Chapter 4
Dynamical Models

4.1 Cartesian Shallow Water Model

The purpose of the lecture and practical this week is for you to become familiar with some key concepts in the study of *atmospheric dynamics*, which is the study of the atmosphere as a fluid—yes the atmosphere is a fluid, just not very dense! We will address forces acting on fluid elements and discuss their origins. In addition we will look at *balanced flow* approximations in the atmosphere, where forces are in balance and there are no accelerations, before extending this to look at *accelerating flows* and *wave phenomena*. You will become familiar with the science involved in the study of the atmosphere and performing *weather forecasts*. But first we need to review some of the concepts behind Newtonian mechanics...

4.2 Newton's mechanics

4.2.1 Newton's laws (equations) of motion

Let us briefly review Newton's three laws of motion for the motion of objects with constant mass.

First law: if the *net force* is zero then the *velocity* of the object is constant. This applies in a vector sense:

$$\sum \mathbf{F} = 0 \Leftrightarrow \frac{d\mathbf{v}}{dt} = 0 \tag{4.1}$$

Note that the $\sum$ symbol means to add up all forces acting on the object.

Second law: Perhaps the most famous law states that the *net force* acting on an object is equal to the product of mass and acceleration:

$$\begin{aligned} \mathbf{F} &= m\mathbf{a} && (4.2) \\ &= m\frac{d\mathbf{v}}{dt} && (4.3) \end{aligned}$$

Third law: All forces exist in pairs: if one object A exerts a force $\mathbf{F}_A$ on a second object B, then B simultaneously exerts a force $\mathbf{F}_B$ on A and the two forces are equal size and opposite sign: $\mathbf{F}_A = -\mathbf{F}_B$.

4.2.2 Examples of Forces

First we consider an objects weight, W, which is an example of a force directed towards the centre of earth.

- If the object's position is in mid-air then, applying Newton's Second Law, this *downward force* gives rise to an *acceleration towards* the centre of earth.
- If the object is placed on a table its position will be fixed. The reason is because the weight force is still there, but the table will also provide an *upward force* to cancel out the weight, resulting in a *zero net force*. Essentially, this cancelling force is an *action-reaction* force provided by the table's contact with the ground—see Newton's Third Law.
- If the object falls through air then the object will have a *drag force*, which opposes its motion and is due to the air pushing past it. When a certain speed is reached the drag force will equal the weight force and therefore there will be *no net force* acting on the falling object. The object will still be travelling towards the ground but, in accordance with Newton's First and Second Laws, not accelerating.

Next, let us consider pressure differences (pressure gradients) giving rise to forces. Consider an air-tight room where the pressure inside the room is equal to the pressure outside the room and that there is a window, area A, on one of the walls of the room. At this point there will be *no net force* on the window; however, if we *increase* the pressure inside the room there will be a *pressure gradient* across the window. This results in a force on the window, perpendicular to the plane of the window and directed outwards:

$$F = \Delta P \times A \tag{4.4}$$

The key points here are that both an objects *weight* and also *pressure gradients* are associated with *forces*, which, through Newton's Second Law, can result in *accelerations*.

4.2.3 Uniform circular motion

Fluids tend to swirl around in ways that can be roughly approximated by circles. Here we briefly examine the requirements for *uniform circular motion*, which is motion in a circle at constant tangential speed.

First let's consider an object on a string moving in a uniform circle, with one end of the string at the centre, radius R, with angular frequency, $\omega = \frac{2\pi}{T}$, where T is the time it takes for one complete revolution. Mathematically its position versus time can be described as:

$$x = R\cos\omega t \tag{4.5}$$
$$y = R\sin\omega t \tag{4.6}$$

As said above objects in motion must obey *Newton's laws of motion*; hence, we may determine the *net force* that must act on the mass to give rise to this circular motion: we just have to find the acceleration and put the result into Newton's second law. The acceleration is the 2nd derivative of position:

