The mathematics of motion camouflage

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Motion camouflage is a strategy whereby an aggressor moves towards a target while appearing stationary to the target except for the inevitable perceived change in size of the aggressor as it approaches. The strategy has been observed in insects, and mathematical models using discrete time or neural network control have been used to simulate the behaviour. Here, the differential equations for motion camouflage are derived and some simple cases are analysed. These equations are easy to simulate numerically, and simulations indicate that motion camouflage is more efficient than the classical pursuit strategy ('move directly towards the target').

Keywords: motion camouflage; stealth strategy; pursuit path; dragonfly; chaotic pursuit

1. INTRODUCTION

Motivated by observations of mating hoverflies, Srinivasan & Davey (1995) describe a new form of stealth strategy that can be used by one creature—the shadower, or aggressor, to approach another—the shadowee, or target. In motion camouflage, the aggressor moves so that it is always on the line segment from the target to a given fixed point. If the effect of size with distance is ignored then this means that the target is unable to discern that the aggressor is moving—the aggressor appears to be at its initial position, or is camouflaged by a stationary object in the background. There is now strong evidence that dragonflies use this strategy in territorial disputes (Mizutani *et al.* 2003) and that humans can be tricked in the same way (Anderson & McOwan 2003*b*). Anderson & McOwan (2003*a*) show that an aggressor can achieve a good approximation to motion camouflage using a neural net control system, and use this to reproduce motion camouflage trajectories. Srinivasan & Davey (1995) describe several algorithms by which approximate motion camouflage might be achieved and present numerical simulations of these algorithms.

These approaches to simulating motion camouflage paths are approximate; they rely either on control methods or on a set of discrete time observations. It is clearly important to be able to determine the accuracy and efficiency of these simulations, and in § 2 a differential equation is derived, which gives the ideal motion camouflage paths for an aggressor moving with constant speed. This differential equation makes it possible to compute accurate motion camouflage paths and to compare these with other strategies or with other algorithms for motion camouflage. As a first step towards a better understanding of motion camouflage the standard test case of a target moving with constant velocity is treated mathematically at the end of § 2. In § 3 this test case is investigated numerically and the solutions are compared with those of the classic pursuit strategy, 'travel at constant speed directly towards the target', which goes back (possibly) to da Vinci (Davis 1962). In § 4, the strategy is applied to the pursuit of a chaotic target in three dimensions, and in § 5 variants of the ideal motion camouflage equations are discussed.

The simulations of §§ 3 and 4 suggest that motion camouflage is more efficient than the classical pursuit strategy in the following sense. If the aggressor is quicker than the target then the motion camouflage strategy captures the target faster than the classical pursuit strategy, while if the aggressor is slower than the target then motion camouflage is more likely to capture the target than the classical pursuit strategy. This result means that some creatures may have evolved motion camouflage as an attack strategy independent of the camouflage effect. By contrast, motion camouflage may be less robust than the classical pursuit strategy; if the target can manoeuvre itself between the aggressor and the reference point then, either the aggressor must try to move towards the reference point, or a new reference point must be chosen. The results described here should make it possible to investigate these issues in greater depth.

2. THE IDEAL MOTION CAMOUFLAGE EQUATIONS

Suppose that the position of the target is $z(t)$ and that of the aggressor is $r(t)$, where $z(t)$ is given and $r(t)$ is to be found and both lie either in a plane or three-dimensional Euclidean space. If the aggressor uses motion camouflage then $r(t)$ lies on the line connecting the target and some fixed reference point r_0 as shown in figure 1. This means that

$$r(t) = r(0) + n(t)(z(t) - r_0), \tag{2.1}$$

where $u(t)$ is a real function with $u(0) = 0$. An initial consistency condition must also hold:

$$r(0) \times (z(0) - r_0) = r_0 \times z(0), \tag{2.2}$$

which ensures that the aggressor starts on the connecting line. This condition is automatically satisfied if $r(0) = r_0$, i.e. if the fixed reference point is the beginning of the aggressor's attack. To simplify some manipulations this assumption will be made throughout the remainder of this paper. In particular, this assumption implies that the aggressor and target are at the same place at time T if $u(T) = 1$. Such a time, if it exists, is called the capture time.

