Contents

Part I

Topics in cloud physics

Topic 1

Properties of clouds

1.1 Cloud types

According to the World Meteorological Organisation (WMO) there are ten basic cloud types. Like plants and animals, clouds are classified by type (genus), species and variety. We will only describe the ten genera here (for more detail on the 14 species and nine varieties see Dunlop, 2008):

1. Cirrus
2. Cirro Stratus
3. Cirro-Cumulus
4. Alto-Stratus
5. Alto-Cumulus
6. Cumulo-Nimbus
7. Strato-Cumulus
8. Stratus
9. Nimbo-Stratus
10. Cumulus

These can be broadly categorised into high, medium and low altitude clouds.

1.1.1 High-altitude clouds

These form near the tropopause at around 10 km (in the mid-latitudes).

Cirrus: are usually associated with fair weather, but could indicate an approaching warm front. They consist of only ice particles and have a feathery or wispy form. They are abbreviated Ci.

Cirro Stratus: are layer clouds which cover most of the sky. They usually herald bad weather. They are abbreviated Cs.

Cirro-Cumulus: are ice clouds usually with irregular attractive patterns. They are a heaped form of cirrus. They are abbreviated Cc.

1.1.2 Medium-altitude clouds

These develop at heights between about 2500 and 5500m.

Alto-Cumulus: Are flattened globules of cloud which are a mixture of ice and super-cooled water. They are white and grey in colour and are often the first sign that thunderstorms will follow. They are abbreviated Ac.

Alto-stratus: Are layer clouds of a dull grey colour. They are often the first sign that steady rain from nimbo-stratus is to follow. They are abbreviated As.

Nimbo-stratus: These are thicker, lower versions of altostratus and are always associated with rainfall or snow. They can be so thick that they make the day appear dark. They are abbreviated Ns.

1.1.3 Low-altitude clouds

These have bases at around 1-2km.

Stratocumulus: These clouds are usually composed of liquid water (but can be mixed-phase in the Arctic) and are grey and white with darker areas inside. Their apperance is rounded and rolled. They are abbreviated Sc.

Stratus: Usually liquid water clouds which are grey in colour. They are the lowest clouds in the sky and in fact form 'fog' if over hills or coasts. They are abbreviated St.

Cumulus: These heaped clouds are often described as 'cauliflower-like' in appearance. They are grey at the base and white at the top. They are abbreviated Cu.

Cumulo-nimbus: These are the classic anvil-headed clouds which can reach as high as 18-20 km in the tropics. They form low down and usually consist of liquid water, but at higher altitudes this water freezes to form water ice. They are abbreviated Cb.

There are other classifications of clouds such as: Lenticular clouds, Nacreous clouds, Noctilucent clouds, Mountain wave clouds, Contrails and distrails and fog/mist. For a none technical introduction to these cloud types see Lloyd (2007); Dunlop (2008).

There is another very important cloud type for the atmospheric physicist, not mentioned so far: the cloud chamber. These are used to investigate important microphysical processes occurring in clouds under controlled conditions. There are several around the world, most notably the large Aerosol Interactions and Dynamics in the Atmosphere (AIDA) chamber in Karlsruhe, Germany; the Cloud project at CERN `http://public.web.cern.ch/public/en/research/CLOUD-en.html`; and of course the Manchester Ice Cloud Chamber (MICC) `http://data.cas.manchester.ac.uk/micc/micc.htm`. Throughout this course I will use examples in several of these cloud types.

1.2 Cloud-scale motions

As covered in the meteorology part of the course the momentum equation is:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho}\nabla P - g\hat{\mathbf{k}} + \ldots \tag{1.1}$$

In the meteorology part of the course it was argued that the atmosphere was roughly in hydrostatic balance, which is true on the synoptic scale, and so vertical motion was ignored. On the scale of clouds and smaller convective elements this is not true. The way we often deal with this is by applying perturbation theory to the momentum equation. That is we assume that the pressure and density can be defined by a reference state that is in hydrostatic balance and small perturbations from that reference state: $P = P_0 + \delta P$ and $\rho = \rho_0 + \delta\rho$.

Therefore in the $\hat{\mathbf{k}}$ direction, the right hand side of Equation 1.1 becomes:

$$\begin{aligned} &= -\frac{1}{\rho_0 + \delta\rho}\frac{\partial P_0 + \delta P}{\partial z} - g + \ldots && (1.2)\\ &= -\frac{1}{\rho_0 + \delta\rho}\left(\frac{\partial P_0}{\partial z} + \frac{\partial \delta P}{\partial z}\right) - g + \ldots && (1.3)\end{aligned}$$

but $\frac{\partial P_0}{\partial z} = -\rho_0 g$ (hydrostatic relation), therefore:

$$\begin{aligned} &= -\frac{1}{\rho_0 + \delta\rho}\left(-\rho_0 g + \frac{\partial \delta P}{\partial z}\right) - g + \ldots && (1.4)\\ &= -\frac{1}{\rho_0 + \delta\rho}\frac{\partial \delta P}{\partial z} + \left(\frac{\rho_0}{\rho_0 + \delta\rho} - 1\right) g + \ldots && (1.5)\\ &\cong -\frac{1}{\rho_0}\frac{\partial \delta P}{\partial z} - \frac{\delta\rho_0}{\rho_0} g + \ldots && (1.6)\end{aligned}$$

In the final equation we can see that the term associated with gravity is equal to the ‘Buoyancy’ force (as derived previously in the Meteorology part of the course also equal to $\frac{\delta\theta}{\theta_0}g$). Hence positive vertical motion can be (and often is) the result of air being less dense than the surroundings ($\delta\rho$ is negative).

Once the air starts to move in the vertical more air will replace it (through continuity). The continuity equation is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \tag{1.7}$$

and must be solved simultaneously with the momentum equation.

Under the assumptions above this reduces to:

$$\nabla \cdot \rho_0 \mathbf{v} = 0 \tag{1.8}$$

This means that through continuity of mass, vertical motion will also result in horizontal motion.

The processes can still be adiabatic and hence conserve potential temperature, but the winds will move θ by advection.

$$\frac{d\theta}{dt} = F_{diabatic} \tag{1.9}$$

where $F_{diabatic}$ are processes that change θ (e.g. condensation, radiation and so on). Cloud and precpitation processes result in non-adiabatic processes and will now be discussed.

Thermal convection can therefore be thought of as follows:

Density anomaly	Also a potential temperature anomaly that results in a buoyancy force and causes vertical acceleration / motion of the air.
Mass continuity	The requirement of mass continuity means that a pressure gradient is set up that results in compensating horizontal and downward motion to replace the air rising in the thermal.
Advection	The resulting wind fields advect potential temperature (under adiabatic conditions).
Toroidal circulation	is set up due to the compensating downward and horizonal acceleration of air due to the pressure gradient. This causes a lot of mixing between the environmental air and the thermal air. This mixing process is usually referred to as entrainment.

Non-adiabatic processes (such as cloud formation) can significantly alter the dynamics of thermal convection through the release of latent heat.

1.3 Cloud formation mechanisms

You have covered what happens when moist air rises in the companion lectures to this part of the course. When the air is below saturation the air cools while conserving dry potential temperature, θ. When the air becomes saturated with water, the condensation of water from the vapour phase is accompanied by latent heat of vapourisation / condensation, which heats the air slightly.

The majority of clouds all form in basically the same manner when humid air is cooled below its dew point. This occurs when air comes in contact with a cold surface (and cools by conduction) or when it is forced to rise. The first process can occur at night, when the surface cools by radiating heat away to space or it can occur in coastal regions when warm air over land advects over a colder water mass. In the second process, air may be forced to rise in the atmosphere by four different mechanisms:

- Through heating of the ground during the day, which creates bubbles of warm air that break away from the surface as ‘thermals’, in the process known as convection;
- Through being forced to rise over a mountain range or similar barrier, which is known as orographic uplift;
- Through uplift at frontal surfaces in a depression: frontal uplift.
- Through a process known as convergence, where air flows into an area from different directions. When this occurs at the surface, there is only one way in which the accumulation of air can escape (due to continuity) and so it is forced to rise.

There is also another way that clouds can form in cloud chambers. Usually the method is to either hold the chamber at 1000 hPa in a pressure vessel, with a liquid or ice coating on the walls of the chamber. Then we suck the air out of the chamber using a fast vacuum pump to around 750 hPa, typically. Another way would be to introduce water vapour using a boiler (a kettle also works) until the air saturates with water vapour. Both these methods have their pros and cons.

1.4 Some definitions

Cloud formation occurs when air becomes saturated with water vapour. The Clausius-Clapeyron equation governs the equilibrium vapour pressure of water vapour, which you will have covered in thermodynamics (see `http://en.wikipedia.org/wiki/Clausius%E2%80%93Clapeyron_relation`):

$$\frac{de}{dT} = \frac{L_v}{T\Delta V} \tag{1.10}$$

where e is the pressure of water vapour, T is the temperature, L_v the latent heat of vapourisation and $\Delta V \cong \frac{R_v T}{e}$ the change in specific volume from the initial phase to the final phase. R_v is the specific gas constant for water vapour. Solving this equation yields for the vapour pressure over a liquid surface:

$$e_{sw} = 610.7 \exp\left(\frac{L_v}{R_v}\left[\frac{1}{273.15} - \frac{1}{T}\right]\right) \tag{1.11}$$

and for over an ice surface:

$$e_{si} = 610.7 \exp\left(\frac{L_s}{R_v}\left[\frac{1}{273.15} - \frac{1}{T}\right]\right) \tag{1.12}$$

where here L_s is the latent heat of sublimation (phase change of ice to vapour).

Mixing ratio has already been defined, but we define saturated vapour mixing ratio

$$w_s(T, P) = \frac{\epsilon e_s(T)}{P} \tag{1.13}$$

which can be defined for either liquid water or ice with the corresponding saturation vapour pressure (where do you think this comes from?).

Previously relative humidity was defined as e/e_s. We will stay with this definition, but also call this saturation ratio. We will refer to saturation ratio over liquid s_l and over ice s_i.

The supersaturation is not considered in these first few lectures, but we will cover it later. It is defined as:

$$S_l = s_l - 1 \tag{1.14}$$

$$S_i = s_i - 1 \tag{1.15}$$

I will use these terms throughout the lectures.

> Important note for this course: from this point forward many of the examples in this course will use parcel theory as the equations of motion are more the subject of PhD in atmospheric physics. A type of cloud where parcel theory can prove useful is stratocumulus or a cloud chamber.

1.5 Adiabatic liquid water mixing ratio

We first consider liquid only clouds here. Once clouds start to form in an expanding air parcel the temperature change no longer conserves dry potential temperature, θ since heat is added by the latent heat of vapourisation.

In the accompanying lectures you have derived an expression for the pseudo-adiabatic lapse rate. This was derived assuming that liquid water is removed from the parcel immediately (and so it did not change the heat capacity of the air). In practice the change in heat capacity of the air due to liquid water is negligible for most applications.

It can be shown that during a moist adiabatic process where liquid water alters the temperature of the air such that the quantity:

$$\theta_{q,sat} = T\left(\frac{100kPa}{P}\right)^{R'/(c_p+c_w\times Q)} \exp\left[\frac{w_s L_v}{T(c_p + c_w Q)}\right] \tag{1.16}$$

is conserved. Those interested in a derivation can look at Rogers and Yau (1989). Here, c_w is the heat capacity of liquid water and Q is the total water mixing ratio, which is conserved in an adiabatic parcel.

