Lecture 1

Introduction
- Elementary economics background
- What is financial mathematics?
- The role of SDE’s and PDE’s

2 Time Value of Money

3 Continuous Model for Stock Price
MATH20912 Introduction to Financial Mathematics

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Assessment:

Test in week 6, 1pm Friday 8th March: 20%
2 hours examination: 80%
The exam format will change this year to be “Answer all questions”, so see the past papers on my website that have been adjusted to look as they would have done under the new format.
General Information

Lectures 2hr pw (total 21hrs):

Monday 9am-10am Stopford Building Theatre 1
Friday 12pm-1pm Stopford Building Theatre 1

Podcasts for all lectures will be available

Notes will be provided in advance of lectures. There are gaps in the notes that we will fill in on the visualiser in the lecture. It is probably helpful to print the notes off before the lecture but consistent page/example numbering should help you use written notes.

Tutorials 1hr pw starting week 2:

- 50% additional examples or detailed explanation on board
- 50% time to ask questions about examples sheets
Financial Markets are where financial contracts are bought and sold.

- **Stock Markets**, such as London Stock Exchange, etc.
- **Bond Markets**, where participants buy and sell debt securities.
- **Futures and Option Markets**, where derivatives are traded.
**Elementary Economics Background**

- **Financial Contract** is a written agreement between **two** parties to exchange payments according to some specified criteria.

- **Holder** is normally the buyer of a contract, who **pays** money at the beginning in exchange for receiving some payments at a later date.

- **Seller** holds the opposite position to the holder, which means they **receive** money at the beginning in exchange for giving out some payments at a later date.

**Example 1.1: Writing a Contract**

Draw up a contract to sell a phone for a fixed price at some future date.
Elementary Economics Background

Shares have been around for hundreds of years
as have bonds
and are still being issued today.
What is value?

Oscar Wilde
“A man who knows the price of everything and the value of nothing...”

- The market price of a contract is objective – it is the price quoted on the market.
- The values of a contract is subjective – two people may value something differently.
- We assume that the value of a contract is the price another investor would be willing to pay in exchange for the contract,
- then value and price are the same and can be used interchangeably.
Money is the circulating medium of exchange as secured by the government (and hence its citizens).

- We assume throughout that the value of the money in the future will be worth less than it is now,
- capitalistic government ensure this through monetary policy.
How do we value money in the future? There are several ways...

**Definition: Simple interest rate**
For the interest rate \( r \) the value \( V(T) \) at time \( T \) of holding \( P \) units of currency starting at time \( t = 0 \):

\[
V(T) = (1 + rT)P
\]  
(1)

where \( T \) is expressed in years.

**Definition: Compound interest rate**

\[
V(T) = \left(1 + \frac{r}{m}\right)^{mT} P
\]  
(2)

where \( m \) is the number interest payments made per annum.
**Time value of money**

**Definition: Continuous compounding**

For a constant interest rate \( r \) the time value of money under continuous compounding is given by:

\[
V(T) = e^{rT}P.
\] (3)

In the limit \( m \to \infty \), we obtain the results above since

\[
e = \lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^z.
\]

Throughout this course we assume that the interest rate \( r \) will be continuously compounded.
Write down the value of an initial investment of £100 at $t = 2$ years with:

- a simple interest rate with $r = 0.05$
- a compound interest rate with $r = 0.05$ and two payments per year
- a continuously compounded interest rate with $r = 0.05$
Investors are primarily interested in the return on investment.

**Definition: Return**

\[
\text{return} = \frac{\text{change in value over a period of time}}{\text{initial investment}}
\]  

(4)

**Example 1.3: Why Buy Shares?**

Consider an investor has £1000 to spend. They see Apple shares are trading £100 and they think it will go up to £125 by the end of the year, and Google shares are trading at £50 and they think it will go up to £65 by the end of the year. Which shares should they invest in?
A set of historical returns for the SMP-500 over nearly 50 years with trends removed. We see noisy or stochastic behaviour in the returns.
Let $S(t)$ represent the share price at time $t$. How to write a simple model for this quantity?

• Return:

\[
\frac{\Delta S}{S} = \frac{S(t + \delta t) - S(t)}{S(t)}
\]

where $\Delta S = S(t + \delta t) - S(t)$

In the limit $\delta t \to 0$:

\[
\frac{dS}{S}
\]
Return:

\[
\frac{dS}{S} = \mu dt + \sigma dW
\]  \hspace{1cm} (7)

- \(\mu dt\) is a measure of the **deterministic** expected rate of growth of the share price. In general, \(\mu = \mu(S, t)\). In simple models \(\mu\) is taken to be constant (\(\mu = 0.1 \text{ yr}^{-1} = 10 \text{ %yr}^{-1}\)).

- \(\sigma dW\) describes the **stochastic** change in the share price, where \(dW\) stands for
  \[
  \Delta W = W(t + \Delta t) - W(t)
  \]
  as \(\Delta t \to 0\)

- \(W(t)\) is a **Wiener process**

- \(\sigma\) is the volatility (\(\sigma = 0.2 \text{ yr}^{-\frac{1}{2}} = 20 \text{ %yr}^{-\frac{1}{2}}\))
Example 1.4: Modelling Return

What does this model mean in words? What should be the properties of the Weiner process to make it consistent with the real data?
Stochastic differential equation for share price

On the left we show computer simulations with $\mu = 0.08$ and $\sigma = 0.1$, on the right is the real data from yahoo finance.

$$dS = \mu S dt + \sigma S dW$$
Lecture 2

1. Random Walks
2. Properties of Wiener Process
3. Approximation for Stock Price Equation
In general to create a random walk we take successive draws from a random distribution and add them together.

In a Markov random walk, each new draw should be independent from the last.

Let a random distribution be defined as $\Delta W$, and define $W_k$ as the value of the random walk at the $k$th step, then we can write

$$W_{k+1} = W_k + \Delta W$$

or

$$W_k = \sum_{i=1}^{k} \Delta W.$$
Random Walk

- To model stocks we use a **normal distribution** as the random distribution.
- We choose that normal distribution to be

  \[ \Delta W \sim N(0, \Delta t). \]

**Example 2.1: Random Walks**

Sketch this random walk with \( \Delta t = 1 \). You can use https://www.random.org to generate random numbers.
Random Walk

Consider our discrete random walk as

$$W_k = \sum_k \Delta W$$

it is trivial to show that

$$W_k \sim N(0, k\Delta t)$$

from standard properties of normal distributions.

Obviously if we write \( t = k\Delta t \) we have

$$W(t) \sim N(0, t).$$

It can be shown in the limit \( \Delta t \rightarrow 0 \) we can write it as the integral

$$W(t) = \int_0^t dW$$

where

$$dW \sim N(0, dt).$$
The standard Wiener process $W(t)$ is a Gaussian random walk process such that:

- $W(0) = 0$
- $W(t)$ has independent increments: if $u \leq v \leq s \leq t$, then $W(t) - W(s)$ and $W(v) - W(u)$ are independent
- $W(s + t) - W(s) \sim N(0, t)$
- $W(t)$ has continuous paths
Example 2.2

Show that the following results hold:

\[ \mathbb{E}[W] = 0; \]
\[ \mathbb{E}[W^2] = t; \]
\[ \mathbb{E}[\Delta W] = 0; \]
\[ \mathbb{E}[(\Delta W)^2] = \Delta t; \]

\[ \Delta W = X (\Delta t)^{\frac{1}{2}}, \quad \text{where} \quad X \sim N(0, 1). \]
The probability density function for $W(t)$ is

$$p(y, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right)$$

and $\mathbb{P}(a \leq W(t) \leq b) = \int_a^b p(y, t) \, dt$

- Simulations of a Wiener process:
Example 2.3: Wiener process pdf

Draw the probability distribution for the Weiner process. How does this distribution change over time?
• The increment $\Delta W = W(t + \Delta t) - W(t)$ can be written as $\Delta W = X (\Delta t)^{\frac{1}{2}}$, where $X$ is a random variable with normal distribution with zero mean and unit variance: $X \sim N(0, 1)$

• $\mathbb{E}[(\Delta W)] = 0$ and $\mathbb{E}[(\Delta W)^2] = \Delta t$.