$$\ddot{x} = -\omega^2 x \tag{4.7}$$
$$\ddot{y} = -\omega^2 y \tag{4.8}$$

For the object on a string the only thing that can cause acceleration is the tension in the string, which causes an acceleration towards the centre of the circle. We can calculate acceleration towards the centre using Pythagoras' theorem $a_r = \sqrt{\ddot{x}^2 + \ddot{y}^2}$, i.e.:

$$a_r = -\omega^2 R \tag{4.9}$$
$$= -\frac{v_t^2}{R} \tag{4.10}$$

where $R = \sqrt{x^2 + y^2}$. So the important point is that the acceleration of an object moving in circular motion is $\frac{v_t^2}{R}$ *directed towards the centre*.
We have also made use of the fact that, for a constant radius circle, the time it takes for one revolution is the distance, $2\pi R$, divided by the speed, v_t: $T = \frac{2\pi R}{v_t}$; hence, $\frac{v_t}{R} = \omega$.

4.3 Equation of motion in the atmosphere

Newton's 2nd law of motion, $m\mathbf{a} = \mathbf{F}$ may be applied to a 2-D fluid on a rotating body (i.e. earth):

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho f v - \frac{\partial P}{\partial x} \tag{4.11}$$
$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\rho f u - \frac{\partial P}{\partial y} \tag{4.12}$$

where all terms are divided by a unit volume of fluid. The left terms is the acceleration of the fluid per unit volume and the right terms are the forces acting on the fluid per unit volume. Here, ρ is the density of the fluid; x, y are positions in the fluid; u, v are the speeds of the fluid in the x and y directions respectively and f is the Coriolis parameter (see Section 4.3.2).

4.3.1 Pressure gradient force

The last terms on the right of Equations 4.11 and 4.12, $-\frac{\partial P}{\partial x}$ and $-\frac{\partial P}{\partial y}$ are pressure gradient forces per unit volume, PGF. They are negative because a negative pressure gradient gives rise to a positive force (air moves from high to low pressure). They can be estimated using:

$$PGF = \frac{|\Delta P|}{\Delta x} \tag{4.13}$$

where ΔP is the change in pressure and Δx or Δy is the change in distance.

This is shown in Figure 4.1. In the absence of any other forces air will move from high pressure to low pressure in a straight line.

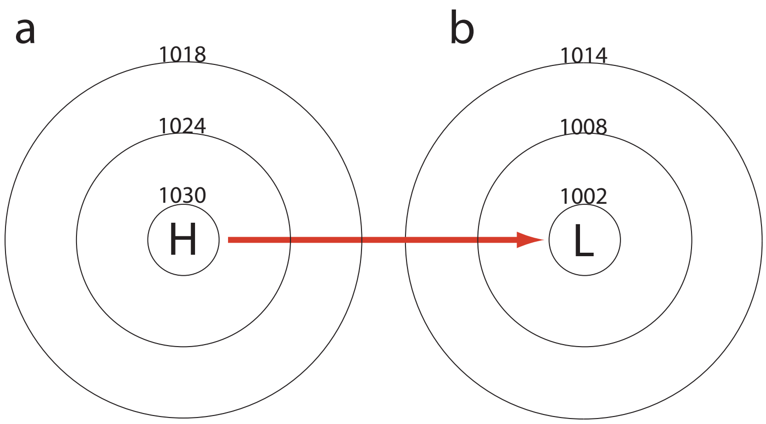


Figure 4.1: Schematic depicting the pressure gradient force from a high pressure system (a) to a low pressure system (b). The circles are lines of constant pressure. This is in a non-rotating system.