The adiabatic liquid water mixing ratio (kg/kg) can be defined as the difference between the saturated vapour mixing ratio at the point of interest and the saturated vapour mixing ratio at cloud base (why is this?):

$$ALMR = w_s(T_1, P_1) - w_s(T_2, P_2) \tag{1.17}$$

where T_1 and P_1 are the cloud base temperature and pressure and T_2 and P_2 are the temperature and pressure where the adiabatic liquid water mixing ratio is to be evaluated; $w_s(T, P)$ is the saturation vapour mixing ratio, which has been previously defined.

From this, the saturation vapour mixing ratio (Equation 1.13) and the saturation vapour pressure (Equation 1.11) we can calculate the adiabatic liquid water mixing ratio.

$$ALMR \cong \frac{\epsilon e_{sl}(T_1)}{P_1} - \frac{\epsilon e_{sl}(T_2)}{P_2} \tag{1.18}$$

The main problem in solving for $ALMR$ is that when you equate $\theta_{q,sat}$ at cloud base at P_1, T_1 and $\theta_{q,sat}$ within the cloud at P_2 it is not possible to solve analytically for T_2 (the temperature in the cloud). Iteration can be used or the following approximation can be used.

1.6 Solving the adiabatic liquid water mixing ratio equation

Firstly we can approximate Equation 1.16 relation (by using the usual approximation $\exp(x) \cong 1 + x$) as:

$$\theta_{q,sat} \cong T\Pi\left(1 + \frac{w_s(T, P)L_v}{Tc_p}\right) \tag{1.19}$$

where the approximation $\exp(x) \cong 1 + x$ has been applied and $\Pi = \left(\frac{100kPa}{P}\right)^{R/c_p}$ Secondly, substituting Equation 1.13 into Equation 1.19 yields:

$$\theta_{q,sat} \cong T\Pi\left(1 + \frac{\epsilon e_s(T)L_v}{PTc_p}\right) \tag{1.20}$$

or

$$\theta_{q,sat} \cong T\Pi\left(1 + \frac{1.5473 \times 10^3 e_s(T)}{PT}\right) \tag{1.21}$$

The saturation vapour pressure (Equation 1.11) can be approximated to:

$$\begin{aligned} e_{sw} &= 610.7 \exp\left(\frac{L_v}{R_v}\left[\frac{1}{273.15} - \frac{1}{T}\right]\right) & (1.22) \\ &\cong 610.7 \exp\left(\frac{L_v}{R_v}\left[\frac{T - 273.15}{273.15^2}\right]\right) & (1.23) \end{aligned}$$

because $273.15^2 \cong 273.15T$.

Thirdly, making the usual approximation to Equation 1.11 (e.g. $\exp(x) \cong 1 + x$) we have that:

$$\begin{aligned} e_{sw} &\cong 610.7 \exp\left(\frac{L_v}{R_v}\left[\frac{T - 273.15}{273.15^2}\right]\right) & (1.24) \\ &\cong 610.7\left(1 + \frac{L_v}{R_v}\left[\frac{T - 273.15}{273.15^2}\right]\right) & (1.25) \end{aligned}$$

and substituting this in Equation 1.20 and rearranging yields:

$$\frac{\theta_{q,sat}}{\Pi_2} \cong T_2 + \frac{\epsilon A L_v}{P_2 c_p}\left(1 + \frac{L_v}{R_v}\left[\frac{T_2 - 273.15}{273.15^2}\right]\right) \tag{1.26}$$

where $A = 610.7$, P_2 and T_2 are the pressure and temperature in the cloud and Π_2 is Π evalulated at this pressure. Fouthly Equation 1.26 can be rearranged to a linear equation in T_2:

$$0 \cong T_2 + \frac{\epsilon A L_v}{P_2 c_p} - \frac{\theta_{q,sat}}{\Pi_2} - \frac{\epsilon A L_v^2}{P_2 c_p R_v \times 273.15} + \frac{\epsilon A L_v^2 T_2}{P_2 c_p R_v \times 273.15^2} \tag{1.27}$$

which simplifies to:

$$T_2 \cong \frac{\frac{1.7815\times10^7}{P_2} + \frac{\theta_{q,sat}}{\Pi_2}}{1 + \frac{6.868\times10^4}{P_2}} \tag{1.28}$$

where T_2 and P_2 are the temperature and pressure at the point we are interested in. Calculation of the adiabatic liquid water mixing ratio can then be done by substituting T_2 into Equation 1.18.

Example 1.1 *A stratocumulus cloud has* $T = 280K$ *and* $P = 900$ *hPa at cloud base. Cloud top is at a pressure of* 870 *hPa. What is the adiabatic liquid water mixing ratio at cloud top?*

- What percentage of the total water mixing ratio is the adiabatic liquid water mixing ratio?
- If the number concentration of particles is 200 cm^{-3} what is the mean diameter, assuming all drops are the same size?
- If the liquid water mixing ratio is sub-adiabatic, say only 70% of the adiabatic value, what is the mean diameter assuming the drop concentration is 100 cm^{-3}?
- If the same initial temperature and pressure are used, but the cloud top is 200 m above cloud base what is the adiabatic liquid water mixing ratio assuming a lapse rate of 6.5K km^{-1}? (hint you will need to use the hydrostatic relation).

We had to specify the number of cloud drops here and assume they were all the same size. In later lectures we will learn how to predict the number of cloud drops and use tools to deal with cases where drops are not all the same size but are distributed in size.

THE KEY POINTS TO TAKE HOME HERE ARE:

- If a parcel of air ascends and does not saturate the temperature variation with changing pressure will conserve potential temperature, θ.
- Once the parcel of air saturates the temperature variation with changing pressure will conserve $\theta_{q,sat}$.
- Another key point is the derivation of the approximate form and the ability to calculate the temperature variation.

In order to help with the calculations see
`http://130.88.66.117/~mccikpc2/utilities/adiabatic01.html`
username: inuit
password: braunfels

1.7 Particle size distributions

Until now we have only considered the bulk thermodynamics of clouds; real clouds contain particles. In some of the earlier problems you were asked what the size of the particles in the cloud were, given a total liquid water mass and a total number concentration of particles—you will have assumed they were the same size. In real clouds the particles are not all the same size but are distributed in size. You already know this because you know that sometimes clouds rain! The reason they rain is because some cloud particles grow preferentially in size and become rain drops: the drag force of drops falling in air is proportial to their projected area, while their weight is proportial to their volume, hence their terminal velocity increases with size. We need to become familiar with describing particle distributions this because, as we shall see, the rate that particles (drops / ice crystals) grow within a cloud depends strongly on their size.

Some reasons cloud particles become distributed in size are:

- Not all cloud drops / ice crystals form at the same time so they are not given the same growth time.
- Real clouds have small variations in the temperature and vapour field within them, meaning that some particles grow from the vapour quicker than others.
- Certain processes that occur in clouds happen in only localised regions of the cloud.

1.7.1 Measurements of cloud particle size distributions

There are several ways of measuring the size-distribution of particles within a cloud. The most common way is to use an instrument that consists of a continuous wave laser which falls onto a detector (or dump spot). When a particle passes through the beam the intensity of the signal is recorded and this can be converted to particle size. Therefore the number of particles breaking the beam per second and the corresponding size of the particle is noted. Gradually a size distribution is built up from these individual particle events.

Different instruments have different intervals between the sizes that they measure and so to make comparisons between the instruments fair we tend to divide by the interval width, hence we define a size distribution as $\frac{dN(D)}{dD}$, where $dN(D)$ is the number of particles between D and $D + dD$ and D is the diameter. This has units of m^{-4}.

1.7.2 Parameterisation of cloud particle size distributions

A useful way of representing size distributions is with the generalised gamma distribution:

$$\frac{dN(D)}{dD} = n_0 D^{\mu} \exp\left(-\lambda_0 D\right) \tag{1.29}$$

here n_0, μ and λ_0 are constant parameters depending on the distribution being parameterised. By comparison to a straight line n_0 (m^{-4}) is called the intercept and λ_0 (m^{-1}) is called the slope of the distribution.

It has been found that on many occasions exponential spectra (i.e. $\mu = 0$) do a reasonable job of explaining the size distribution of particles within clouds. This is useful because exponential spectra can easily be integrated analytically.

1.7.3 Moments of cloud particle size distributions

The moment of a distribution is the integral of the product of the distribution and the independent variable, in this case D. E.g. the zeroth moment, M_0, is just the total number concentration (m^{-3})

$$M_0 = \int_0^\infty \frac{dN(D)}{dD} dD \tag{1.30}$$

and the 2nd moment, M_2 is proportional to the total surface area of the drops per m^3 of air:

$$M_2 = \int_0^\infty \frac{dN(D)}{dD} \times D^2 dD \tag{1.31}$$

specifically, the total integrated area, $Area = \frac{\pi}{4} M_2$. The 3rd moment, M_3 is proportional to total mass of the water drops per m^3 of air:

$$M_3 = \int_0^\infty \frac{dN(D)}{dD} \times D^3 dD \tag{1.32}$$

specifically, the liquid water mixing ratio, $w_l = \frac{\pi\rho}{6} M_3$.

The integral of the generalized gamma distribution is defined as:

$$\int_0^\infty n_0 D^\mu \exp\left(-\lambda_0 D\right) dD = n_0 \frac{\mu!}{\lambda_0^{\mu+1}} \tag{1.33}$$

therefore if we know n_0 and λ_0 we can calculate any moment of the distribution. For example the zeroth moment:

$$M_0 = \int_0^\infty n_0 \exp\left(-\lambda_0 D\right) dD = n_0 \frac{0!}{\lambda_0^1} \tag{1.34}$$

and the 2nd moment:

$$M_2 = \int_0^\infty n_0 D^2 \exp\left(-\lambda_0 D\right) dD = n_0 \frac{2!}{\lambda_0^3} \tag{1.35}$$

and the 3rd moment:

$$M_3 = \int_0^\infty n_0 D^3 \exp\left(-\lambda_0 D\right) dD = n_0 \frac{3!}{\lambda_0^4} \tag{1.36}$$

and so on.

Calculation of the fifth moment is important in precipitation forecasting—why?

1.7.4 Relationship between the particle size distribution and albedo

The cloud albedo is the fraction of total incident radiation that is reflected / scattered back to space by the cloud. We will not cover the physics of scattering of light by cloud particles in this course. Instead we will just quote the result that an excellent approximation to the albedo, A_c, of a liquid water cloud can be calculated using:

$$A_c \cong \frac{\pi M_2 \Delta Z}{\pi M_2 \Delta Z + 15.4} \tag{1.37}$$

1.8 Latham and Salter's cloud brightening scheme

Cloud brightening `http://en.wikipedia.org/wiki/Cloud_reflectivity_modification` is one of the favoured options for mitigating the warming that occurs through a doubling of CO_2. It is a so called geoengineering scheme.

The idea here is to seed maritime stratocumulus clouds with small sea salt particles; doing so increases the cloud albedo, A, because it increases the number concentration of particles in the cloud. It does not increase the mass in the cloud, which is defined by bulk thermodynamics (Section 1.5).

So if in the case of no seeding we have the zeroth moment, $M_0 = n_0 \frac{1}{\lambda_0}$ and the third moment $M_3 = n_0 \frac{6}{\lambda_0^4}$ then taking the ratio of these we have $\lambda_0 = \left(6\frac{M_0}{M_3}\right)^{1/3}$ and thus $n_0 = M_0 \left(6\frac{M_0}{M_3}\right)^{1/3}$.

