-free investment, like a US government bond, today and tomorrow it will be worth more than a dollar. There are a few possible conventions as to how much money is worth after a certain amount of time. Assume a time scale of 1 year, given a yearly interest rate, r, then if A is invested today, at the end of the year it will be worth $A(1+r)$. However, if it is invested for only 6 months at the same quoted yearly rate and the new total is then invested for another 6 months we have a compounding process such that after 1 year A will be worth $A(1+\frac{r}{2})^2$. This can be extended to the continuously compounded case, which is used throughout this course, in which the money is reinvested m times giving $A(1+\frac{r}{m})^m$ after one year. As $m \to \infty$ then

$$(1+\frac{r}{m})^m \to e^r. \tag{1.2}$$

Thus, an amount, A, invested at a continually compounded rate of r for t years is worth Ae^{rt}. This convention is used primarily because it makes the mathematics far simpler than using the cumbersome discretely compounded formula. This time value of money is mainly used in its inverted form to determine what an expected amount in the future is worth today. This is known as **discounting**.

Example 1.1 (Discounting)**.** An investor is going to receive \$100 at some time in the future T, how much is it worth at time t?

Solution 1.1.

For more realistic models, where the interest rate can be a function of time, an amount A invested for t years is worth

$$Ae^{\int_0^t r(t)dt}.$$

Interest rates and discounting is used extensively in developing option pricing techniques.

Definition 1.2 (Bonds). A bond is a debt guarantee in which the holder pays now to receive a *guaranteed* (in as much as anything can ever be guaranteed) fixed payment in the future. The return on a bond is given by

$$\frac{dB}{dt} = rB.$$

1.2 Forwards and Futures

Although it is always possible to buy a share or commodity today or to exchange currency at a particular rate, investors or companies often want to arrange a deal for some time in the future. A

forward contract is one in which one party (in the **long position**) agrees to buy an underlying asset at a certain price (the **delivery price** F) at a certain future time, T. The other party (in the **short position**) agrees to sell the asset at time T at this price F.

A **futures contract** is a standardised forward contract in that parties can enter into long or short positions on *an exchange* where the delivery prices and dates are set by the exchange. In a futures contract the opposing parties (long and short) do not necessarily know each other and so the exchange ensures that the contracts are honoured.

Definition 1.3 (Futures Contract). The futures contract $F_{t,T}$ is the market agreed purchase price at time t to deliver the underlying asset S to the holder at time T.

Example 1.2. Why would anyone want to use such a derivative? Suppose Easyjet know that on 14th September 2020 they will have to pay €1 million to various airports in Europe. They are worried about exchange rates after Brexit – what can they do? Look on the European Options exchange for a suitable contract.

Solution 1.2.

In general, if the underlying asset has value S_t at time t then the **payoff** at the delivery time, T, to the party in the long position is

$$S_T - F$$

where F is the delivery price. Similarly, for the party in the short position the payoff is

$$F - S_T$$

The key thing here is that we have not yet determined what would be a suitable choice of value for F; this is discussed shortly after reminding ourselves of the definition for an option.

Example 1.3. Sketch the payoff for the long position and the short position of a futures contract.

Solution 1.3.

Figure 1.1: Payoff from the **long** position in a futures contract

See figure 1.1 and figure 1.2.

Figure 1.2: Payoff from the **short** position in a futures contract

Lecture 2

Options

In Example 1.2 if the euro weakens against the pound (exchange rate goes down), then Easyjet lose out in the above forward contract. Ideally, they would like it so that if the exchange rate drops then they can walk away from the contract and buy their €1m for less than the agreed delivery price. This is exactly the freedom which an option would give them, they would have the **option** of whether to take the delivery price or to walk away and take the favourable current price. Obviously, if the exchange rate has increased then they can still pay the agreed delivery pric for their €1m. Crucially, unlike in forward contracts (which are free to enter) the party who buys the option must **pay some premium** to obtain the option. The main thrust of this course is to determine how much she should pay.

There are two principal types of options:

Definition 2.1 (Call options). A call option gives the holder the right, but not the obligation to buy the underlying, S, at (or before) a certain date, T, for a certain price, known as the exercise (or strike) price, X.

Definition 2.2 (Put options). A put option gives the holder the right, but not the obligation to sell the underlying, S, at (or before) a certain date, T, for a certain price, known as the exercise (or strike) price, X.

There are also two main genres of options:

Definition 2.3 (European options). A European option can can only be exercised at the expiration date T.

Definition 2.4 (American options). An American option can be exercised *at any time* up to and including the expiry date, T.

