MATH39032
(Mathematical modelling of finance): Examples 5

PDE Transformations: Lectures 10-12

1. Suppose that $u(x,\tau)$ satisfies the following initial value problem on a semi-infinite interval:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad \tau > 0,$$

with

$$u(x,0) = u_0(x), \quad x > 0, \quad u(0,\tau) = 0, \quad \tau > 0.$$

Define a new function $v(x,\tau)$ by reflection in the line $x = 0$, so that

$$v(x,\tau) = u(x,\tau) \quad \text{if} \quad x > 0,$$

$$v(x,\tau) = -u(-x,\tau) \quad \text{if} \quad x < 0.$$

Show that $v(0,\tau) = 0$, and use the result that

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-(x-s)^2/4\tau} ds,$$

to show that

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{0}^{\infty} u_0(s) \left[e^{-(x-s)^2/4\tau} - e^{-(x+s)^2/4\tau} \right] ds.$$

The function multiplying $u_0(s)$ here is called the **Green's function** for this initial value problem.

2. Find similarity solutions to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + F(x), x > 0, \quad \tau > 0,$$

with

$$u(x,0) = 0, \quad x > 0, \quad u(0,\tau) = 0, \quad \tau > 0,$$

in the two cases (a) $F(x) = x$; (b) $F(x) = 1$.

Extend the case (b) by letting $u(0,\tau) = \tau$.

[Hint: In (a) and (b), look for solutions of the form $u(x,\tau) = \tau^\alpha \hat{U}(\xi)$, where α is to be determined in each case, and $\xi = x/\sqrt{\tau}$; you should obtain an ordinary differential equation for $\hat{U}(\xi)$ which is tricky to solve (but not impossible!).]

3. Suppose that a and b are constants. Show that the parabolic equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + bu$$

can always be reduced to the diffusion equation. Use a change of time variable to show that the same is true for the equation

$$c(\tau)\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

where $c(\tau) > 0$. Suppose that σ^2 and r in the Black-Scholes equation are both functions of t, but that r/σ^2 is a constant. Derive the Black-Scholes formulae in this case.