If $u(t)$ can be found then equation (2.1) determines r, and any continuous function $u(t)$ which takes values less than one represents a motion camouflage path. If both
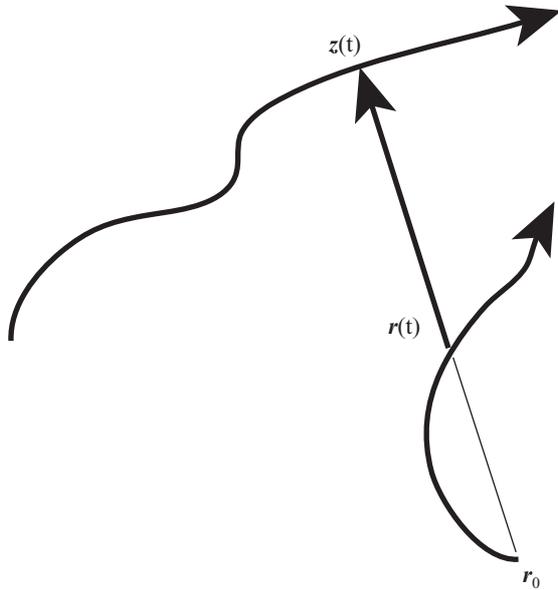
Figure 1. Schematic diagram of part of a motion camouflage solution $r(t)$ and target path $z(t)$ showing the geometry described by equation (2.1). The vector $z(t) - r(t)$ connecting the two paths at a given time is shown, and this extends back to the reference point r_0 (faint line).

the aggressor and the target move with constant speed (a standard assumption) then a unique aggressive path is determined, although there is also a defensive solution. The constant speed constraint is $|\dot{z}| = v$ and $|\dot{r}| = c$ with $v, c > 0$, where the dot indicates differentiation with respect to time. Differentiating equation (2.1) gives $\dot{r} = \dot{u}(z(t) - r_0) + u(t)\dot{z}$, and the unknown velocity $\dot{r}$ can be eliminated by squaring both sides to obtain the equation

$$c^2 = \dot{u}^2|z(t) - r_0|^2 + 2u(t)\dot{u}[(z(t) - r_0).\dot{z}] + v^2u^2(t).$$

This is a quadratic equation for $\dot{u}$ and the standard quadratic formula with the plus sign on the discriminant gives the differential equation for $u(t)$:

$$\dot{u} = \frac{-[(z(t) - r_0).\dot{z}]u + \sqrt{[(z(t) - r_0).\dot{z}]^2u^2 - (v^2u^2 - c^2)z(t) - r_0|^2}}{|z(t) - r_0|^2}$$

(2.3)

with the initial condition $n(0) = 0$.

Taking the negative sign of the discriminant when solving equation (2.2) for $\dot{u}$ gives a defensive solution; an equation for a stealthy retreat. It would be interesting to know whether any creature uses this retreat strategy.

Equation (2.3) is the general equation that determines the ideal motion camouflage path of the aggressor, and it is equally valid for motion in two or in three dimensions. The equation is simple to integrate numerically for arbitrary $z(t)$ (see below), but in the special case of the target moving with constant velocity in the plane it is possible to get a little further mathematically.

