lecture 3

Weather forecasts need to consider flows on synoptic scales ~1000km down to droplet growth scales ~ 1μm ⇒ 1nm scale. (1)

Why do drops form & grow? [ what are molecules doing? ]

Why do gases move towards drops?

Are cloud drops [growing] warmer of colder than surrounding air?

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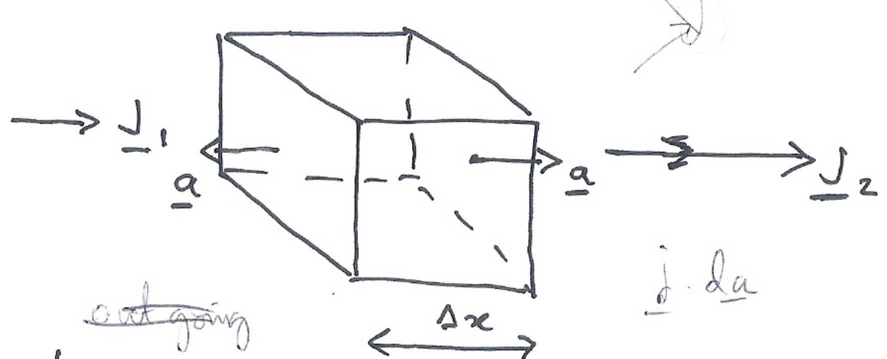
Consider flow of heat, $\underline{J}_h$ into a box [ $Wm^{-2}$ ] [ $Js^{-1}m^{-2}$ ]

$\longrightarrow x$




$\underline{j} \cdot d\underline{a}$

[ $|\underline{J}_2| > |\underline{J}_1|$ ? ]

=> A divergence of heat, box gets colder

$a \times dx = V$

~~outgoing~~

$$-\underline{J}_1 \cdot \underline{a} + \underline{J}_2 \cdot \underline{a} = J_2 a - J_1 a$$

$$= \cancel{\frac{\partial}{\partial x}} \frac{(J_2 - J_1)}{\Delta x} V$$

or in 3-D : Gauss' Divergence Theorem :-

$$\oint_S \underline{j} \cdot d\underline{a} = \int_V \nabla \cdot \underline{j} \, dV = -\frac{\partial q}{\partial t} = \frac{\partial}{\partial t}(mc\Delta T) = \frac{\partial(\rho V c \Delta T)}{\partial t} = \frac{\partial \rho c}{\partial t}\int \Delta T \, dV$$

from $q = mc\Delta T$

[ $m$ = mass
$c$ = heat capacity
$q$ = heat added ]

$$\left(\left(\rho_w c \int dV\right) \times \frac{\partial T}{\partial t}\right)$$

$$\frac{\partial q}{\partial t} = \int_V \rho c \frac{\partial T}{\partial t} dV$$

$$\therefore \rho c \int_V \frac{\partial T}{\partial t} dV = -\int_V \nabla \cdot \underline{j} \, dV$$

$$\therefore \frac{\partial T}{\partial t} = \frac{-1}{\rho c} \nabla \cdot \underline{j} \qquad (1)$$

Fourier's 1st law :

$$\underline{j}_h = -k \nabla T \qquad (2)$$

[ $k$ ≡ thermal conductivity $W K^{-1} m^{-1}$ ]

Insert (2) in (1)