There are also many exotic types of options such as Asian, Russian, Parisian, Bermudan, Lookback, Barrier etc. which have different exercise conditions and are not considered fully in this course. This course mainly deals with the valuation of European options.

2.1 European Call Options

Denote the value of a European call option by $C(S,t)$ where S is the value of the underlying asset at time t. If the strike price of the option is X then at the expiry of the option, $t = T$, the holder of the option has the right, but not the obligation, to buy the underlying, of value S at $t = T$ at this price, X. Clearly if $S > X$ then the holder of the option would exercise the option and buy the underlying (worth S) for X. This would yield the holder of the option a profit of $S - X$. If $S \leq X$ then there is no point in exercising the option as the holder can buy the underlying on the market for less than X.

Hence at expiry ($t = T$) the value of *the call option* is

$$C(S,T) = \max(S - X, 0).$$

2.2 European Put Options

In a similar way to call options, denote the value of a put option by $P(S,t)$. Again the option has a strike price of X and at expiry the holder of the option has the right, but not the obligation, to *sell* the underlying asset at this price. With a put option at expiry ($t = T$) $S < X$ then the holder of the option would exercise as she can sell the underlying for more than she could on the market, and the option would then be worth $X - S$. If, however, $S > X$ the the holder of the options could sell the underlying for more than X and thus it would not be worth exercising the option.

Hence, at $t = T$ the value of a put option is

$$P(S,T) = \max(X - S, 0)$$

Example 2.1 (Option profit)**.** An investor buys a European Call option to buy 100 Hewlett Packard shares with a strike price, X, of \$95. The current stock price is \$100, the expiration date is 6 months and the cost of the call option to buy one option is \$10. Plot a graph of profit from buying the option against underlying asset value in 6 months time, you can assume interest rate $r \approx 0$.

Solution 2.1.

Figure 2.1: Profit/loss from purchasing **one** call option as in Example 2.1

See figure 2.1 for the profit from one option.

See the notes from MATH20912 for a reminder on drawing profit diagrams.

2.3 Why options?

Why is the options market such a big deal. Options appeal to three main types of investors - hedgers, speculators and arbitrageurs.

2.3.1 Options for hedging

This was how we introduced the idea of forward and option contracts. If a company or investor requires a certain amount of goods or currency in a certain amount of time then options provide insurance for cases where there are adverse market moves. It is sometimes possible to hedge against movements in a market which will affect your business. As an example, if jet fuel goes up then it costs BA more money to run their aircraft, so if they buy call options in jet fuel then if the price goes up then they make money to offset their operating losses. If the price has gone down then they're happy because their operating costs are low.

2.3.2 Options for speculating

If an investor has a hunch about which way a market is moving then he can obtain more leverage by using options.

Example 2.2 (Option Speculation)**.** Consider an investor that feels that Barclays is likely to increase in value over the next three months and has $5000 to invest. The current stock price is $20 and call options with a strike price of $25 are available for three months at the cost of $1. Consider two possible alternatives, the stock price goes up to $35 or down to $15.

Solution 2.2.

2.3.3 Options for arbitrage

The principle of arbitrage is an important one in option pricing theory and will be defined and expanded more fully later. However, an arbitrage opportunity is one in which it is possible to lock in a risk free profit. An arbitrageur will look for anomalies in the market, which by definition, exist for only short periods of time and lock in these profits. A good example is in spread betting, if one book has the spread 8-10 and another 12-14 then you can buy at 10 in the first and sell at 12 in the second and make a risk-free profit.

2.3.4 Buying from the Exchange

At the beginning of this lecture, we noted that in Example 1.2, Easyjet may prefer to have the option to cancel their futures contract if the exchange rate moves unfavourably against them. To do this, one way is to buy an options contract instead.

Example 2.3 (Options on the Market)**.** Consider again the Easyjet scenario:

1. They don't want to pay more than £860,000 for the required Euros, how much does such an option cost to buy? Look on the European Options exchange to find the cost of the portfolio.
2. Calculate the total cost of delivering €1m at maturity (you can assume interest rate $r \approx 0$) if $S_T = 0.84$ or $S_T = 0.88$?

Solution 2.3.

Lecture 3

No arbitrage principle

3.1 Trading Strategies

First consider a simple trading strategy, in which we are allowed to buy or sell stocks and bonds, and trading is only allowed at certain times. Let the trading times be a set of fixed intervals between t and T, such that $t \in \{0, \epsilon, 2\epsilon, \ldots, T-\epsilon, T\}$. Then the number of stocks and bonds held by the trading strategy will be held fixed over each interval $(\ (k-1)\epsilon, k\epsilon\ ]$. We denote

- Δ_t:= the number of shares we hold during $(k-1)\epsilon < t \leq k\epsilon$,
- N_t:= the number of bonds we hold during the same period.