In this special case with $c = v = 1$ we may take $z = (0,t)$, so setting $r_0 = (x_0, y_0)$ equation (2.3) becomes

$$\dot{u} = \frac{-(t - y_0)u + \sqrt{x_0^2 + (t - y_0)^2 - x_0^2u^2}}{x_0^2 + (t - y_0)^2},$$

(2.4)

with initial condition $u(0) = 0$. Unfortunately, this equ-

ation does not have a known solution. To see this set $s = t - y_0$ and define a new variable U by

$$u = \frac{(s^2 + x_0^2)^{\frac{1}{2}}U}{x_0(1 + U^2)^{\frac{1}{2}}}.$$

(2.5)

Then equation (2.4) implies that U satisfies the differential equation

$$U' = (1 + U^2)\left(\frac{s_0}{(s^2 - x_0^2)} - \frac{2_0}{(s^2 + x_0^2)}U\right),$$

(2.6)

where the prime denotes differentiation with respect to s and with $U(-y_0) = 0$. This is an Abel equation of the first kind (Murphy 1962●●4●●) for which no analytic solution is available in the literature. A survey of what is known about solutions to Abel's equation can be found in Cheb-Terrab & Roche (2000). Our best approximation is

$$u(t) = \frac{(x_0^2 + (t - y_0)^2)^{\frac{1}{2}}U(t - y_0)}{x_0(1 + [U(t - y_0)]^2)^{\frac{1}{2}}},$$

(2.7)

where $U(s)$ is the solution of equation (2.6) with $U(-y_0) = 0$. Of course, the lack of an analytic solution is no barrier to numerical simulations. Some solutions together with the corresponding motion camouflage paths are shown in figure 2.

3. MOTION CAMOUFLAGE AND CLASSICAL PURSUIT CURVES

In classical pursuit strategies predators move directly towards their prey at each instant, and the differential equations modelling this movement are well established. Davis (1962) ascribes the first mathematical treatment to Bouguer in 1732. If the prey has position $z(t)$ then the predator moves on the curve $r(t)$ so that at each instant velocity is in the direction of the line from $r(t)$ to $z(t)$. If the predator has (constant) speed, $c > 0$, then the differential equation for the motion is

$$\dot{r} = c\frac{z - r}{|z - r|}.$$

(3.1)

If the prey is assumed to move in a straight line in the plane with unit speed and $c = 1$ (Bouguer's problem) then the equation can be solved explicitly, although the paths are given in terms of special functions (e.g. Davis 1962).

Figure 3a shows a solution of the classical pursuit problem (P) together with the corresponding motion camouflage curve (M) with $c = 1.2$ and $v = 1$, so the aggressor moves faster than the target. Although the classical pursuit path looks more direct, this is an illusion. Figure 3b shows that it is only in the final phase of the motion that there is a significant difference between the distance to the target in the two strategies, and that it is the motion camouflage path which captures the target first. (In the simulation shown, capture is interpreted as being within 0.001 units of the target, but the qualitative statement that motion camouflage captures first appears to be robust.) Indeed, further numerical simulations suggest that this is true more generally; the distance between the aggressor and target decreases initially with the same linear behaviour in both the classical pursuit strategy and the motion camouflage strategy, but as shown in figure 3c, the simu-
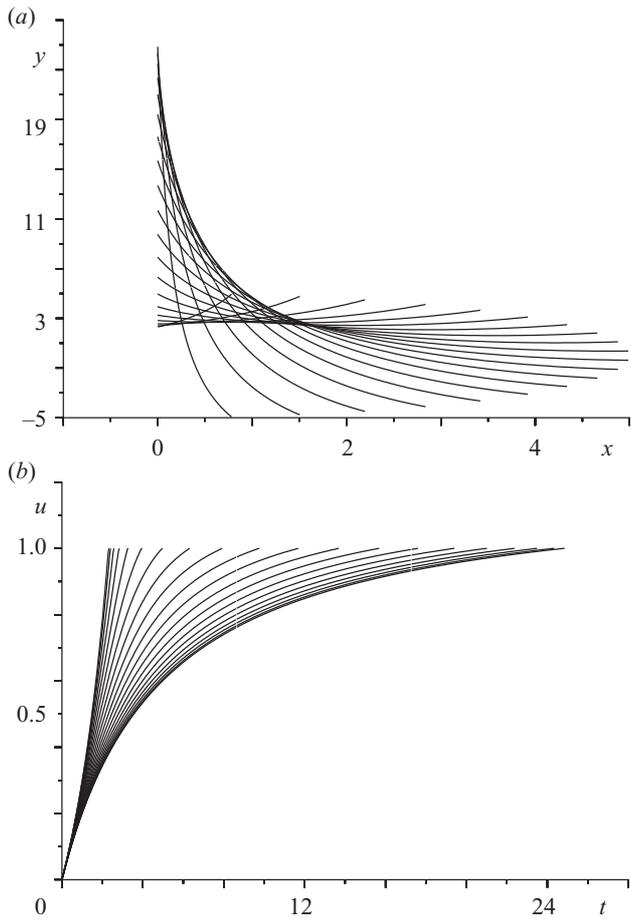
(a)