From this we can calculate that the second moment, M_2 is:

$$M_2 = M_0 \left(6\frac{M_0}{M_3}\right)^{1/3} \frac{2}{\left(6\frac{M_0}{M_3}\right)} \cong 0.6057 M_0^{1/3} M_3^{2/3} \tag{1.38}$$

checking this is dimensionally correct we see that the third moment ($\cong D^3$) is raised to the power $2/3$ which leaves $\cong D^2$, which is what we want. From this we see that if we increase the number of cloud drops the second moment, M_2, of the distribution increases as $M_0^{(1/3)}$. Also from Equation 1.37 we see that increasing M_2 also increases the albedo A_c.

The change in albedo that can be achieved with this method is highest when the number of drops in the cloud is naturally low. If conditions are already polluted then the scheme loses its effectiveness.

Example 1.2 *A stratocumulus cloud, 100m thick, has a liquid water mixing ratio of* 0.5 *g m*$^{-3}$ *and a number concentration of* 50 *drops cm*$^{-3}$ *of air. Over a period of time you are able to increase the concentration of salt particles entering the cloud base to* 500 *drops cm*$^{-3}$ *of air. In both cases the size distribution can be considered to be exponentially distributed. What is the change in the cloud albedo?*

THE KEY POINTS TO TAKE HOME HERE ARE:

- Particle size distributions are parameterised using exponential spectra.
- Knowledge of two moments is enough to fit a moment conserving distribution.
- Any moment of the distribution can then be calculated.
- Albedo of stratocumulus clouds can be increased by addition of salt particles to the air entering the cloud.
- The susceptibility is highest for pristine clouds (i.e. clouds with low natural numbers of aerosol).
- You should be able to describe the principles involved in the cloud brightening scheme using the physics discussed in these lectures.
- You should be able to calculate the change in albedo resulting from adding sea water particles to the cloud.

In order to help with the calculations Excel spreadsheets have been put on Blackboard. The YouTube videos below guide you through how to use them:

Geoengineering clouds:	Showing how to use a spreadsheet to calculate the change in albedo due to seeding with CCN `http://www.youtube.com/watch?v=pmu0Rmuw_3A`

1.9 More details on geoengineering clouds

It is thought that reflecting ~ 3.7 W m^{-2} of energy back to space will be enough to stop catastrophic warming over the next 50 years due to the rising levels of CO_2.

Question: how much sea-spray would we need to do this?

1.9.1 From Latham et al (2008, Phil Trans)

The average solar irradiance F (W m^{-2}) received at the Earth's surface is:

$$F = 0.25F_0\left(1 - A_p\right) \tag{1.39}$$

Where, F_0 is the solar flux at the top of the atmosphere and A_p is the planetary albedo. This can be derived by considering the solar flux at the Earth's surface, $F_0 = 1370$ W m^{-2}, falling on a disk with the radius of Earth and the average solar irradiance, F being over the whole planet.

A change in albedo, ΔA_p, produces a forcing ΔF of:

$$\Delta F \cong -340\Delta A_p \tag{1.40}$$

If $f_1 = 0.7$ is the fraction of Earth's surface covered by ocean, $f_2 = 0.25$ is the fraction of the ocean surface covered by marine stratocumulus clouds and f_3 is the fraction of those clouds that are seeded, then the average change in cloud albedo, ΔA_c required for a change in planetary albedo, ΔA_p is:

$$\Delta A_c = \Delta A_p/(f_1 f_2 f_3) \cong -\Delta F/60 f_3 \tag{1.41}$$

which if $f_3 = 1$ then we need to change the cloud albedo by 0.062 or the planetary albedo by 0.011.

I've shown how to calculate change in albedo previously, but a simpler relation can be shown to be

$$\Delta A_c \cong 0.075 \ln\left(N/N_0\right) \tag{1.42}$$

where N is the drop concentration in seeded clouds and N_0 the same but in unseeded clouds.

Equations 1.41 and 1.42 together can be written:

$$-\Delta F \cong 4.5 f_3 \ln\left(N/N_0\right) \tag{1.43}$$

or

$$\left(N/N_0\right) \cong \exp\left(-\Delta F/\left[4.5 f_3\right]\right) \tag{1.44}$$

which if $-\Delta F = 3.7$ W m−2 is about 2.3.

Aerosol concentrations stay relatively constant over certain regions because there is a balance between being injected into the atmosphere and being removed.

We can estimate how much we would need to spray into a boundary layer to increase the levels up to N with N_0 the natural concentration level. The number of droplets per second that would need to be sprayed into the atmosphere is:

$$\begin{aligned} \frac{dn_{spray}}{dt} &= (N - N_0)A_E H f_1 f_2 f_3/\tau && (1.45) \\ &= N_0(N/N_0 - 1)A_E H f_1 f_2 f_3/\tau && (1.46) \end{aligned}$$

where A_E is the surface area of Earth $\sim 5{\times}10^{14}$m^2 and H is the depth of the boundary layer (typically 1000 m) and τ is the residence time of aerosols in the atmosphere $\sim$ 3 days or 2.6×10^5 seconds.

In order to estimate the volume of sea spray that must be introduced assume that the drops are all the same size and are about $a \cong 0.135\mu$m in radius; $N_0 \cong 100$ cm^{-3} or $\sim 10^8$ m^{-3}.

The volume spraying rate is therefore:

$$\begin{aligned} \frac{dV_{spray}}{dt} &= \tfrac{4\pi}{3}\rho_w a^3 \times \tfrac{dn}{dt} && (1.47) \\ &= \tfrac{4\pi}{3}\rho_w a^3 \times N_0(N/N_0 - 1)A_E H f_1 f_2 f_3/\tau && (1.48) \end{aligned}$$

which is approximately 460 m^3 per second!!

Topic 2

Physics of cloud formation

Now we have covered the necessary basic thermodynamics and definitions for cloud formation and also noted how particle sizes are distributed in clouds we will now start to consider the necessary physics of how cloud particles form and grow into the sizes observed in clouds. We will first cover how cloud drops are formed, then we will cover the subsequent growth of cloud drops and ice crystals.

2.1 Fick's law of diffusion and Fourier's law of heat transfer

These two laws can be used to great effect in describing the initial growth of water drops and ice crystals.

2.1.1 Diffusion of water vapour

If ρ_v represents the concentration of water vapour mass then the current density of water vapour is given by Fick's 1st law:

$$\bar{\mathbf{j}} = -D_v \nabla \rho_v \tag{2.1}$$

where D_v is the diffusivity of water vapour in air (m^{2}s^{-1}) and $\bar{\mathbf{j}}$ is the diffusion flux (kg m^{-2}s^{-1}). This law describes the observation that vapour moves from high to low concentrations in order to smooth out gradients. Note that D_v depends on T and P: $D_v = 2.11 \times 10^{-5} \left(\frac{T}{273.15}\right)^{1.94} \left(\frac{101325}{P}\right)$.

Another thing to note is that the vapour mass is conserved, so similar to Maxwell's equations (for an incompressible flow) we have a continuity equation:

$$\frac{\partial \rho_v}{\partial t} = -\nabla \cdot \bar{\mathbf{j}} \tag{2.2}$$

this states that the local time rate of change of the vapour field is equal to the net outflow and inflow of vapour mass, $\bar{\mathbf{j}}$ (current density).

Substitution of Eq 2.1 into Eq 2.2 yields Fick's second law of mass diffusion:

$$\frac{\partial \rho_v}{\partial t} = D_v \nabla^2 \rho_v \tag{2.3}$$

So the law governing diffusion of mass shows that the local time rate of change of mass in a volume is proportional to the second gradient of the mass field.

2.1.2 Diffusion of heat

Perhaps not unsurprisingly because the transfer of heat by conduction is governed by movement of particles (vibrations of molecules and electrons in a solid and collisions and diffusion of molecules during random motion in a gas) the law governing heat transport by conduction is very similar to mass transfer by diffusion. It is called Fourier's law:

$$\bar{\mathbf{j}}_h = -k\nabla T \tag{2.4}$$

where k is the thermal conductivity of the air and $\bar{\mathbf{j}}_h$ is the current density of heat. Note that k depends on T: $k = (4.40 + 0.071T) \times 10^{-3}$

Assuming the air is relatively stationary then the local change of temperature of the air will depend on the current density of heat to the air. If this heat is added at a constant pressure (usually does) then the rate of change of temperature due to the addition of heat will be:

$$\frac{\partial T}{\partial t} = -\frac{\nabla \cdot \bar{\mathbf{j}}_h}{\rho c_p} \tag{2.5}$$

substituting Eq 2.4 in Eq 2.5 gives us:

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T \tag{2.6}$$

where $\kappa = \frac{k}{\rho c_p}$.

2.2 Cloud condensation nuclei

One thing to note is that in the atmosphere practically all water drops within a cloud are formed on aerosol particles known as Cloud Condensation Nuclei (CCN). In a rising parcel, which therefore cools and saturates water will only condense if there is something for it to condense onto. CCN are the sites that water condenses onto; however, condensation does not occur *exactly* at the point of saturation, since CCN do not take up water straight away. A slight super-saturation is required for the CCN to become active condensation sites. The higher the super-saturation the more of the CCN become active and grow into cloud drops. Why?

2.2.1 aside Kelvin's equation—derivation not needed

The concept of surface tension, σ, is that it arises due to the force of cohesion between molecules in a liquid. The surface tension is the force exerted parallel to the surface divided by the length over which it acts: $\sigma = \frac{F}{L}$.

Consider the 'thought experiment' in Figure 2.2, it demonstrates *capillary action* of water. If you partly submerse a non-wetting tube in water the water in the tube will form a spherical cap and will sink below the water line outside of the tube[1]. The reason is because the vapour pressure over the drop increases. We can calculate how much it increases by recognising that the pressure at depth h is the same both inside and outside of the capillary. The increase in vapour pressure is provided by the extra weight of vapour on top of the spherical cap, but the remaining pressure increase is due to the surface tension (squeezing the molecules inside the drop so they repel each other). The pressure increase due to the surface tension is a function of the drop radius, so the drop will sink until the weight of air above

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[1] If the tube is wettable you will get a concave 'cap' in the tube and the water level will rise.

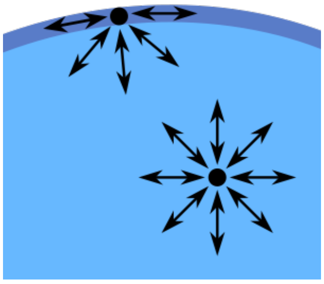

Figure 2.1: Taken from Wikipedia. Molecules inside the body of liquid experience forces pulling them in all direction (hence no net force). Molecules on the surface only experience the cohesive forces on one side and so are pulled inwards, eventually resulting in internal pressure (and force balance). If a molecule is moved away the lateral forces acting will tend to pull the molecule back into the bulk of the liquid.

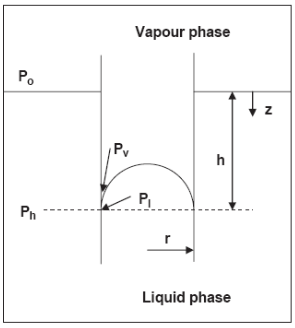

Figure 2.2: Taken from Galvin (2005). A non-wetting tube is inserted into the liquid, which causes a spherical cap to form. The vapour pressure far from the drop is e_0 and the pressure at h is the weight of water plus vapour above it.

it, plus the pressure increase due to surface tension, equal the weight of water in a column outside of the tube. We will now derive it.