Recall: equation for the stock price is

$$dS = \mu S dt + \sigma S dW,$$

then

$$\Delta S \approx \mu S \Delta t + \sigma S X (\Delta t)^{\frac{1}{2}}$$

It means $\Delta S \sim N (\mu S \Delta t, \sigma^2 S^2 \Delta t)$
**Example 2.4**

Consider a stock that has volatility 30% and provides expected return of 15% p.a. Find the increase in stock price for one week if the initial stock price is 100.

Answer: \( \Delta S = 0.288 + 4.16X \)

**Example 2.5**

Given the result from Example 2.4, what is the distribution of the stock price in that case after one week?
Example 2.6

Show that the return $\frac{\Delta S}{S}$ is normally distributed with mean $\mu \Delta t$ and variance $\sigma^2 \Delta t$.
1. Itô’s Lemma
2. Distribution for $\ln S(t)$
3. Solution to Stochastic Differential Equation for Stock Price
4. Examples
We assume that $f(S, t)$ is a smooth function of $S$ and $t$.

Find $df$ if $dS = \mu S dt + \sigma S dW$

- Volatility $\sigma = 0$

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS = \left( \frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} \right) dt$$

- Volatility $\sigma \neq 0$

Itô's Lemma:

$$df = \left( \frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S \frac{\partial f}{\partial S} dW$$
Example 3.1: Itô’s Lemma

Find the SDE satisfied by $f = S^2$. 
Example 3.2: Itô’s Lemma

Show that the stochastic differential equation (SDE) for $f = \ln S$ is:

$$\ln(S(t)) = \ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)$$

Hint: remember that adding together normal distributions results in another normal distribution. This means that constant coefficient SDE’s can be integrated using the result

$$\int_0^t dW = W(t).$$
Now we have found that

$$\ln S(t) - \ln S_0 = \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W(t)$$

where $S_0 = S(0)$ is the initial stock price.

This means $\ln S(t)$ has a normal distribution with mean $\ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) t$ and variance $\sigma^2 t$. 
Example 3.3: Log normal distributions

Consider a stock with an initial price of 40, an expected return of 16% and a volatility of 20%.

Find the probability distribution of \( \ln S \) in six months.

Answer: \( \ln S(0.5) \sim N(3.759, 0.020) \)
Recall that if the random variable $X$ has a normal distribution with mean $\mu$ and variance $\sigma^2$, then the probability density function is

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$

The probability density function of $X = \ln S(t)$ is

$$\frac{1}{\sqrt{2\pi\sigma^2t}} \exp\left(-\frac{(x - \ln S_0 - (\mu - \sigma^2/2)t)^2}{2\sigma^2t}\right)$$
**Exact expression for stock price** $S(t)$

**Definition**

The model of a stock $dS = \mu S dt + \sigma S dW$ is known as geometric Brownian motion.

The random function $S(t)$ can be found from

\[
\ln\left(\frac{S(t)}{S_0}\right) = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W(t)
\]

Stock price at time $t$: $S(t) = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}$
Example 3.4: Exact Formula
Write down a formula for $S(t)$ in terms of the standard normal distribution.

Example 3.5: Sketching the pdf
Try to draw a sketch of the log-normal distribution for $S(t)$. 
Below we plot the lognormal distribution function for \( \mu = 0, \sigma = 0.4 \) and \( t = 1 \).

\[
\ln \mathcal{N} \left( \left( \mu - \frac{\sigma^2}{2} \right) t, \sigma^2 t \right)
\]
Financial Derivatives

European Call and Put Options

Payoff Diagrams, Short Selling and Profit
A Derivative is a financial contract whose value depends on the values of other underlying variables. Other names are financial derivative, derivative security, derivative product. A share option, for example, is a derivative whose value is dependent on the share price.

Examples: forwards, futures, options, swaps, CDS, etc.

Options are very attractive to investors, both for speculation and for hedging risk.
A Spanish farmer signs a contract with Tescos to forward sell 100 Tons of oranges that must be delivered in June 2017 for the agreed price €600 per Ton. Two possible outcomes:

- weather is good, crop yields are high, and as a result market price of oranges in June could be much lower than €600 per Ton.
- weather is bad, crop yields are low, and as a result market price of oranges in June could be much higher than €600 per Ton.

Discussion Point
How would you think the parties agree on the price of oranges in this contract?
**Example 4.1: Selling a phone in the future**

Draw a cartoon to illustrate the buying and selling of a contract outlined in Lecture 1 - Paul to sell his phone for £260 on 5th May 2019.

**Example 4.2: Who wins?**

Now on 5th May 2019, the p10 phone is trading at either £200 or £300. Who is happy in each case?

**Example 4.3: Adding in an option**

Now the holder wishes to include an option to cancel the purchase. What happens when this new contract is agreed in the two scenarios above?
**Definition**

European call option gives the holder the right (not obligation) to buy the underlying asset at a prescribed time $T$ (expiry date/maturity) for a specified (exercise/strike) price $E$.

European put option gives its holder the right (not obligation) to sell underlying asset at the expiry time $T$ for the exercise price $E$. 
Example 4.4: Payoff at expiry

Consider a three-month European call option on a BP share with an exercise price $E = 15$ ($T = 0.25$). If you enter into this contract you have the right but not the obligation to buy one share for $E = 15$ in a three months time.

What happens at expiry if:

1. The share price is £25?
2. The share price is £5?
We denote by $C(S, t)$ the value of European call option and $P(S, t)$ the value of European put option.

**Definition**

Payoff Diagram is a graph of the value of the option position at expiration $t = T$ as a function of the underlying share price $S$.

Call price at $t = T$:

$$C(S, T) = \max (S - E, 0)$$

$$= \begin{cases} 
0, & S \leq E, \\
S - E, & S > E.
\end{cases}$$
Put price at $t = T$:

$$P(S, T) = \max (E - S, 0)$$

$$= \begin{cases} E - S, & S \leq E, \\ 0, & S > E. \end{cases}$$

**Example 4.5: Payoff Diagram**

Draw the payoff diagrams for a call option and a put option.
The profit (gain) of a call option holder (buyer) at time $T$ is

$$\max (S - E, 0) - C_0 e^{rT},$$

where $C_0$ is the initial call option price at $t = 0$.

**Example 4.6: Option profits**

Find the share price on the expiry date in three months, for a European call option with an exercise price of £10 to give a gain (profit) of £14 if the option is bought for £2.25, financed by a loan with continuously compounded interest rate of 5%.
1. Portfolios: Straddle, Bull Spread, etc.
2. Bond and Risk-Free Interest Rate
3. No Arbitrage Principle
**Definition**

Short selling is the practice of selling assets that have been borrowed from a broker with the intention of buying the same assets back at a later date to return to the broker.

This technique is used by investors who try to profit from the falling price of a stock.

**Example 5.1:**

Starting from zero, create a portfolio that is long or short one share by borrowing or investing money.
**Definition**

Portfolio is a combination of assets, options and bonds.

We denote by $\Pi$ the value of a portfolio. Example:

$$\Pi = 2S + 4C - 5P.$$ 

It means that the portfolio consists of long position in two shares, long position in four call options and a short position in five put options.