4.3.2 Coriolis effect or 'force'

The first term on the right of Equation 4.11 and 4.12: $\rho f v$ and $-\rho f u$, describe the *Coriolis force per unit volume*. Since the earth is rotating, air that moves in a straight line will appear to be deflected relative to an observer on the ground. The effect is zero at the equator. Consider air moving from east to west in the northern mid-latitudes. It will be deflected to the right of its path and hence an observer may think it has been acted on by some force. This is the *Coriolis effect*: a fictitious force used to describe the path of an object relative to a rotating observer. Equation 4.14 shows that f depends on the latitude and is related to the rotation of the earth, Ω, and the latitude, ϕ as follows:

$$f = 2\Omega \sin \phi \tag{4.14}$$

where $\Omega = 7.3 \times 10^{-5}$ rad s^{-1} and ϕ is the latitude. The forces per unit volume, in the x and y directions, are related to f and the velocity of the air:

$$F_x = \rho f v \tag{4.15}$$
$$F_y = -\rho f u \tag{4.16}$$

In the northern hemisphere the *Coriolis force* acts *to the right* of the motion of the air and vice-versa. The size of the Coriolis force is equal to the product of the *Coriolis parameter* and the *speed of the air*. This is the reason why, in the northern hemisphere, air flows *anti-clockwise* around *low pressure* systems and *clockwise* around *high pressure* systems. In Figure 4.2 the air starts to move from *high pressure* to *low pressure*, but the *Coriolis force* acts *to the right* of the air's motion.

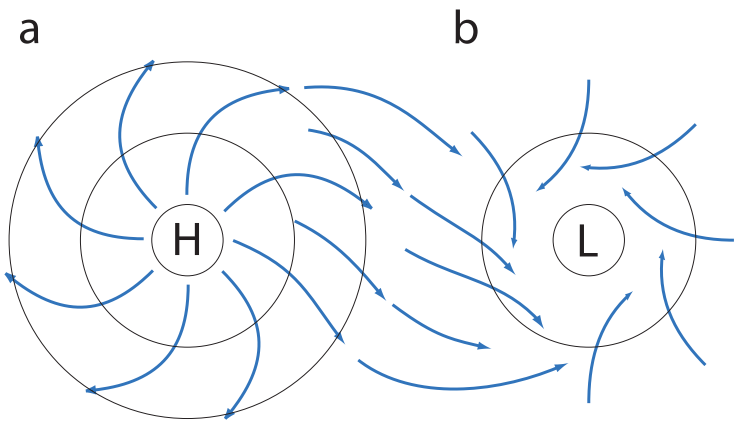


Figure 4.2: Schematic depicting why air flows clockwise around a high pressure system (a) and anticlockwise around a low pressure system (b). Note this diagram is for the northern hemisphere, it is the opposite in the southern hemisphere.

4.3.3 Acceleration of fluid

Do not worry too much about why the acceleration of a fluid is given by the left hand side of Equations 4.11 and 4.12. It is important, but we do not need to go into the details for this course.

4.4 Balanced flow

We consider two types of flow. One where there is no acceleration to the flow, so called *Geostrophic balance*, and that where there is a constant acceleration of the flow, so called *Gradient wind balance*. For Geostrophic balance we neglect the acceleration terms in the Equations of motion:

4.4.1 Geostrophic wind

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} = fv - \frac{1}{\rho}\frac{\partial P}{\partial x} \tag{4.17}$$

$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} = -fu - \frac{1}{\rho}\frac{\partial P}{\partial y} \tag{4.18}$$

which results in:

$$u_g = -\frac{1}{\rho f}\frac{\partial P}{\partial y} \tag{4.19}$$

$$v_g = \frac{1}{\rho f}\frac{\partial P}{\partial x} \tag{4.20}$$

that is as an approximation to the real winds, the *geostrophic wind*, can be estimated from the pressure gradient, density of air and the Coriolis parameter.