The total pressure due to the liquid and vapour at *h outside the tube* is:

$$e_h = e_0 + \rho_l g h \tag{2.7}$$

where ρ_l is the density of water.

The total pressure at depth *h inside the tube* can also be written

$$e_h = e_0 + (e_v - e_0) + \Delta P_c \tag{2.8}$$

where e_v is the vapour pressure at the drops surface and ΔP_c is the pressure difference between the outside and the inside of the drop. Note that the pressure must increase due to the fact that the drop is curved and has surface tension (i.e. molecules pulling inward, creating an internal pressure, and hence force that must be felt on a surface at h).

The vapour pressure is the weight of vapour above the surface of the drop hemisphere, which is described by the hydrostatic relation: $\frac{de}{dz} = \frac{eg}{R_v T}$. This can be integrated from 0 to the surface of the drop:

$$e_v = e_0 \exp\left(\frac{gh}{R_v T}\right) \tag{2.9}$$

By equating Equations 2.7 and 2.8 we can derive an equation for the depth that the water sinks to:

$$h = \frac{\cancel{e_v} + \Delta P_c - \cancel{e_0}}{\rho_l g} \tag{2.10}$$

since for all practical applications $\Delta P_c >> e_v - e_0$

Since forces are in balance, the pressure difference between drop and air must balance the surface tension.

The force due to the difference in pressure is $\Delta Pc \times r^2 \sin\theta d\theta d\phi$ (θ measured between $\hat{k}$, the unit vector along the z axis) directed radially outwards. However, it is the component directed upward we are interested in as the lateral components cancel out. Hence, multiply by $\cos\theta$ and integrate over the hemisphere:

$$\begin{aligned} F &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \Delta P_c r^2 \sin\theta \cos\theta d\theta d\phi && (2.11) \\ &= \Delta P_c \pi r^2 && (2.12) \end{aligned}$$

which must be balanced by the surface tension, F_σ, which is directed along the circumference of the circle in contact with the surface at h. Thus the surface tension force that balances the pressure force can be calculated as $F_\sigma = 2\pi r\sigma$. Thus we have:

$$\Delta P_c = \frac{2\sigma}{r} \tag{2.13}$$

Sub Equations 2.9 and 2.13 into 2.10 to get Kelvin's equation:

$$e_v = e_{s,drop} = e_0 \exp\left(\frac{2\sigma}{R_v T \rho_l r}\right) \tag{2.14}$$

This says that the smaller the water drop the higher the vapour pressure on the surface of the drop is. Thus *supersaturation* is required to grow small drops into larger drops.

Droplet radius (nm)	1000	100	10	1
e_v/e_0	1.001	1.011	1.114	2.95

2.2.2 Cloud condensation nucleus counter

This is an instrument that measures the number of CCN in the atmosphere. It makes use of Fick's 2nd law and the result of Fouriers law between two flat parallel plates that have different temperatures. In this case there will be a continuous flow of heat and water vapour from one plate the the other so the set-up is usually called a continuous flow diffusion chamber.

- Have two parallel plates adjacent parallel plates held at different temperatures, T_1 and T_2.
- Coat them both with water so that the vapour pressure on both plates is at water saturation for the temperature of the plate $e_s(T_1)$ and $e_s(T_2)$.
- Flow filtered (letting CCN with sizes less than $\cong 1\mu$m through) air with CCN suspended in it through the parallel plates.
- The CCN in the air will be exposed to a supersaturation and will start to grow into cloud drops.
- Measure the size of the particles that come out of the instrument and count them with an optical particle detector if they are bigger than 1μm.

Equations 2.3 and 2.6 can be applied for the steady state case, since neither change with time:

$$\nabla^2 \rho_v = 0 \tag{2.15}$$

$$\nabla^2 T = 0 \tag{2.16}$$

which have solutions:

$$\rho_v(x) = \rho_v(0) + \frac{\rho_v(L) - \rho_v(0)}{L} x \tag{2.17}$$

$$T(x) = T_1 + \frac{T_2 - T_1}{L} x \tag{2.18}$$

where L is the separation distance between the plates. Note that from the ideal gas law $\rho_v = \frac{e}{R_v T}$ so that:

$$\frac{e(x)}{T(x)} = \frac{e_s(T_1)}{T_1} + \frac{e_s(T_2)/T_2 - e_s(T_1)/T_1}{L} x \tag{2.19}$$

and that the saturation vapour pressure between the plates:

$$e_s(x) = e_s(T(x)) \tag{2.20}$$

The CCN counter is usually operated for a range of temperature differences between the plates so that one can derive a number of ccn vs supersaturation plot—so called a CCN activity spectrum.

2.2.3 Measured concentrations of CCN

It is often found that measurements in the atmosphere using a CCN counter can be fitted to a power law:

$$N_c = CS^k \tag{2.21}$$

where N_c is the number concentration of activated CCN and S is the percent supersaturation. For polluted air $C \cong 300 - 3000$ cm^{-3} and $k = 0.2 - 2.0$ and for clean air $C \cong 30 - 300$ cm^{-3} and $k = 0.3 - 1.0$.

It has been found that the number of cloud drops is a function of the vertical wind speed in the cloud (Twomey, 1959), because this determines the rate of cooling and therefore the peak supersaturation in the cloud. The reason for this is even though supersaturation may be reached it takes time for vapour molecules to diffuse to the CCN; hence a higher supersaturation can be achieved for a higher vertical wind:

$$N_c \cong 0.88C^{2/(k+2)}\left[70w^{3/2}\right]^{k/(k+2)} \tag{2.22}$$

where w is the vertical wind speed in m s^{-1}.

Example 2.3 *A CCN counter is used to measure the number of active CCN verses super-saturation at the ground. From analysis of the data a power-law (Equation 2.21) is fitted to the data, and the parameters of the fit are* $C = 200$ *cm*$^{-3}$ *and* $k = 0.50$*. Calculate the number of cloud drops activated at cloud base for a wind speed of* $w = 0.2, 0.5, 1.0$ *and* 5.0 *m s*$^{-1}$.

Other things to consider:

- If you know the surface temperature and pressure and cloud pressure you could also calculate the adiabatic liquid water mixing ratio and thus the various moments of the distribution. This would allow you to calculate the albedo.
- If you add more CCN to the air you could quantify the impact it would have on C and k parameters and therefore the change in albedo.

THE KEY POINTS TO TAKE HOME HERE ARE:

- CCN spectra are parameterised using a power-law.
- Supersaturation is required to move water molecules to a curve surface.
- The higher the curvature the higher the supersaturation required.
- Fick's and Fourier's laws—these will be solved on the boundary of a sphere and then for any 3-D shape later in the course to allow us to calculate the growth rates of drops and ice crystals.

2.3 Growth of single cloud drops

In the case of diffusion of heat and mass between two parallel plates we assumed a steady-state so that the diffusivity and thermal conductivity could be neglected. For growth of droplets in the atmosphere we again assume a steady-state for Equation 2.3, but write down the Laplacian in radially symmetrical coordinates as we are considering diffusion to a sphere:

$$\nabla^2\rho_v = \frac{d^2}{dr^2}(\rho_v) + \frac{2}{r}\frac{d}{dr}(\rho_v) = 0 \tag{2.23}$$

One can see by substitution that a solution to this is $\rho_v(r) = \rho_{v,\infty} + (\rho_{v,a} - \rho_{v,\infty})\frac{a}{r}$, where $\rho_{v,\infty}$ is the vapour density at ∞ and $\rho_{v,a}$ is the vapour density on the boundary of the drop a. To get the growth rate of the drop, we need to compute the current density (or flux) of water vapour entering the sphere per unit time and integrate over the surface of the sphere. From Equation 2.1 (i.e. Fick's 1st law) the current density is:

$$\bar{\mathbf{j}}(r) = -D_v\frac{d}{dr}\rho_v = -D_v\frac{(\rho_{v,\infty} - \rho_{v,a})a}{r^2} \tag{2.24}$$

which is the vapour current density measured away from the sphere (along r). On the sphere's surface this current density or flux is $-D\frac{(\rho_{v,a}-\rho_{v,\infty})}{a}$. Integrated over the whole surface of the sphere we therefore have:

$$\frac{dm}{dt} = 4\pi a D_v(\rho_{v,\infty} - \rho_{v,a}) \tag{2.25}$$

and using the ideal gas law $\rho = e/(R_v \times T)$:

$$\frac{dm}{dt} = \frac{4\pi a D_v}{R_v}\left(\frac{e_\infty}{T_\infty} - \frac{e_s(T_a)}{T_a}\right) \tag{2.26}$$

Using similar arguments for the transfer of heat it can be shown that:

$$\frac{dq}{dt} = 4\pi a k(T_\infty - T_a) \tag{2.27}$$

If we assume that upon condensation the latent heat release is removed by the heat flux away from the drop:

$$L_v\frac{dm}{dt} = -\frac{dq}{dt} \tag{2.28}$$

So substituting Equation 2.27 in Equation 2.28 we obtain:

$$\frac{dq}{dt} = -L_v\frac{dm}{dt} = 4\pi a k(T_\infty - T_a) \tag{2.29}$$

which can be rearranged to give

$$T_a = T_\infty + \frac{L_v}{4\pi a k}\frac{dm}{dt} \tag{2.30}$$

So heat is being added to the drop and being transferred away. The drop heats up because when the vapour molecules become incorporated into the drop they loose their internal energy which is transferred to latent heat of vapourisation. Equations 2.25 and 2.29 were first derived by Maxwell. They along with an equation for the saturation vapour pressure (Equation 1.11) give two equations with two unknowns and can be solved iteratively using a computer.

To form an analytic expression one can make use the Clausius-Clapyron equation:

$$\frac{de_s}{dT} = \frac{L_v e_s}{R_v T^2} \therefore \tag{2.31}$$

$$e_s(T_a) = e_s(T_\infty)\exp\left(\frac{L_v}{R_v}\left(\frac{T_a - T_\infty}{T_a T_\infty}\right)\right) \tag{2.32}$$

$$e_s(T_a) \cong e_s(T_\infty)\left(1 + \frac{L_v}{R_v}\left(\frac{T_a - T_\infty}{T_\infty^2}\right)\right) \tag{2.33}$$

and Equations 2.26 and 2.30 to give:

$$\frac{dm}{dt} \cong \frac{4\pi a D_v e_s(T_\infty)}{R_v T_\infty}\left(s_l - \frac{1 + \frac{L_v \delta}{R_v T_\infty}}{1 + \delta}\right) \tag{2.34}$$

where $\delta = \frac{L_v}{4\pi a k T_\infty}\frac{dm}{dt}$ is very small and so, making approximations $\frac{1}{1+\delta} = 1 - \delta$ and $\frac{\delta}{1+\delta} = \delta$ the expression in the brackets becomes $s_l - 1 + \delta\left(1 - \frac{L_v}{R_v T_\infty}\right)$ or $s_l - 1 + \frac{L_v}{4\pi a k T_\infty}\frac{dm}{dt}\left(1 - \frac{L_v}{R_v T_\infty}\right)$. We can rearrange and make $\frac{dm}{dt}$ the subject to get the result:

$$\frac{dm}{dt}\left(1 + \frac{D_v e_s(T_\infty) L_v}{k R_v T_\infty^2}\left[\frac{L_v}{R_v T_\infty} - 1\right]\right) \cong \frac{4\pi a D_v e_s(T_\infty)}{R_v T_\infty}(s_l - 1) \tag{2.35}$$

$$\frac{dm}{dt} \cong 4\pi a \frac{s_l - 1}{\frac{R_v T_\infty}{e_s(T_\infty) D_v} + \frac{L_v}{T_\infty k}\left(\frac{L_v}{R_v T_\infty} - 1\right)} \tag{2.36}$$

Now since $\frac{dm}{dt} = \frac{d}{dt}4/3\pi a^3 \rho_w = 4\pi a^2 \rho_w \frac{da}{dt}$ we can write:

$$a\frac{da}{dt} \cong \frac{s_l - 1}{\frac{\rho_w R_v T_\infty}{e_s(T_\infty) D_v} + \frac{\rho_w L_v}{T_\infty k}\left(\frac{L_v}{R_v T_\infty} - 1\right)} \tag{2.37}$$

$$a\frac{da}{dt} \cong A \tag{2.38}$$

$$\tag{2.39}$$

where A is a constant depending on temperature and supersaturation. For constant supersaturation and temperature etc the result of this is:

$$a(t) = \sqrt{2At + a_0^2} \tag{2.40}$$

where a_0 is the intial radius of the drop. So after all that the increase in size of a drop with time is a parabola. This equation shows that if a cloud contains some large particles and some small particles the difference between the squares of the particle sizes remains constant with time $a_{0,1}^2 - a_{0,2}^2$ so if they are growing they will become closer together in size. Therefore this growth process alone cannot explain the observed size distribution of drops within a cloud. $\therefore$ other growth processes need to be considered.

