MATH39032 (Mathematical modelling of finance): Examples 8

Bond and Barrier Options: Lectures 17-20

1. Consider the zero-coupon bond pricing equation (as derived in the notes):
$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u-\lambda w)\frac{\partial V}{\partial r} - rV = 0,$$
subject to the final condition $V(r,T) = Z$. If the interest rate r follows the Vasicek model
$$dr = (\eta - \gamma r)dt + \beta^{\frac{1}{2}}dX,$$
and so $u = \eta - \gamma r$, $w = \beta^{\frac{1}{2}}$ and supposing the value of the bond is given by
$$V = Ze^{A(t;T) - rB(t;T)}$$
determine $A(t;T)$ and $B(t;T)$.

2. Consider the Black-Scholes equation for an option $V(S,t)$ (in the usual notation), namely
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$
where the volatility σ and interest rate r are both constants.

   (i) By using the substitution $V(S,t) = S^\alpha V_1(S,t)$, where $\alpha = 1 - 2r/\sigma^2$, find the equation satisfied by $V_1(S,t)$.

   (ii) By using the further substitution $V_1(S,t) = V_2(\xi,t)$, where $\xi = S_d^2/S$, where S_d is constant, show that $V_2(\xi,t)$ satisfies the Black-Scholes equation.

   (iii) Explain why $C_{DO} = AV(S,t) + B(\frac{S}{S_d})^{1-2r/\sigma^2}V(S_d^2/S,t)$, where A and B are constants, must also satisfy the Black-Scholes equation.

   (iv) Consider a down-and-out call option C_{DO}, which, in the usual notation, has a payoff $\max(S-X,0)$, but is worthless if the asset value falls below a prescribed value, S_d, where $S_d < X$ (the strike price). Find appropriate values for the coefficients A and B in (iii) where $V(S,t)$ is a vanilla European call option with strike X, such that C_{DO} values a down-and-out call and confirm that the final condition is satisfied.