4.4.2 Gradient wind

The second type of approximation to estimating the wind is to take into account the acceleration term in the Equations of motion. This is useful to describe flow around *low pressure centres* or *high pressure systems*. We know that these flows are approximately circular. Hence, we set the acceleration to that for uniform circular motion (Equation 4.10), $-\frac{v_{gr}^2}{R}$. This results in quadratic equations to be solved for the wind speed.
For a low pressure system:

$$\frac{v_{gr}^2}{R} = -f v_{gr} + \frac{1}{\rho}\frac{|\Delta P|}{R} \tag{4.21}$$

where v_{gr} is the gradient wind; R is the radius that air travels around. The last term is the pressure gradient force. The gradient wind approximation predicts a smaller wind speed than the geostrophic approximation for flow around a low because the Coriolis force has to be smaller (to provide the inward acceleration).
For a high pressure system:

$$\frac{v_{gr}^2}{R} = f v_{gr} - \frac{1}{\rho}\frac{|\Delta P|}{R} \tag{4.22}$$

The gradient wind approximation, around a high, gives a higher wind speed than geostrophic flow because the Coriolis force must be higher than the pressure gradient force. Worthy of note is that if the pressure gradient is too high there can no flow around a high in certain conditions.

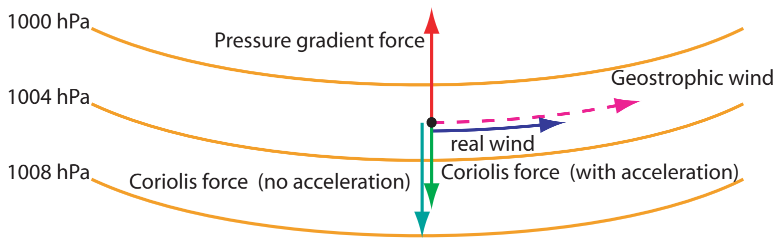


Figure 4.3: Schematic of forces on air flowing around a low pressure system. The pressure gradient force acts towards the low, but the Coriolis forces acts in the opposite direction. For gradient wind flow the pressure gradient must exceed the Coriolis force so that the acceleration toward the centre is large enough for circular motion.

4.5 Vorticity

We define u as the *east-west* component of the wind and v as the *south-north* component. We can write the wind as a vector quantity:

$$\mathbf{v} = \begin{pmatrix} u \\ v \end{pmatrix} \tag{4.23}$$

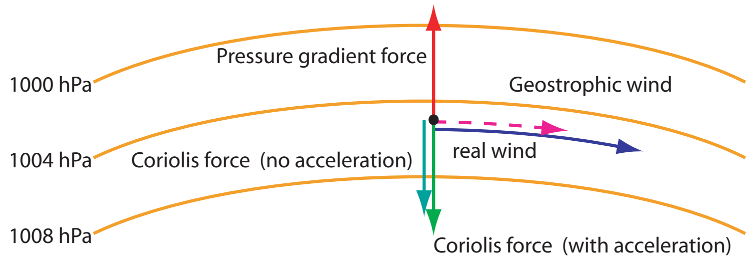


Figure 4.4: Schematic of winds flowing around a high pressure system. In this case the Coriolis force has to exceed the pressure gradient force; hence, winds have to be higher than geostrophic.

where u and v can be defined as functions of position x and y for example (or something else).

Vorticity, ζ, is a way of describing the rotation of air in some way. Air does not have to be rotate around a centre to have vorticity though: sheared flow can also have vorticity. A good way of thinking about it is what would happen to a paddle if it was put into the flow. If the paddle would rotate then the flow at that point has vorticity. In 2-d it is defined as the gradient of v with respect to x minus the gradient of u with respect to y:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{4.24}$$

4.6 Waves in the atmosphere

4.6.1 Gravity waves / Tsunamis

Gravity waves are waves generated in a fluid or at the interface between two media when the force of gravity or buoyancy tries to restore equilibrium. In the atmosphere *gravity waves* are generated in the troposphere by frontal systems or by airflow over mountains. See Figure 4.5.
Shallow fluid gravity waves have a speed (phase speed):

$$c = \sqrt{gh} \tag{4.25}$$

where c is the speed of the wave; $g = 9.8$ m s^{-2} is the acceleration due to gravity; h is the depth of the fluid. It is interesting to see how gravity waves propagate as the depth of the fluid changes, like when a Tsunami approaches the shore. This will be investigated in the practical.