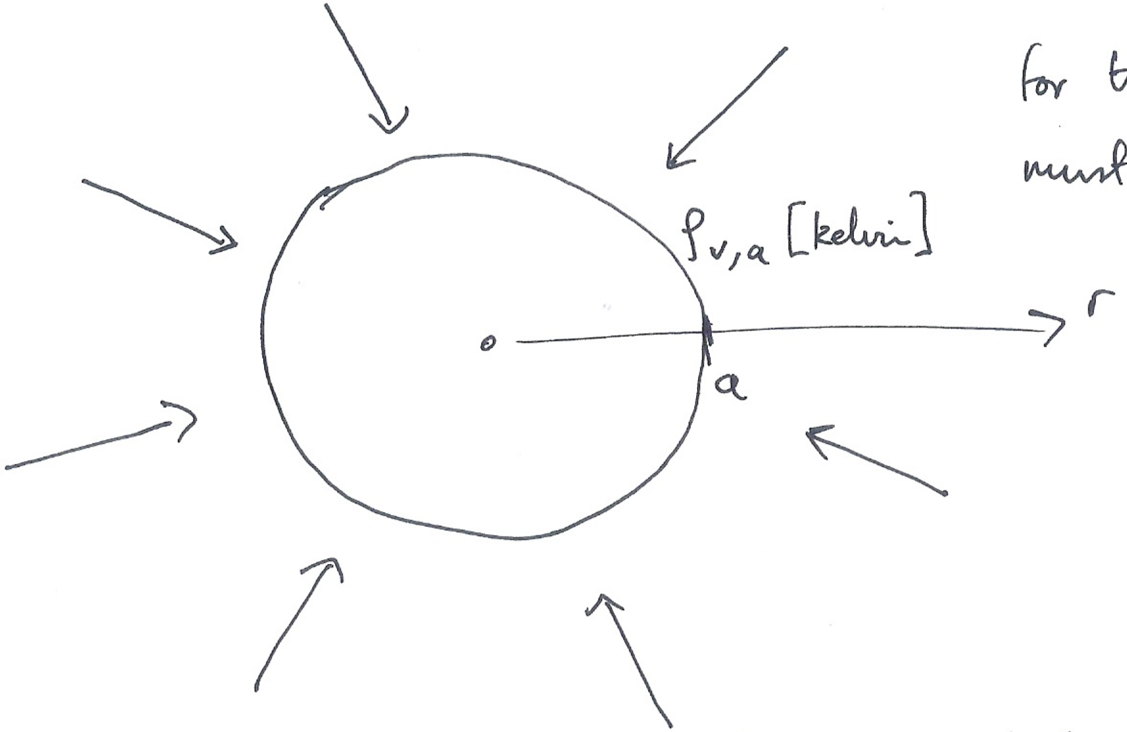
$$\boxed{\frac{\partial T}{\partial t} = \frac{-k}{\rho c} \nabla^2 T}$$

Fourier's 2nd law of diffusion.

For mass :- Fick's 2nd law :-

$$\boxed{\frac{\partial \rho_v}{\partial t} = -D_v \nabla^2 \rho_v}$$

Consider a drop in a vapor field, mass $m$.




for the drop to grow, vapor flux must be towards the drop.

$\rho_{v,\infty}$

$J = -D_v \nabla \rho$

$\nabla^2 \rho_v = 0$

$\Downarrow$

Make assumption that vapor field is <u>steady</u> (ie, $\frac{\partial \rho_v}{\partial t} = 0$).

Does this mean the drop cannot grow? No!

$$\frac{dm}{dt} = \oint \underline{j} \cdot \underline{da} \quad \text{[over surface of sphere]}$$

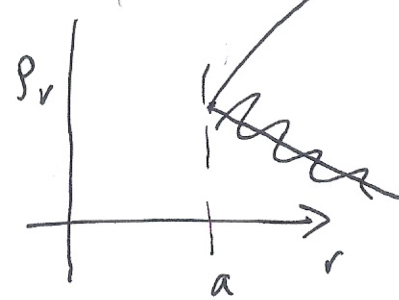
$$\nabla^2 \rho = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \rho}{\partial r}\right) = 0$$

radial vapor field

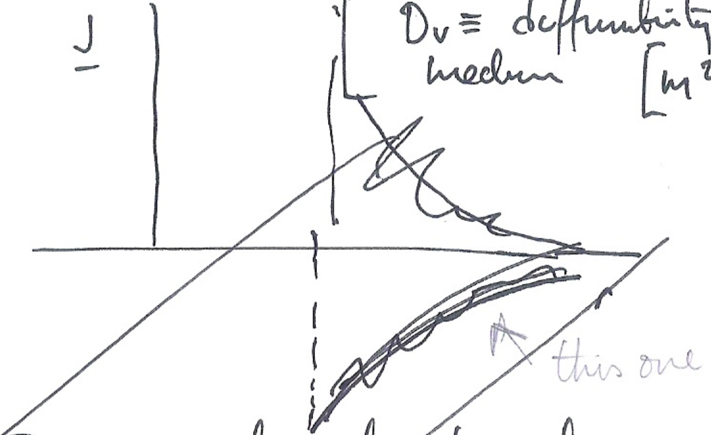
Soln :- $\rho(r) = \rho_{v,\infty} + (\rho_{v,a} - \rho_{v,\infty})\dfrac{a}{r}$