**Example 5.2:**

Draw the payoff diagrams for going both long and short on both a put and call.
Option positions

\[ C(S, T) \text{ Long Call} \]

\[ P(S, T) \text{ Long Put} \]

\[ -C(S, T) \text{ Short Call} \]

\[ -P(S, T) \text{ Short Put} \]
Trading Strategies

- It is common to ask *why* an investor would invest in a strategy in which the payoff at the end is negative.
- A negative payoff implies that the portfolio will be very cheap or even negative in price.
- Investors consider **profit** and **risk**.
- Large profits are normally followed by increased the risk, with large positive (or negative) payoffs.
- Investor wishing to reduce risk are said to hedge their position.

**Hedge**
is something an investor does to **reduce** their risk. This usually means that they are protecting themselves against a large financial loss. However in Financial Mathematics a reduction in risk might mean that both large **losses and profits** are avoided, resulting in a less risky portfolio.
Straddle is the purchase of a call and a put on the same underlying security with the same maturity time $T$ and strike price $E$.

The value of portfolio is $\Pi = C + P$

- Straddle is effective when an investor is confident that a stock price will change dramatically, but is uncertain of the direction of price move.

- Short Straddle, $\Pi = -C - P$, profits when the underlying security changes little in price before the expiration $t = T$.

**Example 5.3:**

Draw the payoff diagram for going long on a straddle and short on a straddle.
**Expected Profits**

The formula for the expected profit at time $T$ is given by (again using a call option as an example)

$$ \text{investors expected profit} = E[C(S, T)] - C_0 e^{rT}. \quad (1) $$

So the expected profit at time $T$ for a portfolio of contracts is

$$ \text{investors expected profit} = E[\Pi(S, T)] - \Pi_0 e^{rT}. \quad (2) $$

**Expected Returns**

The expected return for a portfolio of contracts is

$$ \text{investors expected return} = \frac{E[\Delta \Pi]}{\Pi} = \frac{E[\Pi(S, T)] - \Pi_0}{\Pi_0}. \quad (3) $$
Example 5.4:

\[ S_0 = 40, \ E = 40, \ C_0 = 2, \ P_0 = 2. \]

Can you find the expected return if the stock price at \( T \) is given by the following tree?

\[ S_0 = 40 \]
\[ \begin{array}{c}
60 \\
20 \\
\end{array} \]
\[ p = \frac{3}{4}, \quad p = \frac{1}{4} \]

Ans: 400%
**Bull Spread**

**Bull spread** is a strategy that is designed to profit from a moderate rise in the price of the underlying security.

Let us set up a portfolio consisting of a long position in call with strike price $E_1$ and short position in call with $E_2$ such that $E_1 < E_2$.

The value of this portfolio is $\Pi_t = C_t (E_1) - C_t (E_2)$. At maturity $t = T$

$$\Pi_T = \begin{cases} 0, & S \leq E_1, \\ S - E_1, & E_1 \leq S < E_2, \\ E_2 - E_1, & S \geq E_2 \end{cases}$$

**Example 5.5:**

Sketch the payoff diagram for a bull spread.
We assume the existence of a \textit{risk-free} investment. Examples are US government bonds or deposits in a sound bank. We denote by $B(t)$ the value of this investment.

**Definition**

A \textbf{Bond} is a contract that yields a known amount $F$, called the \textit{face value}, on a known time $T$, called the \textit{maturity date}. The authorised issuer (for example, government) owes the holder a debt and is obliged to repay the face value at maturity and may also pay interest (\textit{the coupon}).
A **Zero-coupon bond** does not pay any coupons and involves only a single payment at $T$.

**Example 5.7:**
Write down the return on a risk free bond if the interest rates are constant, and calculate the value of the bond if $B(t = T) = F$ and $r$ is constant.
One of the key principles of financial mathematics is the No Arbitrage Principle.

- There are never opportunities to make risk-free profit.
- Arbitrage opportunity arises when a zero initial investment $\Pi_0 = 0$ is identified that guarantees non-negative payoff in the future such that $\Pi_T > 0$ with non-zero probability.

Arbitrage opportunities may exist in a real market. But, they cannot last for a long time.
Lecture 6

1. No-Arbitrage Principle

2. Put-Call Parity

3. Upper and Lower Bounds on Call Options
Example 6.1:
Two products are being sold on the market and they offer a payoff according to some future events at time $T$. Everyone on the market knows (and agrees on) the probabilities associated the events. Product $A$ is described by
- Pays £500 with 50% probability at time $T$
- Pays £1500 with 50% probability at time $T$
Product $B$ is described by
- Pays £0 with 99% probability at time $T$
- Pays £100000 with 1% probability at time $T$
Imagine both products are available to buy at £750, which offers the better investment?
**Example 6.2:**

Imagine there are two similar products being sold on the market, where

Product A is described by

- Pays £1000 with 100% probability at time $T$

Product B is described by

- Pays £250 with 100% probability at time $T$

Imagine the products are available to buy at £800 and £200, which offers the better investment?
Example 6.3:

Imagine there are two products being sold on the market, where Product A is described by

- Pays £1000 with 100% probability at time $T$

Product B is described by

- Pays £1000 with 99% probability at time $T$
- Pays £2000 with 1% probability at time $T$

What can we say about the price of these two products?
An **arbitrage opportunity** is a way to make a risk-free profit. Mathematically we define such an opportunity as one which yields a positive payoff (in all circumstances) with a zero investment.

Assume there exists a portfolio $\Pi(S, t)$ consisting of financial contracts that depend on a stock $S$, such that there is zero investment

$$\Pi(S, t = 0) = 0.$$ 

Then an **arbitrage opportunity** exists at time $T$ if

$$\Pi(S, T) > 0 \quad \text{for } S \in [0, \infty).$$
The key principle of financial mathematics is No Arbitrage Principle.

- Arbitrage opportunities cannot exist in the market.
- All risk-free portfolios must have the same rate of return. Let $\Pi$ be the value of a risk-free portfolio, and $d\Pi$ is its increment during a small period of time $dt$.
- Then
  \[
  \frac{d\Pi}{\Pi} = r dt,
  \]
  where $r$ is the risk-free interest rate.
- Let $\Pi_t$ be the value of the portfolio at time $t$. If $\Pi_T \geq 0$, then $\Pi_t \geq 0$ for $t < T$. 
Example 6.4:

Assume that there are two identical risk free products on the market offering to pay the holder £10 on the same date $T$ in the future. Imagine that one of those products, Product A is selling for £5 and the other Product B for £7. What happens? Well, if everyone can make a free choice to buy either product then everyone with choose to buy A, as it offers best value.

So in a market, can you say what happens to the demand (and hence price) of each product?
Now consider the situation when $V$ and $\hat{V}$ are two financial contracts (or a portfolio as in notes) with a payoff that satisfies the following condition under all circumstances (i.e. share prices)

$$V_T \geq \hat{V}_T.$$ 

at maturity $t = T$. If no-arbitrage holds, then $V_t \geq \hat{V}_t$ for all $t$.

- Note that simply taking portfolio $\Pi = V - \hat{V}$, we have $\Pi_T \geq 0$
- and therefore $V_t - \hat{V}_t \geq 0$ is true by no arbitrage.
Example 6.6:
Consider a portfolio of the form

$$\Pi = S + P - C.$$ 

Calculate the payoff at maturity and by using no arbitrage theory the so-called put-call parity:

$$S_t + P_t - C_t = E e^{-r(T-t)}.$$
This relationship between $S_t$, $P_t$ and $C_t$ is called Put-Call Parity which represents an example of complete risk elimination.

- The Put-Call Parity ($t = 0$): $S_0 + P_0 - C_0 = Ee^{-rT}$.
- It shows that the value of European call option can be found from the value of European put option with the same strike price and maturity:

$$C_0 = P_0 + S_0 - Ee^{-rT}.$$ 

**Example 6.7:**

Use this formula for a call option along with the No-Arbitrage Theorem to derive a lower bound for the call option:

$$C_0 \geq S_0 - Ee^{-rT}$$
**Example 6.8(a):**

Find a lower bound for a six month European call option with the strike price £35 when the initial stock price is £40 and the risk-free interest rate is 5% p.a.