Definition 3.1 (Trading Strategy). A (valid) trading strategy is a *predictable* process

$$(\Delta_t, N_t),$$

such that

$$\Pi_t = \Delta_t S_t + N_t B_t.$$

By predictable here we mean that the decisions on what is to be done can only depend on events that have already taken place, and does not require knowledge of future events. In other words it is non-anticipating.

Example 3.1. Give an example of an *anticipating* strategy.

Solution 3.1.

Definition 3.2 (Self-financing condition). A self-financing strategy is a trading strategy such that on each trading time $t = k\epsilon$:

$$\Delta_t S_t + N_t B_t = \Delta_{t+\epsilon} S_t + N_{t+\epsilon} B_t.$$

This can be loosely interpreted as any profit/loss on trading is recorded in the number of bonds that we hold.

Trading strategies can be easily extended to consider any other financial contracts, so for instance if M_t denoted the number of call option contracts we hold, we would have

$$(\Delta_t, N_t, M_t),$$

and

$$\Pi_t = \Delta_t S_t + N_t B_t + M_t C_t.$$

Definition 3.3 (Portfolio). A collection of one or more financial contracts is known as a **portfolio**.

Example 3.2 (Trading Strategies)**.** Consider the case with $\epsilon = 1$ and $T = 2$, write down (Δ_t, N_t) and Π_t for the following self-financing strategies:

1. Buy one stock at $t = 0$.
2. Buy one stock at $t = 0$, then buy another stock at $t = 1$.
3. Buy two stocks at $t = 0$, then sell one stock at $t = 1$.

Solution 3.2.

Definition 3.4 (Arbitrage opportunity). An arbitrage opportunity is a self-financing trading strategy such that:

- No initial cost: $\Pi_0 = 0$,
- Always non-negative final value: $\Pi_T \geq 0$ and
- With the *possibility* of a positive final gain: $E[\Pi_T] > 0$.

So an arbitrage opportunity is one in which it is possible to make an instantaneous, risk free profit.

Although we introduced the idea of making money from arbitrage opportunities one of the principles required for most of the option pricing methodology of this course is that arbitrage opportunities do not exist, or only appear for a very short time. Also associated with the concept of no instantaneous risk-free profit is the idea of a risk-free rate. This is the rate of return (or interest rate) that an investor receives upon making a risk-free investment, such as investing in US treasury bonds.

3.2 Determining forward prices

To determine the correct delivery or forward price on a forward contract it is necessary to invoke the ideas of no arbitrage and the risk-free rate.

Example 3.3. Consider a forward contract to purchase IBM stock, which pays no dividends, in three months time. Suppose that the current share price is \$40 and the current risk-free rate is 5%, also assume that the current delivery price is \$43. Show that an arbitrage opportunity exists.

Solution 3.3.

The principle of no arbitrage says that as soon as this opportunity arises then people like our investor here will rush in to go short on forwards. However, very few people will be willing to go long on forwards with such a high delivery price, so very quickly the price will drop to a fair level and arbitrage opportunities will vanish.

Finding this 'fair price' is simple for forward contracts. Let the current underlying asset price be S, the risk-free rate be r, the time to expiry be T and the delivery price F. We have shown that there are arbitrage opportunities if $F > Se^{rT}$ and similarly (see question in the examples sheet) it is possible to show that there are arbitrage opportunities if $F < Se^{rT}$. Hence the correct delivery price on a forward contract is

$$F = Se^{rT}$$

Note that this derivation does not assume or predict anything about the movement of the underlying asset S but is able to predict a correct value for the forward price. Unfortunately for options it is not this simple.

3.3 The put-call parity

There are a few relationships between option prices which we can determine from basic no arbitrage arguments. One of the most useful is the put-call parity which will eventually enable you to calculate the value of a European put using the value of the call. Consider, two portfolios, A and B which at $t = 0$ consist of the following:

- Portfolio A: A European call option, $C(S, t)$, with exercise price X and expiry date T; and an amount of cash Xe^{-rT}.
- Portfolio B: A European put option, $P(S, t)$, with exercise price X and expiry date T; and one share in the underlying S.

Example 3.4. Evaluate portfolios A and B at maturity to determine the following relationship

$$C(S, t) + Xe^{-r(T-t)} = P(S, t) + S. \tag{3.1}$$

where t is the current time.

Solution 3.4.

It is possible to determine some fairly loose bounds for European options using a similar approach (we went through this extensively in MATH20912) but for accurate valuation it is necessary to model the movements of the underlying asset.