MATH39032 (Mathematical modelling of finance): Examples 2

Stochastic Processes: Lectures 4-6

1. If S follows the following process

$$dS = \mu dt + \sigma dW$$

where dW is the standard Brownian motion, show that the variance of dS over a period dt is $\sigma^2 dt$. (Recall that $Var[X] = E[X^2] - E[X]^2$)

2. If $dS = \mu S dt + \sigma S dW$ and A and n are constants then find the stochastic differential equations satisfied by

   (a) $f(S) = AS$

   (b) $f(S) = S^n$

3. If $f(S) = \log(S)$ and S follows geometric Brownian motion

$$dS = \mu S dt + \sigma S dW$$

use Itô's lemma to show that the process $F = f(S_t)$ follows a generalised Brownian motion. After a period of time t what is the distribution of F and what are the expectation and the variance? What can one conclude about the distribution of S?

4. For a continuous n-times differentiable function $f(S,t)$ perform a Taylor expansion in both t and S to obtain an expression for df. If

$$dS = A(S,t)dt + B(S,t)dW$$

and as $dt \to 0$,

$$dW^2 \to dt \quad \text{and} \quad dtdW = o(dt)$$

then show that the expansion reduces to Itô's lemma, namely

$$df = \left[A(S,t)\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}B^2(S,t)\frac{\partial^2 f}{\partial S^2} \right] dt + B(S,t)\frac{\partial f}{\partial S} dW$$

5. There are n assets satisfying the following SDE:

$$dS_i = \mu_i S_i dt + \sigma_i S_i dW_i.$$

The dW_i are the usual Brownian motions and as such

$$E[dW_i] = 0, \qquad E[dW_i^2] = dt$$

with the additional rule that

$$E[dW_i dW_j] = \rho_{ij} dt$$

where $-1 \leq \rho_{ij} = \rho_{ji} \leq 1$. Assuming the same rules as in question 7 or otherwise, for a continuous, differentiable function $f(S_1, \ldots, S_n)$, determine the process followed by $F = f(S_1, \ldots, S_n)$.
Hint: perform a Taylor expansion in $S_1, \ldots, S_n$ and apply the above rules as in question 7.

6. Consider the mean-reverting process as given by

$$dX = \kappa(\theta - X)dt + \sigma dW.$$

   (a) Given the process

   $$Y = f(X, t) = e^{\kappa t} X,$$

   use Itô's lemma to find an expression for df.

   (b) Simplify your expression for df by eliminating any instances of X or Y.

   (c) Integrate both sides between 0 and T to obtain an expression for Y_T, don't try to evaluate the integral containing dW.

   (d) What is the mean and variance of X_T given X_0 is fixed?
   Hint: You may use the following result that for a known bounded function of time $g(t)$:

   $$E\left[\int_0^T g(t) dW_t\right] = 0,$$

   and

   $$E\left[\left(\int_0^T g(t) dW_t\right)^2\right] = \int_0^T [g(t)]^2 \, dt.$$