MATH39032 (Mathematical modelling of finance): Examples 7

American Options: Lectures 15-16

1. An American put option is a put option which can be exercised at *any* time to receive $X - S$ where X is the exercise price. A perpetual American put option is one which has no expiry date, as such its value is described by the equation

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2P}{dS^2} + rS\frac{dP}{dS} - rP = 0.$$

The boundary conditions are

$$P(S) = 0, \quad \text{as} \quad S \to \infty$$

$$P(S_f) = X - S_f$$

$$\frac{dP}{dS} = -1 \quad \text{at} \quad S = S_f$$

where S_f is the unknown value of S where for $S < S_f$ the option is exercised early and is worth $X - S$ and above S_f the option is valued using the ODE above.

(a) Search for a solution of the form

$$P(S) = AS^\alpha$$

and obtain two values of α. One of these is in contradiction to the boundary condition as $S \to \infty$, which one?

(b) Use the *two* conditions at $S = S_f$ to determine both A and S_f. Show that

$$P(S) = -\frac{1}{\alpha}\left[\frac{\alpha X}{\alpha - 1}\right]^{1-\alpha} S^\alpha$$

where α is the correct value from part (a)

2. Consider American vanilla call and put options, with prices C and P. Derive the following inequalities (the second part of the last inequality is the version of put-call parity result appropriate for American options):

(i) $P \geq \max(X - S, 0)$

(ii) $C \geq S - Xe^{-r(T-t)}$

(iii) $S - X \leq C - P \leq S - Xe^{-r(T-t)}$.