STABILITY OF ASYMMETRIC CLUSTER STATES IN GLOBALLY COUPLED MAPS

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Abstract—*Globally coupled maps can exhibit interesting dynamics. We construct stable cluster states, for which the variables divide into two separately synchronized populations. These stable states coexist with stable synchronized states and explain some observations in the literature.*

I. Introduction

Globally coupled maps have been used to model dynamics in a variety of physical, chemical and biological applications, see [11] for a brief account. The dynamics is defined on N sites labelled by k, so at site k the system is in state $x_n(k)$ at time n, which is taken to be discrete. At each site the time evolution of $x_n(k)$ depends on a local map, f, which is the same for each site, and a coupling term which determines how the state of one site is affected by the states of the other sites. In the literature, coupling via the average of $f(x_n(k))$, $\overline{f}_n = \frac{1}{N}\sum_j f(x_n(j))$, is frequently used [9], [10], [11], [12], giving the equations

$$x_{n+1}(k) = f(x_n(k)) + \epsilon\left(\overline{f}_n - f(x_n(k))\right) \quad (1)$$

$k = 1, \ldots, N$, for the real state variables $x_n(k)$. There is a growing literature on the complicated dynamics which such systems may support. The symmetries of (3) imply that if all the states take the same value, i.e. if they are synchronized, then they remain synchronized and the dynamics is determined by the dynamics of the local map, f. Much of the analysis is concerned with blowout bifurcations of the synchronized state [1], [2], [9]. Here different orbits in a synchronized chaotic state can have different transverse stability properties and there is a cascade of loss of transverse stability as ϵ varies leading, in the supercritical case, to riddled basins of attraction and then loss of stability for the synchronized state. More recently the existence and stability of cluster states has been investigated [10], [11], [12]. In the simplest such states (two cluster states) the variables $x(k)$ divide into two sets, one of size m and the other $N-m$, and each set of variables is synchronized independent of the other, i.e. after a possible relabelling of the variables

$$\begin{array}{ll} x(1) = x(2) = \ldots = x(m) & = y \\ x(m+1) = \ldots = x(N) & = z \end{array} \quad (2)$$

In this note we will study the stability of cluster states in a generalization of (1). Given a smooth real function g (called the *coupling map*) with $g(0) = 0$ we consider maps of the form

$$x_{n+1}(k) = f(x_n(k)) + g\left(\overline{f}_n - f(x_n(k))\right) \quad (3)$$

$k = 1, \ldots, N$. The condition $g(0) = 0$ implies that synchronized states exist and evolve according to the local map. This generalization is analogous to that used by Banaji [3], [4] in the consideration of maps coupled via the average of $x_n(k)$ rather than the average of $f(x_n(k))$. The standard equation (1) corresponds to the choice $g(x) = \epsilon x$. Substituting (2) into (3) we see that the two cluster property (2) is invariant under the map and

$$\begin{array}{l} y_{n+1} = f(y_n) + g\left((1-p)[f(z_n) - f(y_n)]\right) \\ z_{n+1} = f(z_n) + g\left(p[f(y_n) - f(z_n)]\right) \end{array} \quad (4)$$

where $p = \frac{m}{N}$. Following [3], [5], [12] we shall consider p as a continuous parameter although only rational values are relevant to finite dimensional problems. If p is small, so most state variables are synchronized with the z variable, then the cluster state is said to be *asymmetric*, and it is these asymmetric cluster states which concern us here.

Popovych et al [12] make three key observations about two cluster solutions of (1) when p is small and the local map is the logistic map, $f(x) = rx(1-x)$. First, that the behaviour of the variable y in (4) looks almost as though it comes from a one-dimensional map as parameters are varied. In particular, the order of appearance of stable states as the parameter

μ of the local map varies reflects the standard one-deimensional order of the logistic map. Secondly, that the asymmetric two cluster states can appear before the loss of stability of synchronized states. Popovych et al [12] concentrate on parameter values for which (or close to which) there is a stable synchronized state with period three, and their third contribution is a detailed numerical study of the bifurcations in the two cluster equation for such parameters. Our analysis throws some light on these observations in a special case of (3).