(b)

Figure 2. (a) Motion camouflage paths in the (x,y)-plane. The target is moving along the y-axis with $v = 1$ and position $z = (0,t)$ and the aggressor has speed $c = 1.2$. The different curves represent the paths of 20 aggressors with different initial positions. These are regularly spaced on the circle of radius 5 units centred on the starting position of the target. (b) The corresponding functions $u(t)$, which are the solutions of equation (2.4). These curves can be matched to those of (a) by noting that at the point of capture, if $y = Y$ when $x = 0$ in (a) then $u(Y) = 1$ in (b).

lated motion camouflage paths are shorter (and hence more efficient) for all initial conditions equidistant from the starting point of the target.

If the speed of the aggressor is less than the speed of the target then motion camouflage appears to be more efficient than the pursuit strategy, in the sense that capture is possible from a greater range of initial conditions, which are equidistant from the initial position of the target but at different angles to the line of motion of the target. This observation is illustrated further in § 4.

The observation that the paths determined by the two strategies are initially very similar can be made more precise. If the target path is smooth then it can be expanded as a (vector) Taylor series in time: $z(t) = z_0 + tz_1 + t^2z_2 + \ldots$, where z_0, z_1 and z_2 are constant vectors and the dots represent terms of order t^3 and higher which are small when t is small. If a similar ansatz for $r(t)$ is made and substituted into equation (3.1) then a little manipulation gives, on identifying the constant and linear terms in t,
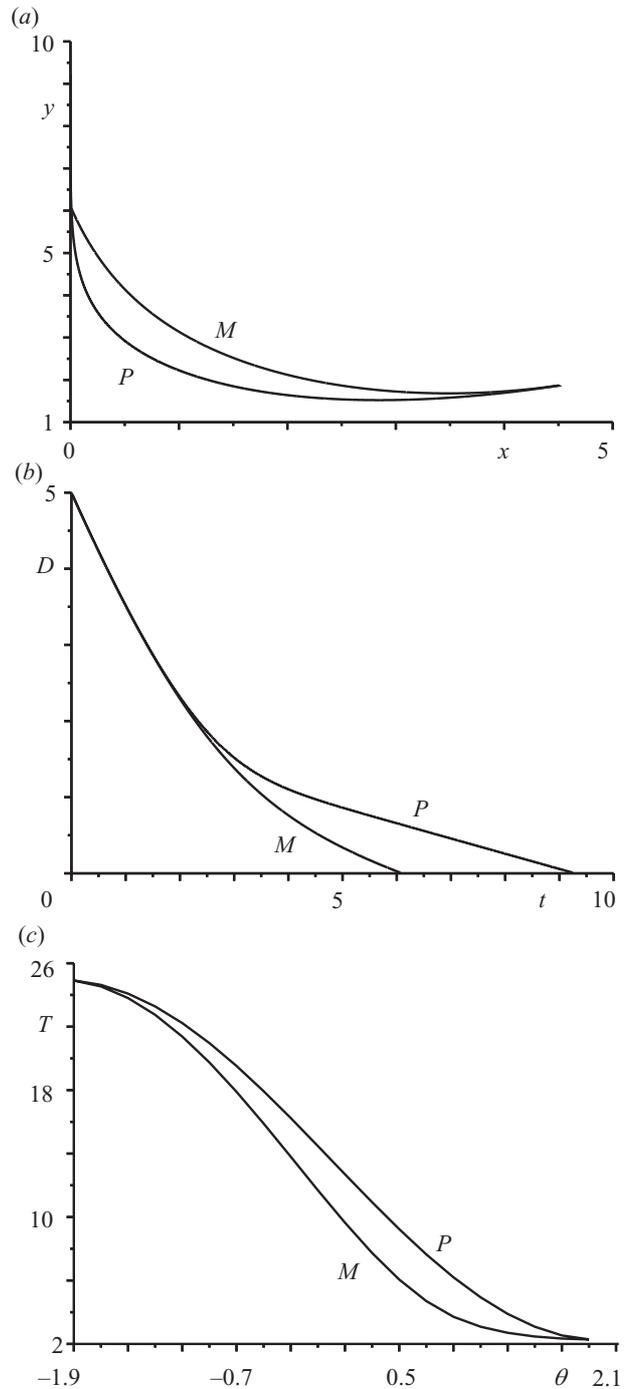
(a)