4.6.2 Waves associated with jets

Vorticity in a fluid tends to persist. However, vorticity does not have to mean air rotates in circles, it can also mean the air has *shear*. The *jet stream* is an example

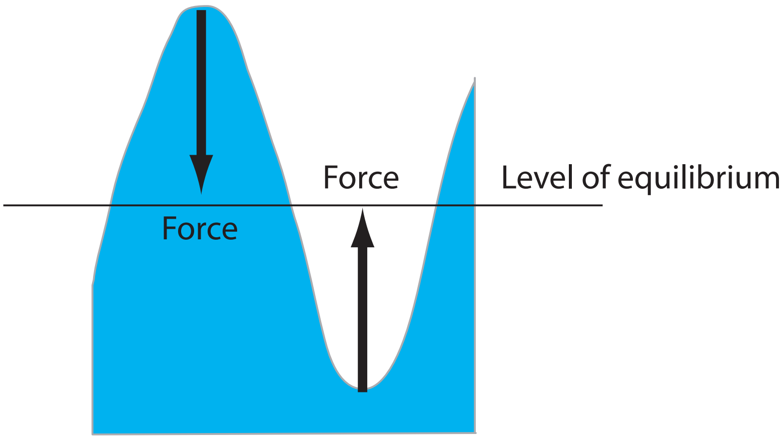


Figure 4.5: Schematic of a water gravity wave. In the peaks gravity provides the force to push the peak back to equilibrium. As gravity pushes the peaks down water moves into the trough and pushes the trough up.

of this: in the jet the air is moving very fast west-east and the winds decrease away from the jet; thus on the north side of the jet the air has positive vorticity and on the south side it has negative vorticity. If air close to the jet moves further away from the jet then it will find itself in a region of lower shear; however, vorticity still needs to maintained. Hence, the air will rotate (anti-clockwise for positive vorticity and clockwise for negative vorticity). This regular spacing of regions of rotating air is known as *barotropic instability* and can give rise to cyclones and anticyclones. See Figure 4.6

4.6.3 Rossby waves

When fluid flows up a mountain any vortices will be *compressed* and spread out side ways. This slows the rotation of the vortex and cause the *relative vorticity* to be negative. As the fluid flows off the mountain any vortices will be *stretched* in the vertical, which leads to an increase in the spin and positive vorticity, thus setting up a wave known as a *Rossby wave* (one that forms because of the conservation of vorticity). The effect is similar to when a ballet dancer starts to spin low down with their arms held out, but increases the rate of spin by standing tall and bringing their arms in towards their centre of mass. Rossby waves are a vital component of our weather.

An important point is air that is far north has a lot of absolute vorticity (because of the spin of earth on its axis). If it moves south, where the atmosphere is deeper, or become stretched in another way, it will conserve this vorticity and start to spin anticlockwise; thus moving north. This would not happen on a rotating cylindrical

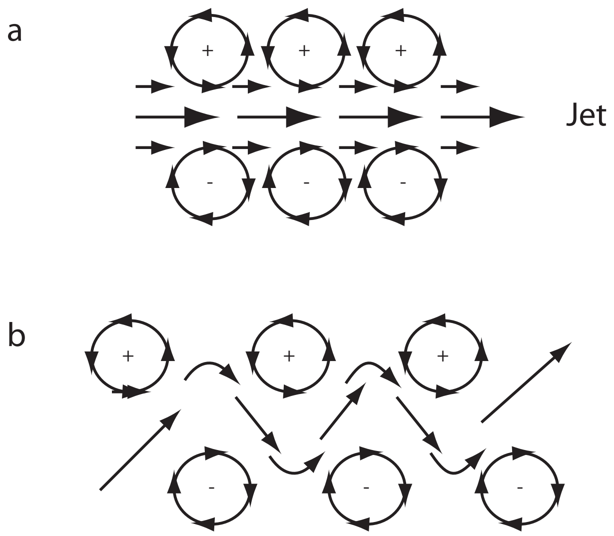


Figure 4.6: Schematic of barotropic instability, where cyclones and anticyclones form either side of a jet. (a) Initial development; (b) later stages of wave.

planet, where the planetary vorticity does not depend on distance from the equator.