II. A HIERARCHY OF EQUATIONS

The first two equations of the hierarchy are the full system (3) and the cluster equations (4). Note that the cluster equations represent the motion on an invariant subspace of the full system, and that the stability of a solution to (4) in its two dimensional phase space does not imply *a priori* anything about the stability of the invariant subspace for the full equations. Our next equation is obtained by letting $p \to 0$ in (4), the asymmetric cluster state limit.

$$\begin{aligned} y_{n+1} &= f(y_n) + g(f(z_n) - f(y_n)) \\ z_{n+1} &= f(z_n) \end{aligned} \tag{5}$$

This map has the structure of a skew product: the z variable evolves independently of the y variable, whilst the dynamics of the y variable depends on both states. The final, and simplest, map in the hierarchy is the local map:

$$z_{n+1} = f(z_n) \tag{6}$$

Equations (3), (4), (5) and (6) represent a hierarchy of simplifications. Starting with the full $N-$dimensional system (3) we move down to the two-dimensional two cluster equations (4), which are exact equations on an invariant manifold of (3). In the asymmetric cluster limit these equations give the simpler skew product equations (5) and moving one further step down in complexity we find the local map equation (6) which is both the driving equation for the skew product and the equation on the fully synchronized subspace of the original equations (3).

In the next section we show how stability properties of some solutions of simple equations in this hierarchy are inherited by corresponding solutions in the next step up the hierarchy.

III. STABILITY: REDUCED EQUATIONS

Suppose that the local map has an stable fixed point w^* with $a = |f'(w^*)| < 1$. Then the full system (3) has a synchronized state with $x(k) = w^*$, $k = 1, \ldots, N$, and the linearization about this solution has eigenvalues $f'(w^*)$ and $(1 - g'(0))f'(w^*)$, with the latter eigenvalue repeated $N - 1$ times [3], [6], [8]. Hence this simple synchronized state is linearly stable provided

$$-a^{-1} + 1 < g'(0) < a^{-1} + 1, \quad a < 1 \tag{7}$$

Suppose that $g'(0)$ lies in this range, and consider the limiting asymmetric cluster state equation (5) with the second (z) variable equal to w^*, which is attracting. In this case the first variable, y, evolves as:

$$y_{n+1} = f(y_n) + g(w^* - f(y_n)) \tag{8}$$

since $f(w^*) = w^*$. So the dynamics is determined by the local map plus a second function, and in particular it is a one-dimensional map. Now suppose that a set of points $\{u_1, \ldots, u_q\}$ is a linearly stable periodic orbit of (8), then if (7) is satisfied,

$$\{(u_1, w^*), (u_2, w^*), \ldots, (u_q, w^*)\} \tag{9}$$

is a linearly stable periodic orbit of (5) and by standard hyperbolicity results (see for example [7]) any sufficiently small perturbation of (5) will also have a linearly stable periodic orbit of period q. In particular, if p is sufficiently small the exact two cluster equation (4) is close to the asymmertric cluster limit (5) and so will have a linearly stable periodic orbit of period q with coordinates close to those of (9):

$$\{(v_1, w_1), (v_2, w_2), \ldots, (v_q, w_q)\} \tag{10}$$

where the v_k are close to u_k and w_k are close to w^*. This establishes the first part of our result:

If (7) is satisfied and (8) has a linearly stable periodic orbit of period q, then for all p sufficiently small there is a linearly stable orbit of period q for the two cluster equations (4).

Note that Popovych et al [12] remark on the fact that the order of appearance of asymmetric two cluster periodic orbits in simulations seems to mimic the one-dimensional Sharkovskii order. This result explains why this might be the case in some related systems: if the synchronized fixed point is stable then for small p the one-dimensional equation (8) determines at least some of the stable periodic orbits of the two cluster equations.