(b)

(c)

Figure 3. (a) A motion camouflage path M and a pursuit path P in the (x,y)-plane. The target is moving along the y-axis with $v = 1$ and position $z = (0,t)$ and the aggressor has speed $c = 1.2$. The initial condition is $(5\cos^{\pi\theta/4}, 5\sin^{\pi\theta/4})$ with $\theta = 0.5$. (b) The distance from the target (D) as a function of time (t) on the two paths shown in (a). (c) The time (T) to capture as a function of angle if the target moves as in (a). The angular variable θ is in units of $\pi/4$ as in (a). The aggressor has speed $c = 1.2$ and capture is interpreted as being within 0.001 units of the target. The times for the motion camouflage strategy are labelled M and the times for the pursuit strategy are labelled P.

$$r = r_0 + \frac{ct}{d_0}d_0 + \frac{ct^2}{2d_0^3}(d_0^2z_1\ldots(d_0.z_1)d_0 + \ldots, \qquad (3.2)$$

where $d_0 = z_0 - r_0$ and $d_0 = |d_0|$. However, if the ansatz

$u(t) = u_1 t + u_2 t^2 + \ldots$ is substituted into the equation for motion camouflage, (2.3), the coefficients u_1 and u_2 can be calculated. Equation (2.1) then implies that for motion camouflage

$$\boldsymbol{r} = \boldsymbol{r}_0 + \frac{ct}{d_0}\boldsymbol{d}_0 + \frac{ct^2}{d_0^3}(d_0^2 \boldsymbol{z}_1 - (\boldsymbol{d}_0.\boldsymbol{z}_1)\boldsymbol{d}_0) + \ldots . \qquad (3.3)$$

Thus the small t behaviour of the paths only differ in the quadratic terms; the two paths are tangential at $t = 0$. This also explains why the distance between the target and the aggressor initially decreases at the same linear rate.

4. CHAOTIC PURSUIT

The examples above model motion in the plane, and the motion of the target is linear. In this section, the same models; motion camouflage given by equation (2.3) and classical pursuit by equation (3.1), are used to investigate more complicated three-dimensional motion of the target.

We will assume that the target moves with constant speed v along the Rössler chaotic attractor (Rössler 1976). In other words, $\boldsymbol{z} = (z_1, z_2, z_3)$, with

$$\dot{z}_1 = (-z_2 - z_3)/\Delta,$$
$$\dot{z}_2 = (z_1 + 0.15 z_2)/\Delta,$$
$$\dot{z}_3 = (0.2 - 10 z_3 + z_1 z_3)/\Delta, \qquad (4.1)$$

where $v^2 \Delta^2 = (z_2 + z_3)^2 + (z_1 + 0.15 z_2)^2 + (0.2 - 10 z_3 + z_1 z_3)^2$ and $\Delta \geqslant 0$. Δ has been chosen so that the target moves with speed v, and $\Delta = 0$ at the stationary points of the differential equation, so equation (4.1) is only valid on the attractor provided it does not contain any stationary points. No direct biological relevance is claimed for this choice of chaotic target path, which is shown in figure 4a, but it is natural to test pursuit strategies against more complicated target paths.