Now we have covered this theory for drops we can use it to determine the growth rate of ice crystals; however, we need to take into account the deviation from spherical geometry.

2.4 Ice nuclei

There has been some suggestions in the literature that because the number of CCN activated depends on supersaturation over liquid water then the number of Ice Nuclei (IN) activated depends on supersaturation over ice. This has led to the development of ice nucleus counters based on this principle.

2.4.1 Ice nucleus counter

Due to the success of the parallel plate continuous flow diffusion chamber for measuring CCN, similar instruments have been made to measure IN. The differences are:

- The temperature of the plates are held below the melting point for ice.
- The plates are coated with ice and not liquid water.
- Supersaturation is reached wrt ice and not necessarily water.

We will talk about some issues and types of IN next lecture.

Example 2.4 *How long does it take for a drop of diameter* $10\mu m$ *to grow to a precipitation-sized drop of* 2 *mm given an in-cloud supersaturation of* 2%*? Assume the temperature is* $290k$*, pressure is* $900hPa$ *and calculate the diffusivity and thermal conductivity from the expressions for* D_v *and* k *in your notes. Use Equation 1.11 for* e_s*.*

Other questions to consider:

- For the problem above what is the drop growth rate after 100 seconds? and therefore what is the temperature of the drop? (hint: differentiate and substitute in Equation 2.29).
- Formation of rain requires particles of different sizes, which have different terminal fall-speeds. Given this can you say anything about why the growth of drops by vapour diffusion is not able to explain the formation of rain in real clouds?

THE KEY POINTS TO TAKE HOME HERE ARE:

- Fick's and Fourier's laws—can be used to solve for diffusion of mass and heat to / from drops.
- Explain why the temperature of a growing drop is warmer than it's surroundings.
- The final result that the growth rate is a parabola.
- Be able to calculate $a(t)$ due to vapour diffusion.
- Understand how continuous flow diffusion chambers work and that CCN and IN counters operate on very similar principles.

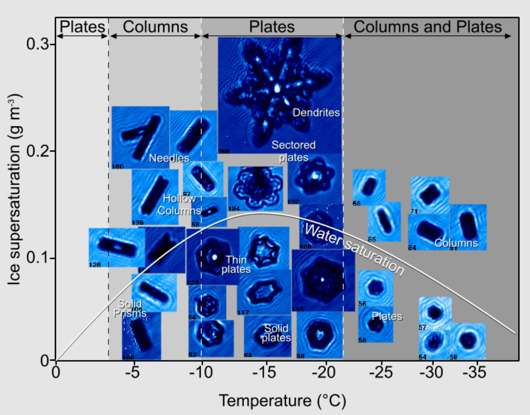

Figure 2.3: Some morphologies grown in lab

2.4.2 Structure of ice and morphology

For this course we just need to consider that, by far, the most common form of water ice in earth's atmosphere is hexagonal ice. This leads to a variety of different crystal morphologies that depend on temperature and supersaturation. The reasons we get these are not well understood, but are due to differences in latent heat release with temperature (and therefore heat flux from the surface of the crystals) and due to rarefaction in the vapour field at different points on the surface of the crystals.

2.4.3 Types of ice nucleation ice nuclei

The general requirement for an IN is that they be in-soluble, perhaps have a similar crystal structure to ice and be quite large so that they have a large surface area for nucleation of a new phase (ice) to occur on. Known ice nuclei are:

- Most mineral particles and desert dust are known to nucleate ice at temperatures of about -20°C (e.g. Connolly et al., 2009)
- Some (but not all) biological particles are known to nucleate ice at warm temperatures, but it is difficult for them to be carried up from the ground into clouds.
- Black carbon is a relatively poor ice nucleus, but there can be quite a lot of it in the atmosphere and so even though the efficiency is low it can be a significant source of ice.
- Fluff!

This is a generalisation, to cover all known ice nuclei is out of the scope of this course. Importantly the number of heterogeneous IN in the atmosphere is about 1×10^5 less that the number of CCN.

2.4.4 Ice nucleation modes

Homogeneous nucleation is the name given to the process where ice forms within a drop that contains no ice nuclei. This occurs due to statistical fluctuations in the bulk of the liquid and can be derived from statistical mechanics—but the derivation is outside of the scope of this course. However, the result is that we can define a nucleation rate, J ($\mathrm{m}^{-3}\mathrm{s}^{-1}$) which is the number of ice 'germs' (or nuclei) forming in the bulk of the supercooled liquid per unit volume every second. An ice germ is when water molecules arrange themselves statistically to have a structure similar to hexagonal ice. This depends on temperature because when it is warm the molecules have too much thermal energy to stay in the hexagonal structure for long enough and it becomes more favourable at colder temperatures if there are too few molecules in the cluster though they will tend to fall apart, an ice germ is the critical size above which addition of another molecule will result in crystalisation of ice in the drop. This can be parameterised as:

$$\log_{10}(J) = -237.4 - 14.75T_c - 0.307T_c^2 - 0.00287T_c^3 - 0.0000102T_c^4 \tag{2.41}$$

which is valid for $-65 \leq T_c \leq -30$. Here, T_c is the temperature in degrees celcius. As this is a *stochastic process*, the *ice crystal formation rate* is then calculated using an equation similar to radioactive decay:

$$\frac{dN_{ice}}{dt} = N_{drops}V_{drop}J(T) \tag{2.42}$$

Heterogeneous ice nucleation occurs when a nucleus acts as a catalyst reducing the temperature needed for ice formation. There are several different modes by which heterogeneous ice nucleation can occur (see Figure 2.4.4):

- Deposition nucleation occurs when water vapour is deposited directly onto a nucleus; it requires supersaturation with respect to ice and temperature below the melting point.
- Condensation-freezing occurs when drops form on ice nuclei and then the ice nucleus becomes active at some temperature.
- Contact nucleation occurs when an ice nucleus comes in contact with a drop at temperatures below the melting point of water.
- Immersion-freezing is pretty much the same as condensation-freezing!

Note that contact nucleation is not very effective within growing clouds and recent data suggest that *condensation-freezing* is the dominant mode of ice nucleation in the atmosphere. Also note that the number of condensation freezing nuclei is often observed to depend strongly of temperature (e.g. $N_{IN} = \alpha \exp(\beta T_c)$; and the number of deposition nuclei is observed to depend strongly on ice supersaturation (e.g. $N_{IN} = A\exp(BS_i)$). We will not cover this in detail in this course though.

2.5 Growth of single ice crystals

Recall that the electrostatic potential function, Φ ($\bar{E} = -\nabla\Phi$) satisfies Laplace's equation, $\nabla^2\Phi = 0$ and that Φ_S is a constant on a conductor and Φ_∞ is a constant

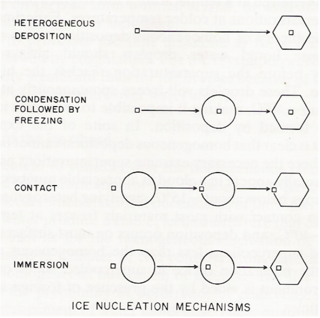

Figure 2.4: Heterogeneous nucleation is thought to occur only different pathways.

at ∞. If we assume a growing or sublimating ice crystal of the same geometry as a conducting body, in the steady-state ρ_v also satisfies Laplace's equation.

Recall Gauss' law from electro-statics, that the flux of $\bar{E}$ through a closed surface is equal to the charge enclosed within the surface (remember $Q = CV$):

$$\int_S \nabla\Phi \cdot \hat{n} dS = -\frac{Q}{\epsilon_0} = -\frac{1}{\epsilon_0} C_e \left(\Phi_S - \Phi_\infty\right) \tag{2.43}$$

Therefore we can draw a complete analogy with the growth of a crystal:

$$\frac{dm}{dt} = \int_S D_v \nabla\rho_v \cdot \hat{n} dS = -D_v \frac{1}{\epsilon_0} C_e \left(\rho_{v,S} - \rho_{v,\infty}\right) \tag{2.44}$$

Note that if the capacitance is that of a spherical conductor ($C_e = 4\pi\epsilon_0 a$) we get the same result as for a drop. We make some changes to the definition of C_e, since there is no need to use ϵ_0 and we also multiply by 4π to get:

$$\frac{dm}{dt} = 4\pi C D_v \left(\rho_{v,\infty} - \rho_{v,S}\right) \tag{2.45}$$

Look similar to the case for a drop? So we can just take the result we derived previously:

$$\frac{dm}{dt} \cong 4\pi C \frac{s_i - 1}{\frac{R_v T_\infty}{e_{si}(T_\infty) D_v} + \frac{L_s}{T_\infty k}\left(\frac{L_s}{R_v T_\infty} - 1\right)} \tag{2.46}$$

Note the changes are that we now use C instead of a, s_i instead of s_l, e_{si} instead of e_{sl} and L_s (latent heat of sublimation) instead of L_v.

For a simple, thin hexagonal plate we calculate the capacitance of a circular disk or radius a, i.e. $C_0 = \frac{2a}{\pi}$. Nowadays computer models can calculate the capacitance of more complex geometries, but McDonald (1963) measured the electrostatic capacitance of brass ice crystal models and came up with shape factors to multiply C_0 by to get the capacitance of other geometries—see Figure 2.5. The shape factor, f, can be used to calculate the capacitance of other crystals by $C = f \times C_0$.