**Example 6.8(b):**

Consider the situation where the European call option is £4. Show that there exists an arbitrage opportunity.
1. Upper and Lower Bounds on Call and Put Options

2. Proof of Put-Call Parity by No-Arbitrage Principle

3. Example on Arbitrage Opportunity
Proving via “No Arbitrage”

No-Arbitrage Principle

“No Arbitrage cannot exist in a market”

- If we set the price of something and it allows for arbitrage to exist, then the price we set is clearly wrong.
- To prove $C_t \geq 0$, first assume opposite is true

$$C_t < 0.$$  

- Then show that it leads to an arbitrage opportunity, this contradicts our no arbitrage assumption.
- Then

$$C_t \geq 0$$

must hold true.
Upper and Lower Bounds on Call and Put Options

(Examples Sheet 3):

\[ S_0 - Ee^{-rT} \leq C_0 \leq S_0 \]
\[ Ee^{-rT} - S_0 \leq P_0 \leq Ee^{-rT} \]

**Example 7.1:**

Sketch the upper and lower bounds for a call

\[ S_t - Ee^{-r(T-t)} \leq C_t \leq S_t \]

and a put option

\[ Ee^{-r(T-t)} - S_t \leq P_t \leq Ee^{-r(T-t)} \]
The value of European put option can be found as

\[ P_0 = C_0 - S_0 + E e^{-rT}. \]

Let us assume that the price of the put option \( P_0 \) is too high \textit{relative} to the portfolio on the right hand side. If this is the case then we can write

\[ P_0 > C_0 - S_0 + E e^{-rT}. \]

Now let us prove this cannot be true because it contradicts the “No-Arbitrage Principle”.

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Example 7.2:
Show that the correct portfolio for arbitrage will be

$$\Pi = C - P - S + B$$

Example 7.3:
Find arbitrage and show that since our initial assumption on the put price was false the following must hold true

$$P_t \leq C_t - S_t + Ee^{-r(T-t)}$$
Proof of Put-Call Parity

- Showing that $P_t > C_t - S_t + E e^{-r(T-t)}$ leads to arbitrage.
- Allows us to prove that $P_t \leq C_t - S_t + E e^{-r(T-t)}$ must hold.
- It is relatively trivial to demonstrate the other side.
- So let us now assume that the put option satisfies the following condition:

$$P_t < C_t - S_t + E e^{-r(T-t)}.$$
**Example 7.4:**

Show that the correct portfolio for arbitrage will be

\[ \Pi = P - C + S - B \]

**Example 7.5:**

Find arbitrage and show that since our initial assumption on the put price was false the following must hold true

\[ P_t \geq C_t - S_t + E e^{-r(T-t)} \]
Example on Arbitrage Opportunity

Taking the previous results together we have that

\[ P_t = C_t - S_t + E e^{-r(T-t)} \]

Example 7.6:

Three month European call and put options with the exercise price £12 are trading at £3 and £6 respectively. The stock price is £8 and interest rate is 5%. Show that there exists arbitrage opportunity.

1. Calculate bounds – what is too high/low?
2. Create portfolio – set to zero initially
3. Evaluate at maturity – profit guaranteed?
1. One-Step Binomial Model for Option Price
2. Risk-Neutral Valuation
3. Examples
Initial stock price is $S_0$. The stock price can either move up from $S_0$ to $S_0u$ or down from $S_0$ to $S_0d$ ($u > 1; d < 1$).

At time $T$, let the option price be $C_u$ if the stock price moves up, and $C_d$ if the stock price moves down.

**Example 8.1:**
Sketch the binominal trees for the stock and option price.

The purpose is to find the current price $C_0$ of a European call option.
One-Step Binomial Model

From the figures we can calculate

$$E[S_T] = qS_0 u + (1 - q)S_0 d.$$ 

and

$$E[C_T] = qC_u + (1 - q)C_d.$$ 

Example 8.2:
Can we use this expectation to price our option?
Now, we set up a portfolio consisting of a long position in $\Delta$ shares and short position in one call.

$$\Pi = \Delta S - C$$

**Example 8.3**

Using this portfolio show that this value for $\Delta$

$$\Delta = \frac{C_u - C_d}{S_0(u - d)}$$

gives a constant payoff $\Pi_T = K$ at maturity.
Can we price this now? With no risk in the payoff the answer is **YES**! If \( \Pi_T = K \) we have

\[
\Pi_0 = Ke^{-rT}
\]

since the payoff is fixed we just discount at the risk-free rate.

**Example 8.4**

Using this value for \( \Pi_0 \) and \( \Delta \), sub back into earlier equations and rearrange to obtain the following formula for the call option price

\[
C_0 = e^{-rT} \left[ pC_u + (1 - p)C_d \right].
\]
By no-arbitrage arguments we derive the current call option price is

\[ C_0 = \Delta S_0 - (\Delta S_0 u - C_u) e^{-rT}, \]

where

\[ \Delta = \frac{C_u - C_d}{S_0 (u - d)} \]

Alternatively, we can interpret this as a Risk-Neutral Valuation

\[ C_0 = e^{-rT} (pC_u + (1 - p)C_d), \]

where

\[ p = \frac{e^{rT} - d}{u - d}. \]
Our subjective probability of up movement $q$ does not appear in the final formula.

This is because $\Pi_T = K$ same value on up or down movement.

The value $p$ appears in the formula and can be thought of as a probability.

It is the probability implied by the market.

Fair price of a call option $C_0$ is equal to the expected value of its future payoff discounted at the risk-free interest rate. For a put option $P_0$ (or in fact any financial contract) we have the same result

$$P_0 = e^{-rT} \left(p P_u + (1 - p) P_d\right).$$
Example 8.5:

A stock price is currently $40. At the end of three months it will be either $44 or $36. The risk-free interest rate is 12%. What is the value of three-month European call option with a strike price of $42? Use no-arbitrage arguments and risk-neutral valuation.
Lecture 9

1. Risk-Neutral Valuation
2. Risk-Neutral World
3. Two-Steps Binomial Tree
Reminder from Lecture 8 . . .

- Call option price:

\[ C_0 = e^{-rT} \left( pC_u + (1 - p)C_d \right), \]

where \( p = \frac{e^{rT} - d}{u - d}. \)

**Example 9.1:**

Show that in order for No-Arbitrage to hold in a one step binomial tree model we require

\[ d < e^{rT} < u \]

and hence

\[ 0 \leq p \leq 1. \]
We interpret the variable $0 \leq p \leq 1$ as the probability of an up movement in the stock price.

This formula is known as a risk-neutral valuation.

The probability of up $q$ or down movement $1 - q$ in the stock price plays no role whatsoever! Why???
Let us find the expected stock price at $t = T$:

$$\mathbb{E}_p [S_T] = pS_0 u + (1-p)S_0 d = \frac{e^{rT} - d}{u-d} S_0 u + (1 - \frac{e^{rT} - d}{u-d}) S_0 d = S_0 e^{rT}. $$

This shows that stock price grows on average at the risk-free interest rate $r$. Since the expected return is $r$, this is a risk-neutral world.

<table>
<thead>
<tr>
<th>In the Real World:</th>
<th>In a Risk-Neutral World:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{E} [S_T] = S_0 e^{\mu T}$</td>
<td>$\mathbb{E}_p [S_T] = S_0 e^{r T}$</td>
</tr>
</tbody>
</table>
Example 9.2:
What is a Risk-Neutral World?