4.6.4 Equatorial waves

Equatorial waves form because the Coriolis parameter goes through zero at the equator. One type is an instability that forms either side of a jet, similar to barotropic instability. However, because the Coriolis parameter changes sign, once the air crosses the equator it starts to slow down and rotate in the opposite direction. This does not happen with barotropic instability. Hence, these waves have a different name: *mixed Rossby-gravity waves*
In the practical you will test this using an *equatorial beta-plane*, which is a set-up that models the region close to the equator using a Coriolis parameter that is positive above the equator and negative below the equator.

Another kind of equatorial wave is the *equatorial Kelvin wave*. These waves form because the Coriolis forces acts to the right of air's motion in the northern hemisphere and to the left of the air's motion in the southern hemisphere. Thus if air / fluid is moving east there will be convergence towards the equator. Kelvin waves are only possible for air that moves towards the east on planets with anticlockwise rotation.

4.7 Questions to go through in class

4.7.1 Equations of motion in the atmosphere

1. In the atmosphere the pressure decreases going east to west from 1018 hPa to 1008 hPa over a distance of 1000 km. Using Equation 4.13 what will be the pressure gradient force per unit volume? Which direction does the force act?
2. Calculate the Coriolis parameter, f, at the following latitudes: $\phi = [50°, 0°, -50°]$.
3. Calculate the Coriolis force at $\phi = 50°$ for $u = 10$ m s^{-1}. Calculate the force for $\phi = -50°$ and $u = 10$ m s^{-1}. (assume $v = 0$ and $\rho = 1$ kg m^{-3}). (use Equation 4.16)
4. Calculate the Coriolis force at $\phi = 50°$ for $v = 10$ m s^{-1}. Calculate the force for $\phi = -50°$ and $v = 10$ m s^{-1} (use Equations 4.15).

4.7.2 Balanced flow

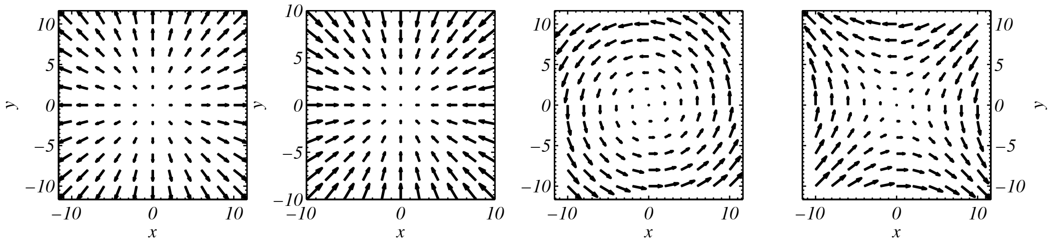
Geostrophic wind: The change in pressure at $\phi = 50°$ north, where the Coriolis parameter is 1×10^{-4} s^{-1}, is $\Delta P = 500$ Pa over a distance $\Delta y = 1000 \times 10^3$ m. Assuming the density of air is $\rho = 1$ kg m^{-3}, calculate the geostrophic wind (use Equation 4.19 and approximate $-\frac{\partial P}{\partial y}$ by dividing ΔP by Δy).

Gradient wind 1: Use the same conditions as above but assume that the air is rotating in a circle, radius Δx, around a low pressure centre. Use Equation 4.21 to estimate the gradient wind.

Gradient wind 2: Use the same conditions as above but assume that the air is rotating in a circle, radius Δx, around a high pressure system. Use Equation 4.22 to estimate the gradient wind.

4.7.3 Vorticity

Calculate the vorticity of the following vector fields using Equations 4.23 and 4.24:



1. $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. (see above, 1st plot on left).
2. $\mathbf{v} = \begin{pmatrix} -x \\ -y \end{pmatrix}$ (see above, 2nd plot from left).
3. $\mathbf{v} = \begin{pmatrix} -y \\ x \end{pmatrix}$ (see above, 3rd plot from left).
4. $\mathbf{v} = \begin{pmatrix} -y \\ -x \end{pmatrix}$ (see above, last plot on right).

