MATH39032
(Mathematical modelling of finance):
Examples 4

PDE Transformations: Lectures 10-12

1. Consider the Black-Scholes equation for an option, $V(S,t)$, in the usual notation
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$
Suppose that the risk-free interest rate $r(t)$ and the volatility $\sigma^2(t)$ are both non-constant known functions of t. Show that the following steps reduce the Black-Scholes equation to the heat-conduction equation:

   (a) Set $S = Xe^x$, $V = Xv$, and put $t = T - t'$ to show that:
$$\frac{\partial v}{\partial t'} = \frac{1}{2}\sigma^2(t')\frac{\partial^2 v}{\partial x^2} + [r(t') - \frac{1}{2}\sigma^2(t')]\frac{\partial v}{\partial x} - r(t')v.$$

   (b) Introduce a new time variable $\tau = \int_0^{t'} \frac{1}{2}\sigma^2(s)ds$, to show that
$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + a(\tau)\frac{\partial v}{\partial x} - b(\tau)v,$$
where you need to determine $a(\tau)$ and $b(\tau)$.

   (c) Consider the first-order partial differential equation obtained by omitting the $\frac{\partial^2 v}{\partial x^2}$ term, namely
$$\frac{\partial v}{\partial \tau} = a(\tau)\frac{\partial v}{\partial x} - b(\tau)v.$$
Verify that the general solution of this equation is
$$v = F(\hat{x})e^{-B(\tau)},$$
where $\hat{x} = x + A(\tau)$ and you need to determine $A(\tau)$ and $B(\tau)$, and where $F()$ is an arbitrary function.

   (d) Now consider the full second-order equation for v in (b); seek a solution of the form
$$v(x,\tau) = e^{-B(\tau)}u(\hat{x},\tau).$$
Show that u satisfies the heat-conduction equation
$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \hat{x}^2}.$$

2. Find a similarity solution to the problem
$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad \tau > 0$$

with

$$u(x,0) = \mathcal{H}(x)$$

where $\mathcal{H}(x)$ is the Heaviside function. Search for a solution of the form $u(x,\tau) = U(\xi)$ where $\xi = x/\sqrt{\tau}$.

3. **Uniqueness proof:** Suppose that $u_1(x,\tau)$ and $u_2(x,\tau)$ are both solutions to the initial value problem

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty, \quad \tau > 0$$

with

$$u(x,0) = u_0(x)$$

for $u_0(x)$ sufficiently well-behaved, and such that as $|x| \to \infty$

$$u(x,\tau)e^{-ax^2} \to 0$$

for any $a > 0$ and $\tau \geq 0$. Show, routinely, that $v(x,\tau) = u_1 - u_2$ is also a solution of the heat conduction equation above, with $v(x,0) = 0$.

Show that if

$$E(\tau) = \int_{-\infty}^{\infty} v^2 dx,$$

then

$$E(\tau) \geq 0, \quad E(0) = 0,$$

and by integrating by parts, that

$$\frac{dE}{d\tau} \leq 0;$$

Thus $E(\tau) \equiv 0$, hence $v(x,\tau) \equiv 0$ and thus, $u_1(x,\tau) \equiv u_2(x,\tau)$ as required.