Lecture 16

Extensions to the Black Scholes Model

16.1 Dividends

Dividend is a sum of money paid regularly (typically annually) by a company to its shareholders out of its profits (or reserves).

In this lecture we are interested in how to model dividends in a stock model, and then how to include them into the Black-Scholes model. We present the most simple possible model for dividends, and that is to assume that in a time $dt$ the underlying stock pays out a cash sum proportional to the stock price $S$ equal to $D_0 S dt$ where $D_0$ is the constant dividend yield.

Example 16.1. Consider a stock is about to pay a dividend, and is trading at £100, and every shareholder will receive a payment of £10 cash for each share they own. What happens to the stock price when the dividend is paid out?

Solution 16.1.
What we notice in this example is that there is a difference between observing the price on the market (which drops by 10) and the value of actually holding the share (which doesn’t change). So since creating a portfolio relies on holding the share, we want to know how to model this.

In reality, dividends are paid out discretely on a regular basis, but we choose to model them as a continuous, proportional payment to make things easier. Let us define the dividend payment
for all shareholder to be

\[ D_0 S dt. \]

**Example 16.2.** Assume that at time \( t = 0 \) we have a portfolio containing a single share

\[ \Pi = S. \]

If the share pays a dividend, how does the value of the portfolio change?

**Solution 16.2.**
If we try to include this in our stock model, we can write

\[ dS = \hat{\mu}Sdt + \sigma SdW \]

where \( \hat{\mu} \) is the new observed growth rate of the stock including dividends. Now consider the portfolio containing a single share again, over a small instant in time the change in this portfolio is

\[ d\Pi = \hat{\mu}Sdt + \sigma SdW + D_0Sdt \]

and rearranging we have

\[ d\Pi = (\hat{\mu} + D_0)Sdt + \sigma SdW. \]

If you want to compare investments in a stock without dividends and a stock with dividends (in which dividend payments are reinvested in stock) then you will want to set \( \hat{\mu} = \mu - D_0 \) so that the dividend payments cancel and the parameter \( \mu \) reflects the real growth rate of the asset. If you download (or observe) historical stock data you will see the term adjusted price which adjusts the stock price as if it were \( \mu \) as opposed to \( \hat{\mu} \).

**NOTE:** in the following section we keep \( dS \) as it is and do not write \( \hat{\mu} \) or anything else for the expected return on the asset. This in a sense does not matter as the \( dS \) terms cancel out from the equation.

## 16.2 Options on Dividend Paying Shares

Now, we set up a portfolio consisting of a long position in one call option and a short position in \( \Delta \) shares. The value is

\[ \Pi = C - \Delta S. \]

**Example 16.3.** Show that the change in value of this portfolio in the time interval \( dt \) is

\[ d\Pi = dC - \Delta dS - \Delta D_0Sdt. \]  \hspace{1cm} (16.1)

**Solution 16.3.**
Example 16.4. Using Itô’s Lemma written like this:

\[
dC = \left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \frac{\partial C}{\partial S} dS,
\]

(16.2)

derive the modified Black-Scholes PDE:

\[
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0) S \frac{\partial C}{\partial S} - r C = 0.
\]

(16.3)

Solution 16.4.
How can we find a solution to this modified Black-Scholes equation? The equation itself looks almost identical to (12.4), except we have an extra term multiplying the first derivative of $S$. Luckily there are a range of tricks we can use to solve a problem like this, by first guessing at the form of a solution and then seeing if the resulting equation is simplified in any way.

To proceed, guess at a solution of the form

$$C(S, t) = e^{-D_0(T-t)}C_1(S, t).$$

After this has been substituted into (16.3), we should find that $C_1$ satisfies the Black-Scholes equation except that the $r$ is replaced by $r - D_0$. Since we have already stated a solution to the Black-Scholes equation, we can then construct a solution to (16.3).
In Examples Sheet 8 you should substitute $C(S, t) = e^{-D_0(T-t)} C_1(S, t)$ into (16.3) to show that the modified Black-Scholes equation has the explicit solution for the European call

$$C(S, t) = S e^{-D_0(T-t)} N(d_{10}) - E e^{-r(T-t)} N(d_{20}),$$

(16.4)

where

$$d_{10} = \frac{\ln(S/E) + (r - D_0 + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}$$

and

$$d_{20} = d_{10} - \sigma \sqrt{T-t}.$$
16.3 Early Exercise

Recall that an American Option is one that may be exercised at any time prior to expiry \( (t = T) \). We have already discussed how this works in a discrete time binomial model, so what happens when we move to continuous time?

**Example 16.5.** How do we price an American option in continuous time?

**Solution 16.5.**

In fact in the continuous limit, the resulting problem can be formulated and solved in several different ways

- **Optimal Stopping Problem:** this is popular with probability academics. The problem can be stated as the optimal time at which to exercise (and hence stop holding the option). Some results can be derived but to get an actual value one of the next two methods must be used.

- **Variational Inequalities:** this formulation is the most robust way to formulate the problem, and there are many numerical techniques (no analytic ones though) available to solve problems of this type.

- **Free Boundary:** this formulation is popular since it can mean that analytic solutions can be derived in some cases. Unfortunately it is not very robust since assumptions have to be made about the existence of the barrier and numerical solutions are difficult to code although they are very accurate.

As such, formulating and then valuing a contract such as this can be very difficult as there are often no explicit analytic solutions.
Example 16.6. Write the American put option problem as a free boundary problem (commonly found in fluid mechanics).

Solution 16.6.