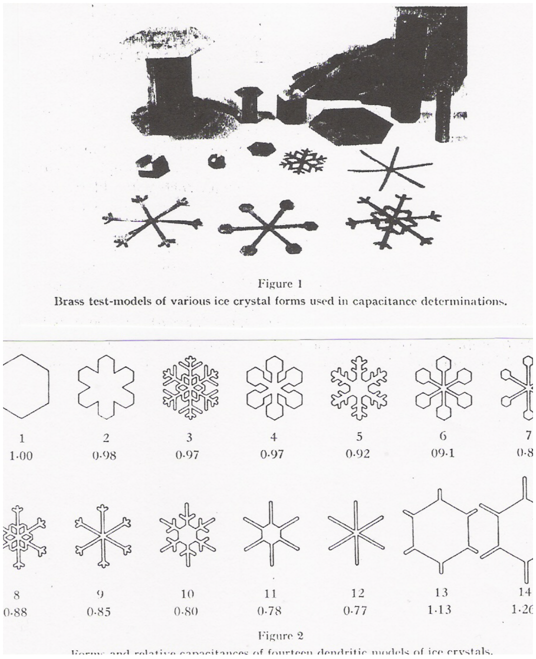

Figure 2.5: McDonald's early work showed that the electro-static analogy works very well for ice crystal growth.

2.6 Vapour growth and nucleation within clouds

Now we have covered CCN, IN and growth by vapour diffusion of single drops and ice crystals lets start to put all of these things together—see Figure 2.6. Note that in general because IN are typically in much lower concentrations than CCN the number of ice crystals formed by heterogeneous ice nucleation is much much less than that formed by homogeneous ice nucleation. This can have an unexpected effect on the ice crystal number.

Example 2.5 *Derive the 'radial' growth vs time of an ice crystal with circular disk morphology, with initial starting radius a_0.*

Other questions to consider:

- For the problem above what is the radial growth rate after 100 seconds? and

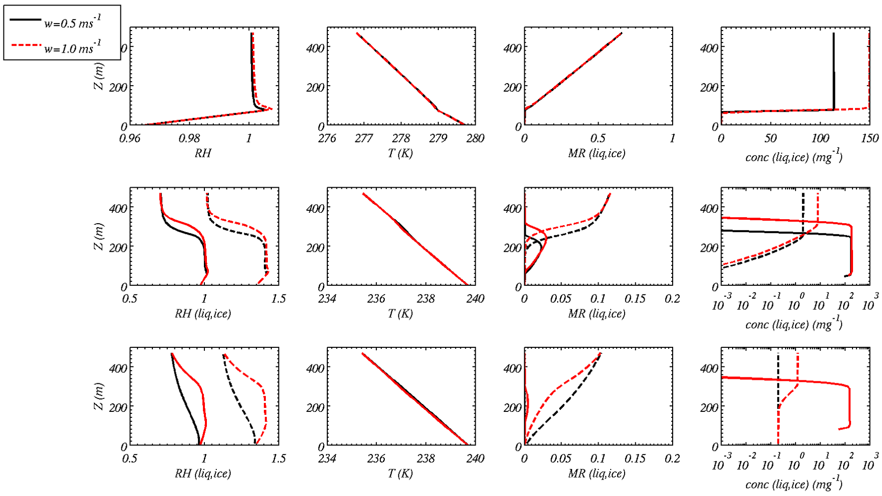

Figure 2.6: Top 4 figures shows a comparison between two model simulations of a warm cloud. It can be seen that the droplet number increases with increasing vertical wind because the supersaturation at cloud base increases; Middle 4 figures shows a comparison between two model simulations of a cold cloud where only homogeneous nucleation can act. It can be seen that liquid water condenses first (supercooled liquid water) and then homogeneous nucleation occurs. The final ice crystal number concentration is highest in the case with the highest updraft. Once ice crystals form they remove supersaturation and conditions approach ice saturation (equilibrium). The evaporation of drops in favour of ice crystal growth is known as the *Bergeron-Findesen Process*; Bottom 4 figures shows a comparison between two model simulations the same cold cloud as middle except that there are some heterogeneous ice nuclei in addition to homogeneous nucleation. In the case of the slow updraft conditions of water saturation are not reached because the low number of ice crystals is enough to deplete the supersaturation so that supercooled liquid water cannot form. In the case with the high updraft speed saturation over liquid water is reached and therefore both heterogeneous and homogeneous nucleation can occur. $\therefore$ adding IN in a cold cloud tends to reduce the total crystal number.

therefore what is the temperature of the ice crystal? (assume the saturation ratio $s_i = 1.10$ and $T_\infty = -15$°C and the intial size is $a_0 = 5\mu$m).

- What is the radial growth rate of a drop under the same conditions?
- If there are 100 mg^{-1} of drops radius 10μ m at -36°C how long in seconds would it take to freeze halve of them by homogeneous freezing?

THE KEY POINTS TO TAKE HOME HERE ARE:

- Be able to apply the equations of ice crystal growth for crystals of different morphology.
- Understand different ice nucleation modes and the variables important to ice nucleation in clouds.
- Be able to calculate the number of ice crystals formed by homogeneous nucleation.
- Understand how these processes interact within clouds.
- Know what the Bergeron–Findeisen (B–F) mechanism is and why it can lead to precipitation.

Topic 3

Supersaturation in clouds

So far when considering the water content within a cloud we have dealt with the bulk thermodynamical considerations within a cloud so when saturation is reached we assumed condensation occured instantaneously. This theory works reasonably well for describing the water content within liquid clouds or even ice clouds, but cannot be applied to mixed-phase clouds. In real clouds the supersaturation is vitally important to the outcome of the cloud and the amount of precipitation it produces because it affects the number of cloud drops and ice crystals that are in the cloud and therefore affects the production of precipitation.

The supersaturation in clouds is still an active area of research (one that I focus on quite a bit), supersaturation is something that is extraordinarily difficult to measure using an aircraft, due to its transient nature, and something that is *apparently* difficult to include within UK Met.Office (Met.Office) models and climate models.

Why do we need to know about supersaturation?

- Because we need to be able to predict the activation of cloud drops and the nucleation of ice crystals by heterogeneous deposition (which depends on supersaturation)
- Because we need to predict the growth of particles from the vapour (which depends on supersaturation)
- Because we need to know how long liquid water persists in a cloud (Bergeron-Findeison process).

3.1 Rate of change of supersaturation in a cloud

A theory describing the supersaturation in a cloud can be derived from definitions we have covered previously. Firstly as previously described the *supersaturation* is given by $S_l = \frac{(e-e_{sat,l})}{e_{sat,l}}$ so the rate of change is (quotient rule):

$$\frac{dS_l}{dt} = \frac{1}{e_{sat,l}}\frac{de}{dt} - \frac{e}{e_{sat,l}^2}\frac{de_{sat,l}}{dt} \tag{3.1}$$

From the previous definition of vapour mixing ratio $w_v = \frac{\epsilon e}{P}$ we can rearrange ($e = \frac{w_v P}{\epsilon}$) and define the rate of change of vapour pressure (product rule):

$$\frac{de}{dt} = \frac{1}{\epsilon}P\frac{dw_v}{dt} + \frac{1}{\epsilon}w_v\frac{dP}{dt} \tag{3.2}$$

Now we can use the Clausius-Clapyeron equation:

$$\frac{de_{sat,l}}{dt} = \frac{de_{sat,l}}{dT}\frac{dT}{dt} = \frac{L_v e_{sat}}{R_v T^2}\frac{dT}{dt} \tag{3.3}$$

and the 1st law of thermodynamics (time derivative):

$$c_p\frac{dT}{dt} - R_a\frac{T}{P}\frac{dP}{dt} - L_v\frac{dw_l}{dt} - L_s\frac{dw_i}{dt} = 0 \tag{3.4}$$

and substitute these into Equation 3.1 to give us:

$$\frac{dS_l}{dt} = \frac{1}{e_{sat,l}}\left(\frac{1}{\epsilon}P\frac{dw_v}{dt} + \frac{1}{\epsilon}w_v\frac{dP}{dt}\right) - \frac{e}{e_{sat,l}}\frac{L_v}{R_vT^2}\left(\frac{R_aT}{c_pP}\frac{dP}{dt} + \frac{L_v}{c_p}\frac{dw_l}{dt} + \frac{L_s}{c_p}\frac{dw_i}{dt}\right) \tag{3.5}$$

Following this we can take the time derivative of the hydrostatic relation for the parcel:

$$\frac{dP}{dt} = -\frac{gP}{R_aT}w \tag{3.6}$$

(where w is the vertical wind) and substitute Equation 3.6into Equation 3.5:

$$\frac{dS_l}{dt} = (S_l + 1)\left[\left(\frac{g}{c_p} - \frac{g}{R_aT}\right)w - \left(\frac{1}{w_v} + \frac{L_v^2}{R_vT^2c_p}\right)\frac{dw_l}{dt} - \left(\frac{1}{w_v} + \frac{L_vL_s}{R_vT^2c_p}\right)\frac{dw_i}{dt}\right] \tag{3.7}$$

where we have also used the fact that total water in the parcel is conserved:

$$\frac{dw_v}{dt} + \frac{dw_l}{dt} + \frac{dw_i}{dt} = 0$$

Recall the growth rates of drops and ice crystals $\frac{dm}{dt} \cong \frac{4\pi a S_l}{A}$, so that the total rate of change of mass (liquid or ice) is $\frac{dw_{l,i}}{dt} = N_{l,i}\frac{dm_{l,i}}{dt}$ and that $w_v = \frac{\epsilon e}{P}$. Thus Equation 3.7 describes the rate of change of supersaturation in a parcel of cloudy air with liquid drops and ice crystals.

The terms in Equation 3.7 are as follows:

Left hand side:	This is the rate of change of supersaturation.
First term on right:	The positive term describes how the saturation ratio increases due to the decrease in temperature of the parcel (conservation of energy), while the negative term describes the change due to the change in pressure of the air, which changes the vapour pressure of water vapour (see Equation 3.2).
Second term on right:	This is a sum of terms describing the reduction of water vapour as it condenses onto liquid water and also the reduction in supersaturation due to latent heating of the air (latent heat release and then conduction to the air).
Third term on right:	This is a sum of terms describing the reduction of water vapour as it condenses onto ice water and also the reduction in supersaturation due to latent heating of the air (latent heat release and then conduction to the air).

Example 3.6 *Explain what all the terms in Equation 3.7 mean.*

THE KEY POINTS TO TAKE HOME HERE ARE:

- Be able to explain the different terms in Equation 3.7.

3.2 Quasi-steady supersaturation and limiting cases

If we set the time-derivative of supersaturation to zero we can define a *steady-state* supersaturation where the expansion of the parcel is rapid enough to maintain a constant supersaturation:

$$0 = \left[\left(\frac{g}{c_p} - \frac{g}{R_a T}\right) w - \left(\frac{1}{w_v} + \frac{L_v^2}{R_v T^2 c_p}\right) N_l \frac{4\pi a_l S_l}{A_1} - \left(\frac{1}{w_v} + \frac{L_v L_s}{R_v T^2 c_p}\right) N_i \frac{4\pi a_i \left\{(S_l + 1) \times \frac{e_{sat,l}}{e_{sat,i}} - 1\right\}}{A_2}\right] \tag{3.8}$$

therefore, rearranging Equation 3.8:

$$\left(\frac{g}{c_p} - \frac{g}{R_a T}\right) w = \left(\frac{1}{w_v} + \frac{L_v^2}{R_v T^2 c_p}\right) N_l \frac{4\pi a_l S_l}{A_1} + \left(\frac{1}{w_v} + \frac{L_v L_i}{R_v T^2 c_p}\right) N_i \frac{4\pi a_i \left\{(S_l + 1) \times \frac{e_{sat,l}}{e_{sat,i}} - 1\right\}}{A_2} \tag{3.9}$$

This can be solved for S_l to find the *steady state supersaturation* for given concentrations and sizes of drops and ice crystals to give a solution of the form:

$$S_l = \frac{a_0 w - b_{1,i} N_i a_i}{b_l N_l a_l + b_{2,i} N_i a_i}$$

Early laboratory measurements have shown that once we have a mixed phase cloud the ice phase grows very rapidly and the liquid phase evaporates. This is known as the B–F process. It originally led to global models assuming that supercooled clouds are very rare in nature. Thankfully this is now being corrected.