Figure 4b shows the path obtained by integrating the coupled set of four differential equations for motion camouflage, equations (2.3) and (4.1), with initial target position equations (2.2), (2.5) and (2.7)●●1●● and initial aggressor position $(3, -1, -500)$ and with $v = 20$ and $c = 18$. The coordinates (x, y, z) correspond to the position vector $\boldsymbol{r}$ of the aggressor determined by equation (2.1) once equation (2.3) is solved. Despite the fact that the target moves faster than the aggressor, the aggressor captures the target after approximately 26.4 time-units. The classic pursuit problem, equations (3.1) and (4.1), results in a six-dimensional set of differential equations and, given the same initial positions and speeds the pursuit path is shown in figure 4c. In this case, the aggressor moves close to the attractor containing the target's path, but does not get within 0.1 units of target in the 200 units of time shown here. Again, this illustrates that motion camouflage is a more efficient interception strategy than classical pursuit.

5. RELATED MODELS

The techniques described here can be used to construct other motion camouflage models incorporating different effects. For example, there is nothing in the derivation of equation (2.3) that requires the target to move with constant speed, and the equation could easily be modified to
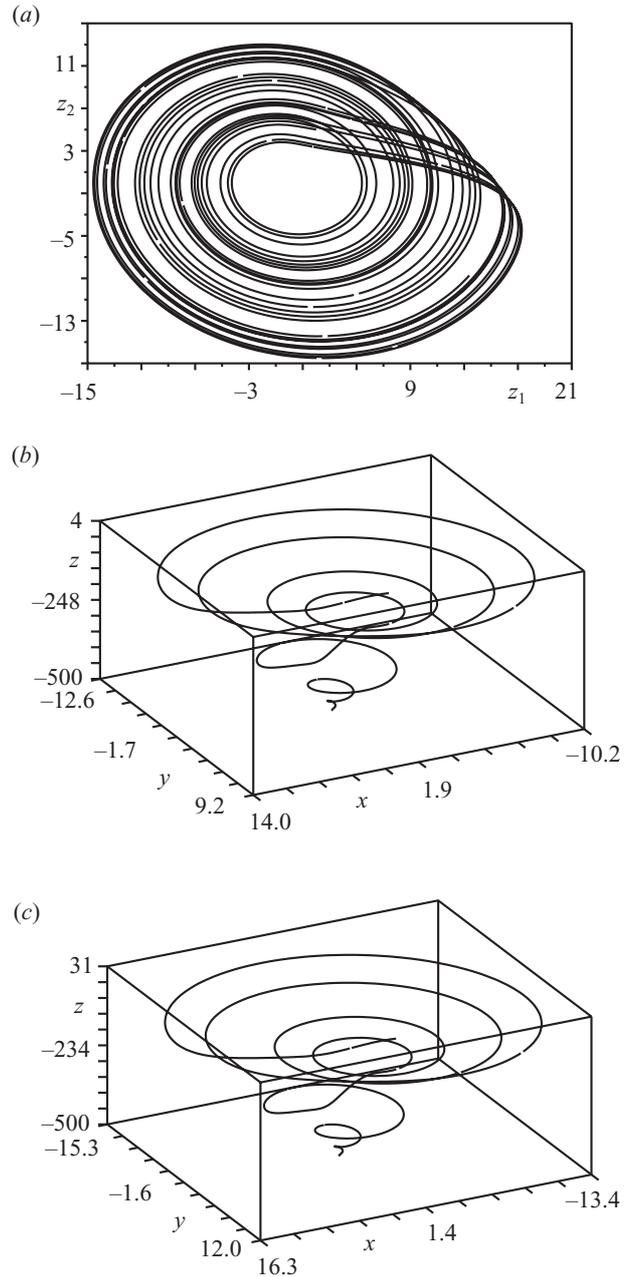
(a)