* $\rho_\infty > \rho_{va}$



$\therefore \quad \underline{J} = -D_v \nabla \rho$

$$= -\frac{D_v(\rho_{v,\infty} - \rho_{v,a})\,a}{r^2}$$




[$D_v \equiv$ diffusivity of substance in medium $[m^2 s^{-1}]$]

Drop :-

$$\frac{dm}{dt} = -\oint \hat{\underline{J}} \cdot \underline{da}$$

[$\underline{da} \equiv$ area element of sphere $\equiv 4\pi a^2$, directed outward]

$$= -\frac{D_v(\rho_{v,\infty} - \rho_{v,a})\,a \times 4\pi a^2}{a^2}$$

$$\boxed{\frac{dm}{dt} = 4\pi a D_v(\rho_{v,a} - \rho_{v,\infty})} = 4\pi a D_v\left(\frac{e_\infty}{R_v T_\infty} - \frac{e_a}{R_v T_a}\right)$$

[ideal gas law $P = \rho R T$]

$$\boxed{\frac{dq}{dt} = 4\pi a k (T_\infty - T_a)}$$

[Rate at which heat conducted away depends on $(T_\infty - T_a)$]

Adding mass to drop causes it to heat up, why?

$$\frac{dq}{dt} = -L_v \frac{dm}{dt}$$

Various approximations... Maxwellian drop growth eq$^n$:-

$$\boxed{\frac{dm}{dt} = \frac{4\pi a\,(s_e - 1)}{\alpha(T,P)}}$$

$\left[ s_e = \frac{e}{e_v} \quad e_v \text{ from Kelvin} \right]$

ie/ drops don't grow if $e \leq e_v$

mass of spherical drop (change of variable)

$$m = \frac{4}{3}\pi a^3 \rho_w$$

$$\left[ \frac{dm}{dt} = 4\pi a^2 \rho_w \frac{da}{dt} \right]$$

$$\therefore \frac{da}{dt} = \frac{(s_e - 1)}{\rho_w\, a\, \alpha(T,P)}$$

$[\, s_e - 1 = S_e$ supersaturation $]$

with sol$^n$:-

$$\boxed{a(t) = \sqrt{a_0^2 + 2(\rho_w\, \alpha(T,P))^{-1}(s_e - 1)t}}$$

Lets consider how 2 drops grow with time. $a(t) = \sqrt{a_0^2 + At}$

| | |
|---|---|
| $a_0 = 1$ | $a_0 = 8$ |
| $At = 80$ | $At = 80$ $\}$ 144 |
| $\therefore a = 9$ | $\therefore a = 12$ |

$12 - 9 = 3$


drops get closer together in size with time

Is this a problem?
(In describing clouds?)

It turns out it is . . . why?

If all drops were same size the clouds would precipitate all at once!

Example.

$$a(t) = \sqrt{a_0^2 + 2\rho_w^{-1}\alpha(T,P)(S_e-1)^{-1}\, t}$$

if $a_0 = 5\,\mu m$ $(5\times10^{-6}\,m)$ $T = 280K$, $P = 900\,hPa$

$S_e = 1.02$

$(\rho_w\alpha)^{-1} = \cancel{1.2557\times10^{7}}\ 8\times10^{-11}$ $(m^5\,kg^{-1}\,s^{-1})$ $[m^2\,s^{-1}]$ $\cancel{8.2\times10^{-6}}$ $1.2\times10^{-7}$

$\rho_w = 1000\,kg\,m^{-3}$

How long does it take for a drop to grow into precip?

$a = 1\times10^{-3}\,m$

$\Rightarrow \dfrac{a^2 - a_0^2}{2(\rho_w\alpha)^{-1}(S_e-1)} \simeq 3\times10^5$ seconds. (about 3-4 days!)

Clouds do not last this long!

$\dfrac{(0.1\times10^{-3})^2}{2\times(\rho_w\alpha)^{-1}S_e} \approx \underline{125\,s}$