3.2.1 Liquid clouds

The supersaturation in a liquid only cloud is usually maximum several 10s of metres above cloud base. The maximum depends on how many there are and the updraft speed (Figure 3.1). Eventually a steady-state will be reached (Figure 3.1), which corresponds to the steady-state superaturation with no ice terms:

$$S_l = \frac{a_0 w}{b_l N_l a_l}$$

as can be seen this depends on the updraft speed, w, and the number of drops, N_l.

3.2.2 Ice clouds

Ice only clouds behave similarly to liquid only clouds the difference is that because there are usually much less ice crystals than liquid drops the supersaturation can go higher initially and take longer to be depleted due to the growing ice crystals (see Figure 3.2).

3.2.3 Mixed phase clouds

In mixed phase clouds the supersaturation usually has to be at or above water saturation. When it drops below saturation the droplets evaporate until we are left with an ice cloud. The rate at which this happens depends on the updraft speed

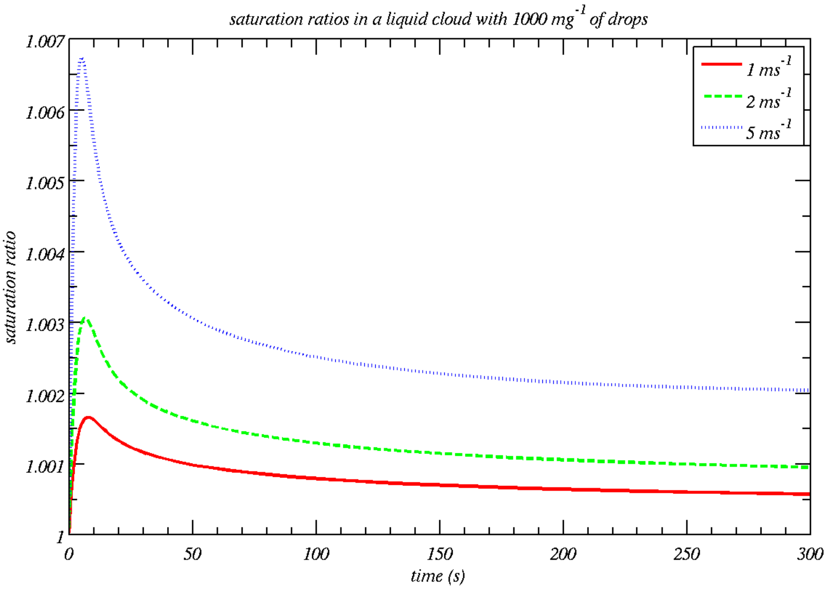

Figure 3.1: Supersaturation in liquid clouds with different updraft speeds.

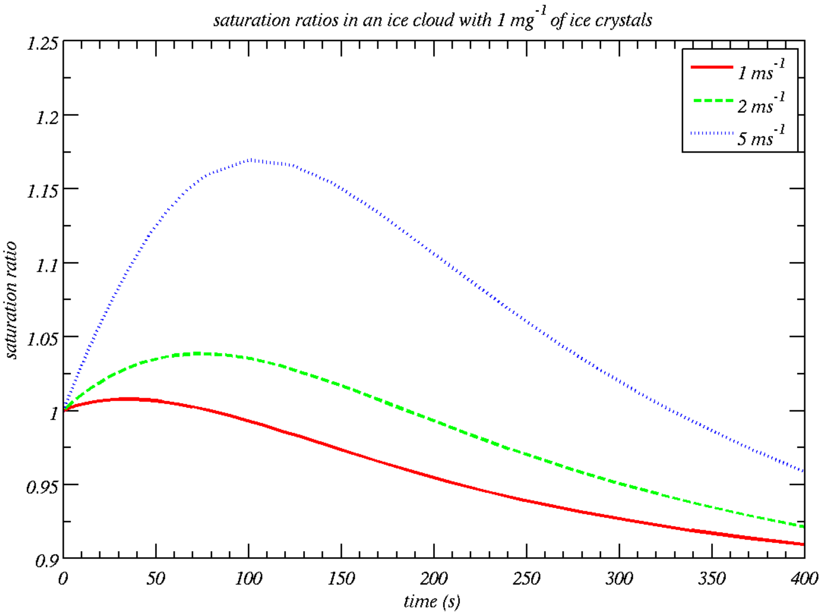

Figure 3.2: Supersaturation in ice clouds with different updraft speeds.

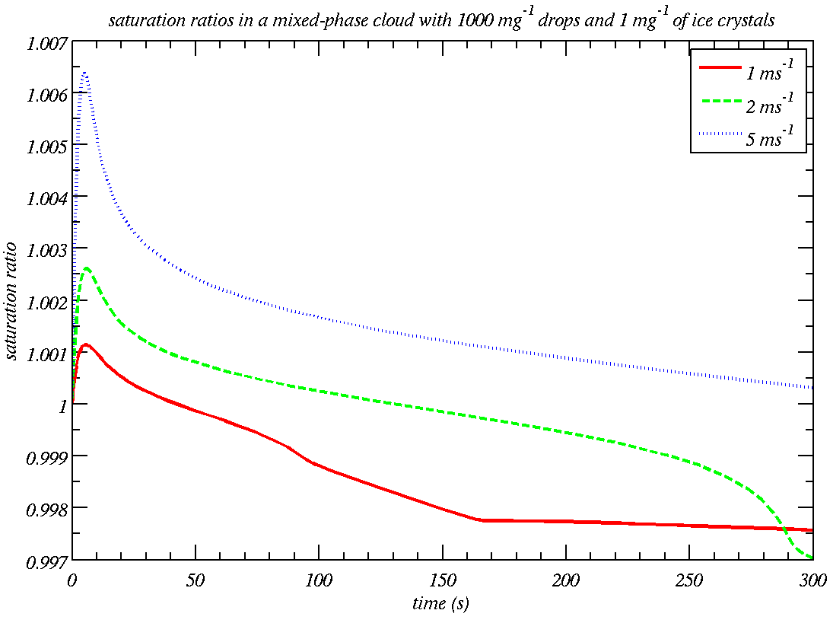

Figure 3.3: Supersaturation in mixed-phase clouds with different updraft speeds.

(Figure 3.3). A steady-state updraft speed can be defined by setting the steady-state supersaturation to 0, which is the minimum updraft required to maintain long lived mixed phase clouds.

$$w = \frac{b_{1,i} N_i a_i}{a_0}$$

3.2.4 Bergeron-Findeisen process

In a mixed-phase cloud eventually the ice crystals will grow so large that they will take up all of the water vapour so that the humidity drops below 1. This happens because the saturation vapour pressure of ice is lower than that for liquid water, so that it can grow when liquid water is evaporating.

3.2.5 Glaciation time-scale

In mixed phase clouds people have often thought about a glaciation time-scale, which is the time it takes the cloud to completely change to an ice only cloud if there were no vertical wind. Glaciation time-scale is depicted in Figure 3.4. It can be seen that the more ice crystals there are the quicker the glaciation time-scale.

3.2.6 Effects of turbulence and overturning

Real clouds are turbulent, they dont necessarily have an organised updraft, but can have turbulent motions driving the cloud formation process. In this case the glaciation time in a mixed-phase cloud does not make that much sense. Recent obser-

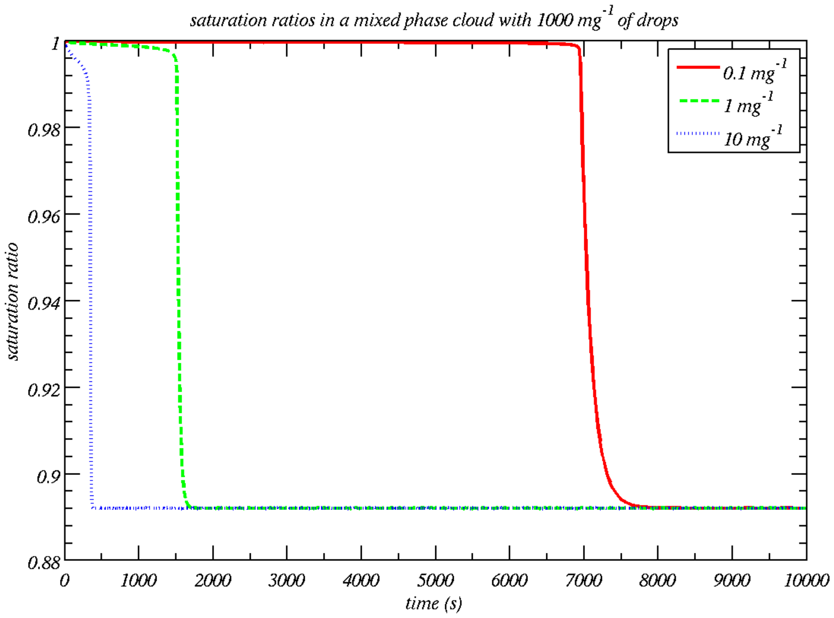

Figure 3.4: Supersaturation in mixed phase clouds with different concentrations of ice and no updraft

vations from the Chilbolton Observatory have shown that supercooled clouds can be very persistent even if they contain ice particles, which is inconsistent with the glaciation time-scale concept. The effects this has are depicted in Figure 3.5.

Questions to consider:

- Explain the Bergeron-Findeisen process.
- Explain what is meant by the concept of glaciation time-scale.
- Explain what is happening in the turbulent case.
- Be able to calculate steady-state supersaturations and critical ascent rates.

THE KEY POINTS TO TAKE HOME HERE ARE:

- Understand how these processes interact within clouds (hint: this may be on the exam).
- Know what the B–F mechanism is and why it can lead to precipitation.
- Explain what is happening in the turbulent overturning case.