In the next section a more precise statement about stability conditions will be needed. The condition for linear stability of (9) in (5) is

$$\prod_{i=1}^{q} |(1 - g'(w^* - u_{i+1}))f'(u_i)| < 1, \quad |f'(w^*)| < 1 \tag{11}$$

where $u_{q+1} = u_1$. To investigate the stability of (10) in (4) set $y_n = v_n + \xi_n$ and $z_n = w_n + \eta_n$. Here and below we interpret the subscript n periodically on the periodic orbit, so $v_r = v_{q+r}$ and $w_r = w_{q+r}$. Setting

$$\Delta_n^p = (1-p)g'((1-p)(w_{n+1} - v_{n+1})) \qquad (12)$$

and $\delta_n = g'(p(v_{n+1} - w_{n+1}))$ these linearized equations, obtained by substituting into (4) and retaining only terms linear in ξ and η, are

$$\begin{aligned} \xi_{n+1} &= (1-\Delta_n^p)f'(v_n)\xi_n + \Delta_n^p f'(w_n)\eta_n \\ \eta_{n+1} &= p\delta_n f'(v_n)\xi_n + (1-p\delta_n)f'(w_n)\eta_n \end{aligned} \qquad (13)$$

and the assumption of linear stability implies that $(\xi_n, \eta_n) \to (0,0)$ as $n \to \infty$. If the Jacobian matrix of (13) is J_n then the exact condition for linear stability is that all the eigenvalues of $J_q J_{q-1} \dots J_1$ lie inside the unit circle. Ignoring terms of order p this is just

$$\prod_{k=1}^{q} |(1-\Delta_k^0)f'(v_k)| < 1, \qquad \prod_{k=1}^{q} |f'(w_k)| < 1 \quad (14)$$

Although these conditions are not sufficient for linear stability we shall assume that they hold on the periodic orbit (10), and indeed, for sufficiently small p, v_n is close to u_n and w_n close to w^* and so (14) holds by continuity.

IV. Stability: full equations

Of course, stability in the reduced equations need not imply stability in the full equations, (3). It is therefore somewhat surprising that there is a close relationship between the two in the asymmetric limit. If (7) is satisfied there is a stable synchronized fixed point with all states equal to a constant, w^*. In addition, if (8) has a stable periodic orbit of period q and p is sufficiently small then there is a stable two cluster state (10) in which m of the state variables are close to the periodic orbit of (8) and $N-m$ are close to w^*; note that $p = \frac{m}{N}$, and in particular, N must be large enough so that the hyperbolicity results used in section III can be applied.

The exact asymmetric two cluster solution, after relabelling the variables if necessary, is

$$\begin{aligned} x_n(1) = x_n(2) = \dots = x_n(m) &= v_n \\ x_n(m+1) = \dots = x_n(N) &= w_n \end{aligned} \qquad (15)$$

with $p = \frac{m}{N} \ll 1$. To linearize about this solution set

$$\begin{aligned} x_n(j) &= v_n + \xi_n(j), \qquad && j = 1, \dots, m; \\ x_n(m+k) &= w_n + \eta_n(k), \qquad && k = 1, \dots, N-m \end{aligned} \qquad (16)$$

where the variables η and ξ are small. Substituting into the full equations (3) and using the fact that (v_n, w_n) is a solution of (4) the linearized equations are

$$\begin{aligned} \xi_{n+1}(i) &= (1-\Delta_n^0)f'(v_n)\xi_n(i) \\ &\quad + \tfrac{\Delta_n^0}{N}\left(f'(v_n)\textstyle\sum_j \xi_n(j) + f'(w_n)\sum_k \eta_n(k)\right) \\ \eta_{n+1}(\ell) &= (1-\delta_n)f'(w_n)\eta_n(\ell) \\ &\quad + \tfrac{\delta_n}{N}\left(f'(v_n)\textstyle\sum_j \xi_n(j) + f'(w_n)\sum_k \eta_n(k)\right) \\ &\quad i = 1, \dots, m; \quad \ell = 1, \dots, N-m \end{aligned} \qquad (17)$$

To show that $\xi_n(i)$ and $\eta_n(\ell)$ tend to zero as n tends to infinity it is enough to show that this is true in N independent invariant directions (corresponding to N eigenvectors of the Jacobian matrix with eigenvalues of modulus less than one). Following [6] we choose the first $N-m-1$ independent choices given by by defining $\eta_1(\ell) = -\eta_1(\ell+1)$ for $\ell \in \{1, \dots, N-m-1\}$ (and all the other variables equal to zero), for which the eigenvalue is

$$(1 - g'(w_{n+1} - v_{n+1}))f'(v_n) + O(p) \qquad (18)$$

and by our assumptions (14) this gives a stable direction. Similarly there are $m-1$ independent choices ($i = 1, \dots, m-1$) with $\xi_1(i) = -\xi_1(i+1)$, and the other variables equal to zero. In this case the eigenvalue is

$$f'(w_n) + O(p) \qquad (19)$$

which also gives a stable direction by the assumptions (14) on the q periodic orbit (w_n, v_n).