- Risk-Neutral Valuation:
  
  \[ C_0 = e^{-rT} \mathbb{E}_p [C_T] \]

- The option price is the expected payoff in a risk-neutral world, discounted at risk-free rate \( r \).
Now the stock price changes twice, each time by either a factor of $u > 1$ or $d < 1$. Assume that the length of the time step is $\Delta t$ such that $T = 2\Delta t$. After two time steps the stock price will be $S_0u^2$, $S_0ud$ or $S_0d^2$. 

**Example 9.3:**
Draw the two step tree for stock prices.
The call option expires after two time steps producing payoffs of $C_{uu}$, $C_{ud}$ and $C_{dd}$ respectively.

**Example 9.4:**
Write down formula for $C_{uu}$, $C_{ud}$ and $C_{dd}$ in terms of $S_0$, $E$, $u$ and $d$. 
EXAMPLE 9.5:

Draw and annotate the two step tree for option prices.

- We apply the risk-neutral valuation **backward in time**:

  \[ C_u = e^{-r \Delta t} (pC_{uu} + (1 - p)C_{ud}) \]

  \[ C_d = e^{-r \Delta t} (pC_{ud} + (1 - p)C_{dd}) \]

- Current option price:

  \[ C_0 = e^{-r \Delta t} (pC_u + (1 - p)C_d) \]
Example 9.6:

Substitution gives

\[ C_0 = e^{-2r\Delta t} \left( p^2 C_{uu} + 2p(1 - p)C_{ud} + (1 - p)^2 C_{dd} \right), \quad (1) \]

for the option price. What do \( p^2 \), \( 2p(1 - p) \) and \( (1 - p)^2 \) represent?
Finally, the current call option price is

\[ C_0 = e^{-rT} \mathbb{E}_p [C_T], \quad T = 2\Delta t. \]

The current put option price can be found in the same way:

\[ P_0 = e^{-2r\Delta t} \left( p^2 P_{uu} + 2p(1 - p)P_{ud} + (1 - p)^2 P_{dd} \right) \]

or

\[ P_0 = e^{-rT} \mathbb{E}_p [P_T]. \]
Example 9.7:
Consider six months European put with a strike price of £32 on a stock with current price £40. There are two time steps and in each time step the stock price either moves up by 20% or moves down by 20%. Risk-free interest rate is 10%. Find the current option price.
1. Binomial Model for Stock Price
2. Option Pricing on Binomial Tree
3. Matching Volatility $\sigma$ with $u$ and $d$
Continuous random model for the stock price:
\[ dS = \mu S dt + \sigma S dW \]

**Example 10.1**

Given current stock price \( S_0 \) at \( t = 0 \), what possible values of the stock price at time \( t \) in this model?
The binominal model for the stock price is a discrete time model:

- The stock price $S$ changes only at discrete times $\Delta t, 2\Delta t, 3\Delta t, ...$

- The price either moves up $S \rightarrow S_u$ or down $S \rightarrow S_d$ with $d < e^{r\Delta t} < u$.

- The probability of up movement is $q$.

**Example 10.2**

If there are $n$ steps in the tree, how many possible stock prices can we observe (at all times)?
Let us build up a tree of possible stock prices. The tree is called a binomial tree, because the stock price will either move up or down at the end of each time period. Each node represents a possible future stock price.

We divide the time to expiration $T$ into several time steps of duration $\Delta t = T/N$, where $N$ is the number of time steps in the tree.

**Example 10.3:**
Sketch the binomial tree for a stock price with $N = 4$. 
We introduce the following notations:

- $S_n^m$ is the $n$-th possible value of stock price at time-step $m\Delta t$.

Then $S_n^m = u^n d^{m-n} S_0^0$, where $n = 0, 1, 2, \ldots, m$.

$S_0^0$ is the stock price at the time $t = 0$. Note that $u$ and $d$ are the same at every node in the tree.

For example, at the third time-step $3 \Delta t$, there are four possible stock prices: $S_0^3 = d^3 S_0^0$, $S_1^3 = u d^2 S_0^0$, $S_2^3 = u^2 d S_0^0$ and $S_3^3 = u^3 S_0^0$.

At the final time-step $N \Delta t$, there are $N + 1$ possible values of stock price.
Example 10.4:
Sketch the binomial tree for a call option price with \( N = 4 \).

We denote by \( C^m_n \) the \( n \)-th possible value of call option at time-step \( m\Delta t \).

- Risk Neutral Valuation (backward in time):
  \[
  C^m_n = e^{-r\Delta t} \left( pC^{m+1}_{n+1} + (1-p)C^{m+1}_n \right).
  \]
  Here \( 0 \leq n \leq m \) and 
  \[
  p = \frac{e^{r\Delta t} - d}{u - d}.
  \]

- Final condition: \( C^N_n = \max( S^N_n - E, 0 ) \), where
  \[
  n = 0, 1, 2, ..., N, \ E \text{ is the strike price}.
  \]
The current option price $C_0^0$ is the expected payoff in a risk-neutral world, discounted at risk-free rate $r$:

$$C_0^0 = e^{-rT}E_p[C_T].$$

**Example 10.5:**
Why not use arbitrage arguments?
Assume we wish that our binomial tree model matches the statistical properties of the continuous stock price model. How to do this?

**Expected Stock Price**

- Continuous Stock Model: \( \mathbb{E}[S] = S_0e^{\mu \Delta t} \).
- Discrete Binomial Tree: \( \mathbb{E}[S] = qS_0u + (1 - q)S_0d \).

**Example 10.6:**
Combine these two results for the first equation needed to match the models.
Matching variance

Variance of the Stock Price
For the Continuous model we have:

\[ \text{var} \left[ \frac{\Delta S}{S} \right] = \sigma^2 \Delta t \]

**Example 10.7:**
Derive the variance for the binomial tree and hence the second equation.
What is Left?

- This gives us two equation for three unknowns, so what to do?
- This means free choice, a popular extra condition is $u = d^{-1}$.
- See Examples Sheet 4 for details on how to derive formula for $u$ and $d$ in terms of $\mu$, $r$, $\sigma$, and $dt$. 
Lecture 11

1. American Put Option Pricing on Binomial Tree
2. Replicating Portfolio
An American Option is one that may be exercised at any time prior to expire \((t = T)\).

We should determine when it is best to exercise the option.

It is not subjective! It can be determined in a systematic way!
Example 11.1: 
Take a look at the historical observation of a stock price $S_t$ in figure below. Can you decide when it would have been best for the holder to exercise the option?

**Figure:** A simulated plot of a stock price $S(t)$ as a function of time. Underneath we have the corresponding payoff at exercise $E - S(t)$ where $E = 10$
American Option

- So the optimal strategy must take into account all of the possible future paths.
- The American put option value can take this into account with the simple condition:

\[ P(S, t) \geq E - S \]

for all \( S \) and \( t \).

**Example 11.2:**
If \( P < \max(E - S, 0) \), then there is obvious arbitrage opportunity.
American Put Option on a Binomial Tree

We denote by $P^m_n$ the $n$-th possible value of put option at time-step $m\Delta t$.

- **European Put Option:**
  
  $$P^m_n = e^{-r\Delta t} \left( pP^m_{n+1} + (1 - p)P^m_{n+1} \right).$$
  
  Here $0 \leq n \leq m$ and the risk-neutral probability
  $p = \frac{e^{r\Delta t} - d}{u - d}$.

- **American Put Option:**
  
  $$P^m_n = \max \left\{ \max (E - S^m_n, 0), e^{-r\Delta t} \left( pP^m_{n+1} + (1 - p)P^m_{n+1} \right) \right\},$$
  
  where $S^m_n$ is the $n$-th possible value of stock price at time-step $m\Delta t$.