(b)

(c)

Figure 4. (a) The, Rössler attractor: projection onto the (z_1, z_2)-plane of solutions to equation (4.1) with initial conditions (equations (2.2), (2.5) and (2.7)). The results of 100 time-units with $v = 20$ are shown. (b) The motion camouflage solution with $v = 20$, $c = 18$. The initial conditions of the aggressor are $(3, -1, -500)$. (c) The corresponding classical pursuit path.

allow the target to change speed. The modification to equation (2.3) on the assumption that there is a time delay τ between the aggressor's observation of the position of the target and its motion can also be derived by replacing $\boldsymbol{z}(t)$ by $\boldsymbol{z}(t - \tau)$ in § 2.

While it is easy to relax the assumption of constant speed for the target, it is more difficult to do the same thing for the aggressor, since it must be replaced by another constraint to give a unique aggressive solution. One possible alternative would be to impose a maximum relative speed of approach. This is natural, since if the aggressor grows too rapidly as perceived by the target then

the motion camouflage is more likely to be detected. It would be interesting to determine which strategy within the space of all motion camouflage strategies is adopted by dragonflies. The framework developed here should help to make this possible.

A related motion camouflage strategy which is also observed in dragonflies (Mizutani *et al.* 2003) involves remaining stationary with respect to distant objects, i.e. remaining on the same bearing as seen by the target. As Mizutani *et al.* (2003) note, this is equivalent to taking the fixed reference point at infinity, and is often cited in sailing manuals (see, for example, Cal Sailing Club 2003) and pilot training as a criterion indicating a potential collision course. Again, this is easy to detect in experiments as the lines connecting the aggressor and the target at different times will be parallel. The differential equation for the motion of the aggressor can be derived by fixing a constant unit vector e and noting that the aggressor must always be in the direction of e as seen from the target at z. Hence

$$r = z + ue, \tag{5.1}$$

which is the starting point for the derivation of another set of equations. In this case, linear motion by the target is met by linear motion from the aggressor if capture is possible.

6. CONCLUSION

Motion camouflage strategies are likely to be encountered in many circumstances; humans seem equally susceptible (Anderson & McOwan 2003*b*). This note provides a simple modelling technique that can be used to assess the strategy numerically using widely available differential equation packages. This has also made it possible to provide accurate pictures of the ideal motion in standard and non-standard cases. Moreover, the theoretical ideas presented are easily modified to incorporate refinements of the constant speed solutions.

If the target moves in a straight line with constant speed, the motion camouflage equations do not have a closed solution in terms of standard special functions. However, simulations show that the motion camouflage strategy is more efficient than the classical pursuit strategy (figure 3*c*), although aggressors moving according to the two strategies initially start to close the gap between themselves and the target at the same linear rate. Furthermore, simulations indicate that even if the aggressor moves slower than the target, capture is possible in cases where the classical pursuit path ends up following the trail of the target, and that this holds whether the target moves on a straight line or on a chaotic attractor.

This paper provides a coherent mathematical framework within which motion camouflage strategies can be analysed. It raises many questions, mathematical and biological: which motion camouflage strategy is adopted by dragonflies and hoverflies? How is the transition between motion camouflage and other pursuit strategies determined in real situations (i.e. when does the illusion break down due to size, and how should the target and aggressor react to this discovery)? What is the relationship between the aggressor's path and the strange attractor in figure 4*c*? These questions and others can at least be given a clear mathematical formulation.

The theory presented here does not pretend to explain *how* an insect might follow an ideal motion camouflage path, but it does make it possible to compute these paths and to gain insight into the strategy given different target movement. These ideal paths can also be compared with experimental measurements and theoretical models that do incorporate realistic biological control mechanisms, as well as other ideal strategies such as the classical pursuit paths discussed here.

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