A.5 Lecture 6

Example 2.5 Derive the 'radial' growth vs time of an ice crystal with circular disk morphology, with initial starting radius $a = a_0$. Use the expression for the capacitance $C_0 = \frac{2a}{\pi}$ and that the mass of a disk is $m = \pi a^2 h \rho_i$, where $h = 2\,\mu$m is the thickness of the disk.

Answer We use the ice crystal growth equation (Equation 2.46). Make a change of variable by differentiating the expression for the mass (above):

$$\frac{dm}{dt} = 2a\pi h\rho_i \frac{da}{dt} \tag{A.1}$$

Then set the RHS equal to the RHS of Equation 2.46. You should have:

$$\frac{da}{dt} = \frac{4}{h\pi}\left(\frac{s_i - 1}{\frac{\rho_i R_v T_\infty}{e_s D_v} + \frac{\rho_i L_v}{kT_\infty}\left(\frac{L_v}{R_v T_\infty} - 1\right)}\right) \tag{A.2}$$

Call the term in brackets on the RHS, A. Integrating yields:

$$a = a_0 + \frac{4}{h\pi} At$$

Example 2.5a For the problem above what is the radial growth rate after 100 seconds? and therefore what is the temperature of the ice crystal? (assume the saturation ratio $s_i = 1.10$ and $T_\infty = -15°$C, $P = 900$hPa and the initial size is $a_0 = 5\mu$m).

Answer Use Equation A.2

- For ice crystals at these ambient conditions you should find that $A = \frac{s_i - 1}{\frac{\rho_i R_v T_\infty}{e_{si}(T_\infty) D_v} + \rho_i \frac{L_s}{T_\infty k}\left(\frac{L_s}{R_v T_\infty} - 1\right)}$ is equal to $\cong 2.53 \times 10^{-12}$ m^2 s^{-1}. Note that $\rho_i = 910$ kg m^{-3} and $L_s = 2.8 \times 10^6$Jkg^{-1}.
- So the growth rate is $\frac{da}{dt} = \frac{4}{2\times10^{-6}\pi} 2.5 \times 10^{-12} \cong 1.6 \times 10^{-6}$ m s^{-1}.
- The 'radius' after 100 seconds is $a(t) = 5 \times 10^{-6} + \frac{4}{2\times10^{-6}\pi} \times 2.5 \times 10^{-12} \times 100 \cong 164\mu$m.
- For the initial temperature of the crystal use the mass growth rate (calculated using Equation A.1) and equate the latent heat of sublimation to the heat lost by Fourier's law.
- The mass growth rate is therefore:

$$\begin{aligned}\frac{dm}{dt} &= 2 \times 5 \times 10^{-6}\pi \times 2 \times 10^{-6} \times 910 \times 1.6 \times 10^{-6} \\ &= 2.9 \times 10^{-14}\end{aligned}$$

- Multiply the mass growth rate by the latent heat of sublimation, $L_s = 2.82 \times 10^6$, and equate to Fourier's law (note, $k \cong 0.023$ W m^{-2}):

$$2.9 \times 10^{-14} \times 2.82 \times 10^6 = 4 \times 2 \times ak\left(T_a - T_\infty\right).$$

- Rearranging for T_a: $T_a - T_\infty \cong 0.09$K. So the ice crystal temperature is elevated above 258.15K by 0.09 K.

Example 2.5b What is the radial growth rate of a drop at the same conditions?

Answer $\frac{da}{dt} = \frac{A}{\left(2At+a_0^2\right)^{1/2}}$, under these conditions, A is the right hand side of Equation 2.37, but the supersaturation is not 0.10 as that is the supersaturation over an ice surface.

- The saturation ratio over ice (1.10 in this case) can be converted to vapour pressure, e by multiplying by the saturation vapour pressure e_{si} from the saturation vapour pressure (Equation 1.12).
- This gives $e = 1.10 \times 610.7 \exp\left(\frac{2.82\times10^6}{461}\left[\frac{1}{273.15} - \frac{1}{258.15}\right]\right) \cong 182$ Pa.
- To get the saturation over liquid water divide the vapour pressure by the saturation vapour pressure over a liquid water surface: $\frac{182}{\exp\left(\frac{2.5\times10^6}{461}\left[\frac{1}{273.15}-\frac{1}{258.15}\right]\right)} \cong 0.95$. So the supersaturation is $0.95 - 1 \cong -0.05$. Meaning the drop will evaporate. The importance of this will become apparent when we cover supersaturation in clouds (Bergeron-Findeison process).
- A in this case is therfore $\cong -8.5 \times 10^{-15}$ ms^{-1}.
- So the growth rate is $\frac{A}{\left(2At+a_0^2\right)^{1/2}} \cong \frac{-8.5\times10^{-15}}{\left(2\times-8.5\times10^{-15}\times100+\left(5\times10^{-6}\right)^2\right)^{1/2}} \cong -1.76 \times 10^{-9}$ ms^{-1}.

Example 2.5c If there are 100 mg^{-1} of drops radius 10μ m at -36°C how long in seconds would it take to freeze halve of them by homogeneous freezing?

Answer Use Equation 2.42 which defines a stochastic process (like radioactive decay) in which drops freeze to form ice crystals.

- It can be recognised that the rate of change of ice crystals is minus the rate of change of drops so Equation 2.42 can be written: $\frac{dN_{drops}}{dt} = N_{drops}V_{drop}J(T)$, which has a solution similar to radioactive decay: $t_{1/2} = \frac{1}{VJ(t)}\ln\left(\frac{N_I}{N_F}\right) = \frac{1}{VJ(t)}\ln(2)$
- The nucleation rate, J, (Equation 2.41) has a value of 3.15×10^{12} at -36°C.
- The volume of a 10μ m drop is $\frac{\pi}{6}\left(10 \times 10^{-6}\right)^3 \cong 5.23 \times 10^{-16}$ m^3.
- The half-life is therefore: $\frac{1}{VJ(t)}\ln(2) \cong \frac{1}{5.23\times10^{-16}\times3.15\times10^{12}} \times 0.693 \cong 420$s