This deals with $N-2$ independent directions, the remaining two correspond to the stability of the periodic orbit in the two cluster equation, which is obtained by setting $\xi_n(j) = \xi_n$ and $\eta_n(k) = \eta_n$. The linearized equations are now just the linearization restricted to the two cluster subspace which is stable by assumption. We have therefore described N independent stable directions, establishing the linear stability of the two cluster solution in the full equation (3). This gives our second result:

Under the assumptions of section III, if the synchronized fixed point is stable and a stable two cluster solution of (5) exists with period q, then for all p sufficiently small the corresponding two cluster state is stable in the full equations (3).

Note that for fixed N, the smallest p that can be realized is $\frac{1}{N}$, and that our result demonstrates that parameter regions in which stable synchronized states and stable cluster states coexist can be expected in the full equations.

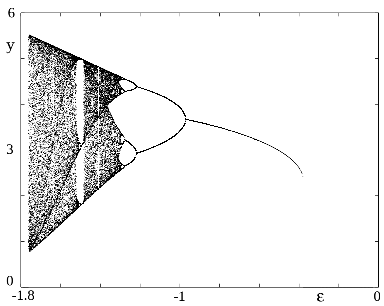

Fig. 1. Non-trivial attractor of (22) as a function of ϵ with $-1.8 < \epsilon < 0$ on the horizontal axis.

V. An Example

The example of Popovych et al [12] does not give interesting results in the fixed point region discussed here, although it is possible that the techniques outlined above could be adapted to explain results in the periodic regime for their example . On the other hand, other simple choices of the local map do give interesting results. We shall consider the case

$$f(x) = 1 - \cos x \quad g(x) = \epsilon x \tag{20}$$

This is a standard choice for g, but the local map f has a superstable fixed point at $x = 0$, which will play the role of w^*, and all orbits tend to this point. Note that f is part of the family of maps $f_\lambda(x) = \lambda(1 - \cos x)$. These maps do have interesting dynamics: $x = 0$ is always superstable, but the full range of unimodal behaviour can coexist with this trivial orbit. The full equations are

$$\begin{aligned} x_{n+1}(k) = \;& 1 - \cos x_n(k) \\ & + \tfrac{\epsilon}{N} \textstyle\sum_{j=1}^{N} [\cos x_n(j) - \cos x_n(k)] \end{aligned} \tag{21}$$

$k = 1, \ldots, N$, and (8) becomes

$$y_{n+1} = (1 - \epsilon)(1 - \cos y_n) \tag{22}$$

As ϵ varies this can have the full range of behaviour exhibited by f_λ. This is shown in Figure 1 for ϵ in the range $(-1.8, 0)$.

The theoretical work of previous sections shows that any of the non-trivial periodic attractors of this map in the region below $\epsilon \approx 0.4$ should give rise to stable asymmetric two cluster states. To test this the full system (21) was simulated with $N = 500$ and two values of ϵ: $\epsilon = -1.1$ for which (22) has a stable orbit of period two, $\epsilon = -1.7$ which is in the chaotic region. For initial conditions $x_1(k) = 0.01$, $k = 1, \ldots, 499$ and $x_1(500) = 4.3$ the system quickly settled into a two cluster state with the first 499 states synchronized very close to zero, and last state either period two ($\epsilon = -1.1$) or apparently chaotic ($\epsilon = -1.7$). Although this chaotic region is not covered, strictly speaking, by the analysis presented here, it is not a great deal of trouble to extend the results to this case.

VI. Conclusion

We have shown how to create asymmetric cluster states which are stable in the full equations and which coexist with stable synchronized fixed point states. It is hoped that this style of argument can be applied to a greater range of behaviour and systems by generalizing to the case of stable periodic synchronized states.

Acknowledgement: The debt this note owes to the work of my (then) PhD student, Murad Banaji [3], [4], [5] should be clear. This note can be seen as an attempt of the supervisor to follow the arguments of the student!

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