- **Final condition:** $P^N_n = \max \left( E - S^N_n, 0 \right)$, where
  
  $n = 0, 1, 2, ..., N$, $E$ is the strike price.
So valuing the American option is easy in binomial trees.
These formula must be applied at each step in the tree.

**Example 11.3:**
Can we still use the simplified formulas (9.4) and (9.5) to price an American option?
Example 11.4:

Evaluation of American Put Option on Two-Step Tree:

- We assume that over each of the next two years the stock price either moves up by 20% or moves down by 20%. The risk-free interest rate is 5%.

- Find the value of a 2-year American put with a strike price of $52 on a stock whose current price is $50.
The aim here is to replicate the value of call option with other financial contracts on the market.

Let us try to do this with a portfolio of stocks and bonds.

At maturity, if the portfolio matches the call we have:

$$\Pi_T = C_T = \max(S - E, 0)$$

To prevent risk-free arbitrage opportunity, the current values should be identical. We say that the portfolio replicates the option.

The Law of One Price:

$$\Pi_T = C_T \implies \Pi_t = C_t.$$
Consider replicating portfolio of $\Delta$ shares held long and $N$ bonds held short.

The value of portfolio: $\Pi = \Delta S - NB$. A pair $(\Delta, N)$ is called a trading strategy.

$\Delta$ and $N$ will be free variables of both asset price and time.

Money is conserved in the portfolio (self-financing), so you must borrow (increase $N$) to buy shares or buy bonds (decrease $N$) with proceeds from share sales.

**Task**

How to find $(\Delta, N)$ such that $\Pi_T = C_T$ and $\Pi_0 = C_0$?
Example 11.5:

For the one-step binomial tree model, can we find \((\Delta, N)\) such that \(\Pi_T = C_T\) and \(\Pi_0 = C_0\)? Hence show that

\[
C_0 = e^{-rT} \left( p C_u + (1 - p) C_d \right),
\]

(1)

where

\[
p = \frac{e^{rT} - d}{u - d}.
\]
The Black-Scholes model for option pricing was developed by Fischer Black, Myron Scholes in the early 1970’s. This model is the most important result in financial mathematics.
The Black-Scholes model is used to calculate a call price using: stock price, strike price $E$, volatility, time to expiration, and risk free interest rate.

The Black-Scholes model involves several explicit assumptions.

Over the years since the first derivation they have all been relaxed to try and make the model more realistic.

**Example 12.1:**
Can this model be used to price any financial contract?
Black - Scholes Assumptions

Assumptions

- One can borrow and lend cash at a constant risk-free interest rate.
- The stock price follows a Geometric Brownian motion with constant expected return and volatility.
- No transaction costs.
- The stock does not pay dividends.
- Securities are perfectly divisible (i.e. one can buy any fraction of a share of stock).
- No restrictions on short selling.
Basic Notation

- We denote by $V(S, t)$ the value of an option. We use the notations $C(S, t)$ and $P(S, t)$ for the value of a call and a put when the distinction is important.

Task

To derive the famous Black-Scholes Equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

- First, we set up a portfolio consisting of a long position in one option and a short position in $\Delta$ shares.
- The value of the portfolio is $\Pi = V - \Delta S$.

Task

To find the number of shares that makes this portfolio risk free.
Example 12.2:

Use Itô’s lemma to obtain

\[
d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} - \Delta \mu S \right) dt \\
+ \left( \sigma S \frac{\partial V}{\partial S} - \Delta \sigma S \right) dW
\]

for the change in portfolio value.
Can we eliminate risk?

Yes, we can choose \( \Delta = \frac{\partial V}{\partial S} \) to remove the \( dW \) terms.

**Example 12.3:**

Put \( \Delta = \frac{\partial V}{\partial S} \) into the equation for \( d\Pi \).
This choice results in a risk-free portfolio $\Pi = V - \frac{\partial V}{\partial S} S$ whose increment is

$$d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

**No-Arbitrage Principle**

The return on a risk-free portfolio **must** be $r dt$.

So we get

$$\frac{d\Pi}{\Pi} = r dt$$

**Example 12.4:**

Use the no-arbitrage principle to derive the Black Scholes equation.
Finally we obtain the Black-Scholes equation

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0
\]

- Scholes received the 1997 Nobel Prize in Economics.
- It was not awarded to Black in 1997, because he died in 1995.
- Black received a Ph.D. in applied mathematics from Harvard University.
Example 12.5:
How do we solve for the price of a financial contract?
If a PDE is of backward type, we must impose a final condition at \( t = T \). For a call option, we have
\[
C(S, t = T) = \max(S - E, 0).
\]

**Figure**: Plot of a European Call Option value against stock price.

\[
\begin{align*}
C(S, t = 0) \\
C(S, t = T)
\end{align*}
\]
1 Boundary Conditions for Call and Put Options

2 Exact Solution to the Black-Scholes Equation
Boundary Conditions

- We use $C(S,t)$ and $P(S,t)$ for call and put options. Boundary conditions are applied for zero stock price $S = 0$ and $S \to \infty$.

**Example 13.1:**
Consider a call option, if the stock price is close to or equal to zero, what is the likelihood that the option will be exercised? Hence show that the value of the option at $S = 0$ is simply

$$C(S = 0, t) = 0$$

**Example 13.2:**
Consider a call option, if the stock price is extremely large, what is the likelihood that the option will be exercised? Hence show that the value of the option as $S \to \infty$ is simply

$$C(S, t) \to S \text{ as } S \to \infty$$
Example 13.3:

In a similar way, show that the boundary conditions for a put are

\[ P(0, t) = Ee^{-r(T-t)} \]

and

\[ P(S, t) \to 0 \quad \text{as} \quad S \to \infty \]
The Black-Scholes equation

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0
\]

with appropriate final and boundary conditions has the explicit solution:

\[
C(S, t) = SN(d_1) - Ee^{-r(T-t)}N(d_2),
\]

where

\[
N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \quad \text{(cumulative normal distribution)}
\]

and

\[
d_1 = \frac{\ln(S/E) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}, \quad d_2 = d_1 - \sigma \sqrt{T-t}.
\]
**Example 13.4:**

Why does the cumulative normal distribution appear in the solution?
Call Price Premium

Actual Value
Intrinsic Value
Speculative Value
Maximum Value

$E$

$S_T$

Maximum Value
Actual Value
Speculative Value
Intrinsic Value
Example 13.5:
Calculate the price of a three-month European call option on a stock with a strike price of £25 when the current stock price is £21.6. The volatility is 35% and the risk-free interest rate is 1% p.a.
Example 13.6:

Find the limit

$$\lim_{\sigma \to \infty} C(S, t).$$

The cumulative normal distribution has the following properties:

1. \( N(-x) = 1 - N(x) \)
2. \( N(-\infty) = 0 \)
3. \( N(0) = \frac{1}{2} \)
4. \( N(\infty) = 1 \)
1. Δ-Hedging

2. The Greeks
In practice, options are traded in a market.
The price of an option is just whatever someone is offering to buy (or sell) at.
So what use is the Black-Scholes model?
Remember how the Black-Scholes model is derived, the first step is to set up portfolio with

$$\Pi = V - \Delta S$$

and choose $\Delta$ to make the portfolio risk free.

**Example 14.1: Real World Trading**

How might a trader use the Black Scholes model in a real stock market?
Let us show that \( \Delta = \frac{\partial C}{\partial S} = N(d_1) \).

**Example 14.2:**

Differentiate (13.5) with respect to \( S \) to obtain

\[
\Delta = N(d_1) + \left( SN'(d_1) - E e^{-r(T-t)} N'(d_2) \right) \frac{\partial d_1}{\partial S}
\]

for the first stage.