4.7.4 Waves

All of these questions require the use of Equation 4.25.

1. A wave in the ocean has depth of 1 km. Calculate its phase speed.
2. The same wave approaches the coast-line where the depth is 2 m. What is its phase speed now?
3. Assuming the depth of the atmosphere is 10 km, how fast do gravity waves propagate?

4.8 Answers to questions

1. Just do $\frac{101800-100800}{1000\times10^3} \cong 1\times10^{-3}$ N m^{-3}.
2. $f = 2\times7.3\times10^{-5}\sin\phi$ (watch out for degree / radians mode). Hence we have: $f = 1.118\times10^{-4}$, $f = 0$ and $f = -1.118\times10^{-4}$ s^{-1}
3. We calculated the Coriolis parameters above for $\phi = 50°$ and $\phi = -50°$; hence multiply these by the density (1 kg m^{-3}) and the velocity to get $F_y = -1\times1.118\times10^{-4}\times10 = -1.118\times10^{-3}$ N m^{-3} and $F_y = -1\times-1.118\times10^{-4}\times10$ N m^{-3} = 1.118×10^{-3}. Hence, in both case the force is towards the equator.
4. We calculated the Coriolis parameters above for $\phi = 50°$ and $\phi = -50°$; hence multiply these by the density (1 kg m^{-3}) and the velocity to get $F_x = 1\times1.118\times10^{-4}\times10 = 1.118\times10^{-3}$ N m^{-3} and $F_x = 1\times-1.118\times10^{-4}\times10$ N m^{-3} = -1.118×10^{-3}. Hence, in the northern hemisphere the force is towards the right of its motion and in the southern it is towards the left of its motion.

4.8.1 Balanced flow

Geostrophic wind: Take the pressure gradient and divide by the Coriolis parameter and the density:

$$\begin{aligned} u_g &= \frac{1}{1\times10^{-4}}\frac{500}{1000\times10^3} \\ &= 5\text{m s}^{-1} \end{aligned}$$

Gradient wind 1: You get:

$$\frac{v_{gr}^2}{1000\times10^3} = -1\times10^{-4}v_{gr} + \frac{500}{1000\times10^3}$$

Solving as a quadratic $ax^2+bx+c=0$ with $a = 1\times10^{-6}$, $b = 1\times10^{-4}$ and $c = -5\times10^{-4}$ gives that $x = -104.77$ or $x = 4.77$. The solution is the positive value, so $v_{gr} = 4.77$ m s^{-1}.

Gradient wind 2: You get:

$$\frac{v_{gr}^2}{1000\times10^3} = 1\times10^{-4}v_{gr} - \frac{500}{1000\times10^3}$$

Solving as a quadratic $ax^2+bx+c=0$ with $a = 1\times10^{-6}$, $b = -1\times10^{-4}$ and $c = 5\times10^{-4}$ gives that $x = 94.72$ or $x = 5.28$. There are two stable solutions here; however, the most likely is $v_{gr} = 5.28$ m s^{-1}.

4.8.2 Vorticity

Answers are:

1. $\zeta = \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} = 0$
2. $\zeta = \frac{\partial -y}{\partial x} - \frac{\partial -x}{\partial y} = 0$
3. $\zeta = \frac{\partial x}{\partial x} - \frac{\partial -y}{\partial y} = 1-(-1) = 2$
4. $\zeta = \frac{\partial -x}{\partial x} - \frac{\partial -y}{\partial y} = -1-(-1) = 0$

4.8.3 Waves

1. $\sqrt{9.8 \times 1000} = 98$ m s^{-1}
2. $\sqrt{9.8 \times 2} = 4.43$ m s^{-1}
3. $\sqrt{9.8 \times 10000} = 313$ m s^{-1}. Close to the speed of sound!

4.8.4 Homework and reading

Work through the questions in this sheet in your own time (answers / workings are provided, but you should try to understand them).