We need to prove

\[
\left( SN'(d_1) - E e^{-r(T-t)} N'(d_2) \right) = 0.
\]

See Examples Sheet 7.
Example 14.3:
Find the value of $\Delta$ for a 6-month European call option on a stock with a strike price equal to the current stock price ($t = 0$). The interest rate is 6% p.a. and the volatility $\sigma = 0.16$. 
**Example 14.4:**

Let us find the Delta for a European put option by using the put-call parity:

\[ S + P - C = E e^{-r(T-t)}. \]

Differentiate it with respect to \( S \) to get

\[ 1 + \frac{\partial P}{\partial S} - \frac{\partial C}{\partial S} = 0. \]

Then rearrange to get

\[ \frac{\partial P}{\partial S} = \frac{\partial C}{\partial S} - 1 = N(d_1) - 1. \]
**Example 14.5:**
Sketch the value of the call option and its corresponding $\Delta$.

**Example 14.6:**
Sketch the value of the put option and its corresponding $\Delta$. 
The option value: \( V = V(S,t; \sigma, r, T) \).

Greeks represent the sensitivities of options to a change in an underlying variable or parameter on which the value of an option is dependent.

- Delta:
  \[ \Delta = \frac{\partial V}{\partial S} \]
  measures the rate of change of option value with respect to changes in the underlying stock price.

- Gamma:
  \[ \Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{\partial \Delta}{\partial S} \]
  measures the rate of change in \( \Delta \) with respect to changes in the underlying stock price.

See Examples Sheet 7. \( \Gamma = \frac{N'(d_1)}{S\sigma\sqrt{T-t}} \).
Greeks

- Vega:

  \[ \frac{\partial V}{\partial \sigma} \] measures the sensitivity to volatility \( \sigma \).

  One can show that \( \frac{\partial V}{\partial \sigma} = S\sqrt{T - t}N'(d_1) \)

- Rho:

  \[ \rho = \frac{\partial V}{\partial r} \] measures the sensitivity to interest rate \( r \).

  One can show that \( \rho = E(T - t)e^{-r(T-t)}N(d_2) \).

The Greeks are important tools in financial risk management. Each Greek measures the sensitivity of the value of derivatives or a portfolio to a small change in a given underlying parameter.
Black-Scholes Equation and Replicating Portfolio
Static and Dynamic Risk-Free Portfolio
Replicating Portfolio

- Contracts with same payoff
- If we replicate the payoff we can find the value of that contract.
- This also allows banks to lock in risk-free profit at the point of sale

Example 15.1

What does it mean to replicate a financial contract?
The aim is to show that the option price $V(S, t)$ satisfies the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

Consider the replicating portfolio

$$\Pi = \Delta S - NB.$$

Recall that a pair $(\Delta, N)$ is called a trading strategy.

- SDE for a stock price $S(t)$: $dS = \mu S dt + \sigma S dW$.
- Equation for a bond price $B(t)$: $dB = r B dt$. 
Example 15.2:

Consider a financial contract $V_t$ written on the underlying asset $S$, use the portfolio $\Pi$ under a self-financing constraint to replicate the contract and derive the Black-Scholes equation

$$
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.
$$

(1)
Example 15.3

Assume that the risk-neutral valuation of a call option is \( C_0 = £10 \), but a trader is able to sell the option for £11. How does a trader use replication to lock in a guaranteed profit now as opposed to at maturity?
Let us remember the Put-Call Parity. We set up the portfolio consisting of a long position in one stock, long position in one put and a short position in one call both with the same maturity and strike price. Then the value of the portfolio is

$$\Pi = S + P - C.$$ 

Example 15.4:

(a) What is the payoff of this portfolio at maturity?
(b) What is the risk when holding this portfolio?

*Hint:* The risk of a portfolio is the variance of the return.
Put-Call Parity is an example of complete risk elimination when we carry out only one transaction in call/put options and underlying security.

Let us consider the dynamics risk elimination procedure.

We could set up a portfolio consisting of a long position in one call option and a short position in $\Delta$ shares.

$$\Pi = C - \Delta S.$$  

We can eliminate the random component in $\Pi$ by choosing

$$\Delta = \frac{\partial C}{\partial S}.$$
Example 15.5:

$\Delta$ is a function of $S$ and $t$, so how does this work in the real world?

This is a $\Delta$-hedging strategy! It required a continuous rebalancing of the number of shares.
Options on Dividend-Paying Stock

American Put Option
A dividend is a sum of money paid regularly (typically annually) by a company to its shareholders out of its profits (or reserves).

Example 16.1:
What happens to the stock price when a dividend is paid out?
We assume that in a time $dt$ the underlying stock pays out a dividend $D_0 S dt$ where $D_0$ is a constant dividend yield.

in our stock model, we get

$$dS = \mu S dt + \sigma S dW - D_0 S dt$$

which we can rewrite as

$$dS = (\mu - D_0) S dt + \sigma S dW.$$
Now, we set up a portfolio consisting of a long position in one call option and a short position in $\Delta$ shares.

The value is $\Pi = C - \Delta S$.

**Example 16.3:**

Show that the change in value of this portfolio in the time interval $dt$ is

$$d\Pi = dC - \Delta dS - \Delta D_0 S dt.$$
**Example 16.4:**

Using Itô’s Lemma written like this:

\[
dC = \left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \frac{\partial C}{\partial S} dS,
\]

derive the modified Black-Scholes PDE:

\[
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0)S \frac{\partial C}{\partial S} - rC = 0.
\]
Let us find the solution to the modified Black-Scholes equation in the form

\[ C(S, t) = e^{-D_0(T-t)}C_1(S, t). \]

**Task**

Prove that \( C_1 \) satisfies the Black-Scholes equation with \( r \) replaced by \( r - D_0 \).

If we substitute \( C(S, t) = e^{-D_0(T-t)}C_1(S, t) \) into the modified Black-Scholes equation, we find the equation for \( C_1 \) in the form

\[
\frac{\partial C_1}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C_1}{\partial S^2} + (r - D_0)S \frac{\partial C_1}{\partial S} - (r - D_0)C_1.
\]

The auxiliary function \( C_1(S, t) \) is the value of a European Call option with interest rate \( r - D_0 \).
Examples Sheet 8: show that the modified Black-Scholes equation has the explicit solution for the European call

\[ C(S, t) = S e^{-D_0(T-t)} N(d_{10}) - E e^{-r(T-t)} N(d_{20}), \]

where

\[ d_{10} = \frac{\ln(S/E) + (r - D_0 + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \]

and

\[ d_{20} = d_{10} - \sigma \sqrt{T - t} \]
Recall that an **American Option** is one that may be exercised at any time prior to expiry \((t = T)\).

**Example 16.5:**

How do we price an American option in continuous time?

The problem can be formulated in several ways

- **Optimal Stopping Problem:** this is popular with probability academics. The problem can be stated as the optimal time at which to exercise.

- **Variational Inequalities:** this formulation is the most robust way to formulate the problem.

- **Free Boundary:** this formulation is popular since it can means that analytic solutions can be derived in some cases.
The American put problem can be written as a free boundary problem (commonly found in fluid mechanics).

**Example 16.6:**
Write the American put option problem as a free boundary problem (commonly found in fluid mechanics).
Bond Pricing with Known Interest Rates and Coupon Payments

Zero-Coupon Bond Pricing
A bond is a contract that yields a known amount (nominal, principal or face value) on the maturity date, $t = T$. The bond may pay a coupon (interest payment) at fixed times.

If there is no coupon payment, the bond is known as a zero-coupon bond.
Let us introduce the following notation:

\( V(t) \) is the value of the bond, \( r(t) \) is the interest rate, and \( K(t) \) is the coupon payment rate.

**Example 17.1:**
Show that a bond with a time varying interest rate \( r(t) \) paying a coupon at the rate \( K(t) \) and face value \( F \) should satisfy the following equation

\[
\frac{dV}{dt} = r(t)V - K(t)
\]

with the final condition

\[
V(T) = F.
\]
Example 17.2:
Sketch the value of a bond as a function of time in the following cases

1. \( r(t) > 0 \) and \( K(t) = 0 \);

2. \( r(t) = r_0 > 0 \) and \( K(t) = K_0 > r_0F \) where \( r_0 \) and \( K_0 \) are constants;

3. \( r(t) = r_0 > 0 \) and \( K(t) \) is single one-off payment of \( K_0 \) at time \( t = T/2 \).
In continuous time we can take advantage of integration methods to write analytic solutions for bond prices.

**Example 17.3:**
Show that the solution of a zero coupon bond with $V(T) = F$ can be written as

$$V(t) = F \exp \left( - \int_t^T r(s) \, ds \right).$$
Example 17.4:

A zero-coupon bond, $V$, issued at time $t = 0$, is worth $V(t = 1) = 1$ at maturity $T = 1$. Find the bond price $V(t)$ at time $t < 1$ and $V(0)$, when the continuous interest rate is

$$r(t) = t^2.$$
The bond pricing equation is a linear first order ODE which can be solved in general using the integrating factor.

In this case the integrating factor we need is

\[ I(t) = e^{-\int^t r(s)ds}. \]

**Example 17.5:**

Using the integrating factor, solve the bond price equation with final condition \( V(T) = F \) to obtain the solution for the coupon bond

\[ V(t) = Fe^{-\int^T_t r(s)ds} + \int^T_t e^{-\int^s_t r(u)du} K(s)ds. \]
Let us consider the case when the continuous coupon payment rate $K(t) > 0$. The solution of the equation $\frac{dV}{dt} = r(t)V - K(t)$ can be written as

$$V(t) = F \exp \left( - \int_t^T r(s)ds \right) + V_1(t),$$

where

$$V_1(t) = C(t) \exp \left( - \int_t^T r(s)ds \right)$$

It can be shown that this gives the explicit solution

$$V(t) = \exp \left( - \int_t^T r(s)ds \right) \left[ F + \int_t^T K(y) \exp \left( \int_y^T r(s)ds \right) dy \right],$$

See See Examples Sheet 9 for details.
1. Measure of Future Values of Interest Rate

2. Term Structure of Interest Rates (Yield Curve)
Bonds with different maturities and face value are sold in the market.

Interest rates are not sold but **implied** from the bond price.

Recall that the solution of the zero-coupon bond is

\[ V(t) = F \exp \left( - \int_t^T r(s) \, ds \right). \]

Now let us introduce the notation \( V(t, T) \) for bond prices. Bond prices are usually quoted at time \( t \) for different value of \( T \).
Measure of Future Value of Interest Rate

We define

\[ Y(t, T) = -\ln(V(t, T)) - \ln(V(T, T)) \frac{T - t}{T - t}, \]

as a measure of the future values of interest rate, where \( V(t, T) \) is taken from financial data.

Then we can write

\[ Y(t, T) = -\ln\left( F \exp\left( -\int_t^T r(s)ds \right) \right) - \ln F \]

so that

\[ Y(t, T) = \frac{1}{T - t} \int_t^T r(s)ds \]
### Measure of Future Value of Interest Rate

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>Maturity Date $T$</th>
<th>Face Value</th>
<th>Current Sale Price $V(t, T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>£100</td>
<td>98.24</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>£250</td>
<td>244.47</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>£100</td>
<td>96.43</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>£1000</td>
<td>939.00</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
<td>£500</td>
<td>448.95</td>
</tr>
</tbody>
</table>

**Example 18.1:**

Calculate the yields $Y(0, 2)$, $Y(0, 3)$ and $Y(0, 5)$ from the table above.

**Example 18.2:**

Using arbitrage arguments, what is the value of a zero coupon bond at time $t = 3$ maturing at $T = 5$ with $F = 100$, denoted $V(3, 5)$, and hence the future yield $Y(3, 5)$?
Example 18.3:

By differentiating the bond solution with respect to $T$, show that the forward interest rate is given by

$$ r(T) = -\frac{1}{V(t,T)} \frac{\partial V(t,T)}{\partial T}. $$

This is the interest rate at the future date $T$ (forward rate).
We can say that \( Y(t, T) \) is the average value of the interest rate \( r(t) \) in the time interval \([t, T]\). Therefore the bond price can be written as

\[
V(t, T) = Fe^{-Y(t,T)(T-t)}
\]

We define the *term structure of interest rates* (yield curve):

\[
Y(0, T) = -\frac{\ln(V(0, T)) - \ln(V(T, T))}{T} = \frac{1}{T} \int_{0}^{T} r(s)ds
\]

as the average value of interest rate in the future.
Example 18.4:

Assume that the instantaneous interest rate $r(t)$ is

$$r(t) = r_0 + at,$$

where $r_0$ and $a$ are positive constants. Calculate the bond price formula $V(t, T)$ and then sketch the term structure of interest rates.
Example 18.6:

In what ways can we extend our model for interest rates?

There exists a risk of a default bond, \( V(t, T) \), when the principal is not paid to lender as promised by the borrower. How can we take this into account?

Consider a 1 year bond, \( V(0, 1) \), that has probability \( p \) of defaulting on repayments.

**Bond Tree:**

\[ V(0, 1) \to 0 \]

\[ p \]

\[ 1 - p \]

\[ F \]

**Price:**

\[
V(0, 1) = e^{-r} \left( F \left(1 - p \right) + 0.p \right)
\]

and therefore the yield is

\[
Y(0, 1) = -\ln \left( e^{-r} F \left(1 - p \right) \right) + \ln F
\]

\[
Y(0, 1) = r - \ln \left(1 - p \right)
\]
In this case the bond has a yield of the form

\[ Y(0, 1) = r + s \]

and the positive parameter \( s \) is called the yield spread w.r.t risk-free interest rate \( r \).

**Example 18.7:**
Show that the yield spread, \( s \), is approximately \( p \).

In fact, if we model default as a Poisson process with intensity \( \lambda(t) \) we find the yield spread is

\[ s(T) = \frac{1}{T} \int_0^T \lambda(s) \, ds \]
1. Asian Options
2. Derivation of a PDE for Asian Options
Exotic Options

Example 19.1:
Given this is a course on finance, is there anything I have learned that will help me get rich?

- Options that are not traded in the market are usually called exotic options.
- They are custom built and sold over the counter (OTC)

Example 19.2:
Why might exotic options present an opportunity to make money?
Some examples of exotic options are:

- Asian
- Bermudan
- Parisian
- ParAsian (cross between Asian and Parisian!)
- Barrier
- Compound
- Lookback

Often these options can be combined or included in other financial contracts.
Example 19.3:
How do we come up with a price for these options?
**Asian Options**

**Definition**

An **Asian Option** is a contract giving the holder the right to buy/sell an underlying asset for **its average price** over some prescribed period.

**Example 19.4:**

What is the average of the stock price over the period $[0, t]$?

The floating strike Asian put option has the final condition:

$$V(S, T) = \max \left( S - \frac{1}{T} \int_0^T S(t) dt, 0 \right).$$

**Example 19.5:**

What is the payoff for a fixed strike Asian call option?
We introduce a new variable:

\[ I(t) = \int_0^t S(t) \, dt \quad \text{or} \quad \frac{dI}{dt} = S(t). \]

**Example 19.6:**
Write down the payoff for a floating strike Asian put option in terms of \( I \).
**Example 19.7:**

Using the adjusted Itô’s lemma:

\[
dV = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial I} dI,
\]

set up a hedging portfolio to eliminate risk and obtain the modified Black-Scholes equation for the Asian option

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV + S \frac{\partial V}{\partial I} = 0.
\]

The value of an Asian option must be calculated numerically (no analytic solution!).