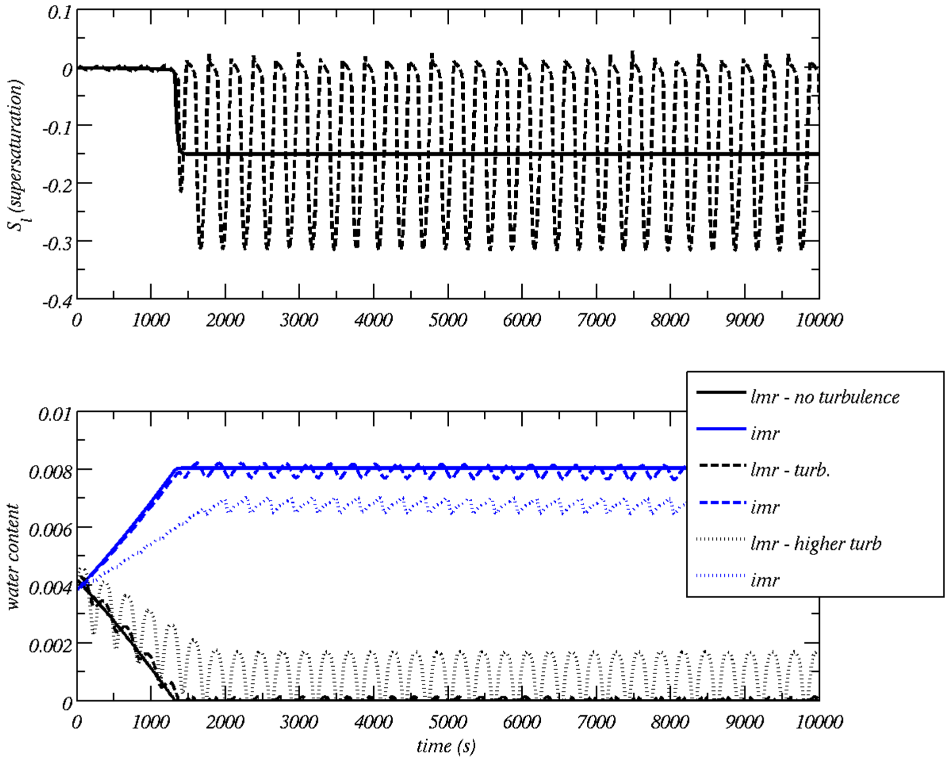

Figure 3.5: Supersaturation in a mixed phase clouds that is turbulent, $w = w1 \times \cos(2\pi t/\tau)$

Topic 4

Growth of precipitation particles

We considered the growth of droplets from the vapour in previous lectures. From the theory we saw that droplets of different sizes growing by vapour diffusion tend to get closer together in size with time. Therefore, after a some time this theory would predict that the droplet size-distribution in a cloud becomes very narrow. We know intuitively that clouds rain, but not all of the cloud falls as rain—some drops remain in the cloud. This must mean that growth by vapour diffusion is not the only growth process in a cloud. We will finish off our study of cloud microphysics by studying processes which can broaden the size distribution in clouds.

4.1 Collision and coalescence

Collision and coalescence is the name given to the process where drops collide with other drops and coalesce within a cloud. You may also hear to it referred to as warm rain autoconversion; however, this is modellers jargon.

For typical drops it can be shown that Brownian motion is not a large factor governing their motion. The motion occurs primarily due to gravitational settling, which is governed by Newton's second law of motion:

$$\frac{1}{2}\rho_a u^2 \frac{\pi}{4} D^2 C_d\left(Re\right) - \frac{\pi\left(\rho_w - \rho_a\right)}{6} D^3 g = \frac{\pi \rho_w}{6} D^3 \ddot{z} \tag{4.1}$$

where $\ddot{z}$ is measured upwards. For Stokes flow, (i.e. non-turbulent) $C_D(Re) = 24/Re = \frac{24\eta}{uD\rho_a}$. Here η is the visocity of air, which depends on temperature, but can be taken to be 1.718×10^{-5} Pa s for this course.

In practice we can assume that the drop is falling at it's terminal velocity. Other researchers have found that turbulent air motions that are often present in clouds can enhance the collision of drops over that expected by pure gravitational settling by perhaps a factor of 2, but it depends on the strength of the turbulence. This arises because the drops have inertia so do not necessarily follow the air streamlines, neglecting these difficulties and assuming the acc^n is zero we have:

$$\frac{24}{8} u \pi D \eta = \frac{\pi\left(\rho_w - \rho_a\right)}{6} D^3 g \tag{4.2}$$

Rearranging for the terminal velocity, u yields:

$$u = \frac{\left(\rho_w - \rho_a\right)}{18\eta} g D^2 = a D^b = 3.129 \times 10^7 D^2 \tag{4.3}$$

so the terminal velocity can be parametrised by a power law in diameter.

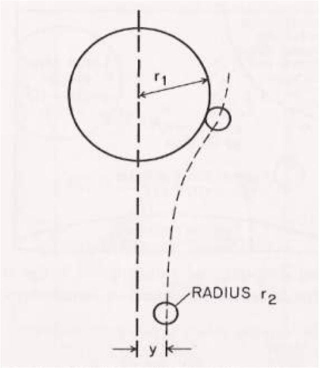

Figure 4.1: A small drop in the path of a large drop will not necessarily collide with it due to inertial and aerodynamic forces.

4.1.1 Collision efficiency

As large drops fall past small drops the small drops may be swept up by the large drops. The air will move around the drop, which causes an acceleration of the air perpendicular to the direction the drop is falling in. This influences the motion of the small drop relative to the large drop, thus moving it away from the large drop. There is a displacement of separation, y, (measured from the two droplet centres) above which the two drops will move far apart enough so that they will not collide. The collision efficiency is defined as the ratio of the total area that would result in a collision πy^2 to the geometrical area of overlap, $\pi (r_1 + r_2)^2$.

$$E(r_1, r_2) = \frac{y^2}{(r_1 + r_2)^2} \tag{4.4}$$

$$E(D, d) = \frac{4y^2}{(D + d)^2} \tag{4.5}$$

$E(D, d)$ is small when the collected drop d is small compared to the collector D. Once the collected drop size is larger than about $10\,\mu$m it can be assumed that $E(D, d) \cong 1$ although this is a generalisation.

4.1.2 Sweep-out kernel

The volume of air swept out per second by the large drop in a reference frame that sets the velocity of the small drop to zero is

$$\frac{\pi}{4}(D + d)^2 \left|u(D) - u(d)\right| \tag{4.6}$$

This is also called the sweep-out kernel.

4.1.3 Continuous growth model

This model considers the growth of a drop that has somehow managed to find itself slightly bigger than the rest of the other drops probably by heterogeneities in the atmosphere or differences in the aerosol size distribution on which the drops grew in the first place. We assume the drops grow continuously in size by collisions with other drops so the number of drops swept out per second by the large drop is:

$$\frac{\pi}{4}(D+d)^2\,|u(D)-u(d)|E(D,d)\frac{dN}{dd}dd \tag{4.7}$$

where $\frac{dN}{dd}$ is the drop size distribution function (see earlier lectures). Therefore the volume growth rate of the large drop is calculated by recognising that each addition of a drop size d adds a volume of $\frac{\pi}{6}d^3$ to the large drop:

$$\frac{dV}{dt}=\int_0^D\frac{\pi}{6}d^3\frac{\pi}{4}(D+d)^2\,|u(D)-u(d)|E(D,d)\frac{dN}{dd}dd \tag{4.8}$$

Making a change of variable $V=\frac{\pi}{6}D^3$ and $\therefore\ \frac{dV}{dt}=\frac{\pi}{2}D^2\frac{dD}{dt}$ enables us to write down the diameter growth rate:

$$\frac{dD}{dt}=\int_0^D\frac{1}{6}d^3\frac{\pi}{2}\frac{(D+d)^2}{D^2}|u(D)-u(d)|E(D,d)\frac{dN}{dd}dd \tag{4.9}$$

We could substitute an exponential form for $\frac{dN}{dd}$ and integrate; however, it is easier to approximate and say that $D+d\cong D$ and that $u(d)\cong 0$ so that Equation 4.9 becomes:

$$\frac{dD}{dt}\cong\int_0^D\frac{1}{6}d^3\frac{\pi}{2}u(D)E(D,d)\frac{dN}{dd}dd \tag{4.10}$$

Note that $\int_0^D\frac{\pi}{6}\rho_w d^3\frac{dN}{dd}dd$ would just be the liquid water content, w_l so integrating the RHS of Equation 4.10:

$$\frac{dD}{dt}\cong\frac{\bar{E}w_l aD^b}{2\rho_w} \tag{4.11}$$

where w_l is the liquid water mixing ratio. This has solution:

$$D(t)=\left(\frac{\bar{E}w_l a(1-b)}{2\rho_w}t+D_0^{1-b}\right)^{\frac{1}{1-b}} \tag{4.12}$$

Please note that the continuous growth equation is an approximation because in reality growth occurs by discrete events of droplet capture and the radius cannot take on any value, but it has to be multiples of values. In practice this leads to even quicker broadening, but we wont cover the relevant equations here—they are more the scope of a PhD.

However, we still have difficulty in explaining the observations that rain size distributions falling from clouds are exponentially distributed—where do the small particles come from if they are constantly growing? The answer is from drop break-up.

4.2 Coalescence and drop breakup

Not all collisions between drops will result in a coalescence event. Sometimes drops will collide and bounce off each other because the surface tension force is quite high compared to the momentum of the drops. This is treated by multiplying Expression 4.7 by another efficiency, the coalescence efficiency, E_{coal}. For this coarse we will not treat this explicitly.

Sometimes collisions between drops will cause the resulting drop to become unstable and breakup—so called collision induced breakup. Once droplets grow to very large sizes ($\cong$ 6mm in diameter) the surface tension of water is insufficient to hold the drop together and they can break-up spontaneously—so called spontaneous raindrop breakup.

There are different 'modes' of breakup:

Neck / filament	originates through glancing blows and is where a filament stretches away from the bulk of the drop and detaches.
Sheet	originates when the large drop is split into two as a droplet collides slightly off centre with it. Half of the large drop is torn off in a sheet.
Disk	originates when the smaller drop collides at the centre of the larger drop. Coalescence occurs temporarily and the drop spreads out into a sheet before disintegrating.

The many 'small' particles in the distribution (which we have said is negative exponential) arise due to droplet break-up. The physics governing this process is beyond the scope of this course, but the final result is that equilibrium size distributions can be approached where there is a dynamic equilibrium between the growth by collision-coalescence and breakup.

4.3 Accretion and aggregation

Analogous equations to Equation 4.12 can be written down to *approximately* describe the growth of ice by accretion of liquid drops and also the aggregation of ice crystals. For accretion there is the added complexity that when droplets freeze onto the ice particles they release latent heat and warm the surface—this can cause the ice crystal to melt or sublime. For aggregation of ice crystals the geometry of the crystals complicates things, but generally the principle is the same. For aggregation the collision efficiency and coalescence efficiency are grouped into one efficiency which we call the aggregation efficiency, $E_{agg} = E \times E_{coal}$ and this is approximately equal to 0.1.

Example 4.7 *Explain why with reference to the solutions of the relevant growth equations why growth by vapour diffusion leads to a narrowing of the distribution, whereas growth by collision and coalescence leads to a broadening.*

Other questions to consider:

- Assuming that the collision efficiency is equal to 1, the liquid water mixing ratio is 1×10^{-3}kg kg^{-1} and the intial diameter is $D_0 = 10\,\mu$m what will the diameter be after 1000 seconds?
- How does this compare to growth at 2% supersaturation in a cloud at $T = 283.15$K?
- Understand the reason exponential rain drop spectra are maintained.

THE KEY POINTS TO TAKE HOME HERE ARE:

- Be able to derive / apply the continuous growth equation.
- Understand the concept of sweep out, collision efficiency and gravitational kernel.

4.4 Hail growth in a cloud

The first cloud physics conference I attended Roland List, the secretary general of IAMAS gave a talk on hail growth within clouds. The talk was about why hail-stones falling through the air precess like a spinning top, but Roland went into lots of detail on how they grow. At the end of the talk we were all puzzled and so a question was asked: "so why do hail-stones precess as they fall?" Roland looked up to the audience and said, "because they do!".

4.4.1 Growth by accretion / riming

Hail-stones or hail-embryos, which are called *graupel* pellets, grow in a cloud by vapour diffusion, aggregation and accretion of liquid water or *riming* as cloud physicists call it. We have already covered the physics of growth of ice from the vapour and the continuous growth model (which can be applied with limitations to consider aggregation and riming) so why do we need to have a separate section for hail growth? Well it turns out that when these growth processes are considered together there are some noteworthy differences.

Firstly we can multiply Equation 4.8 by the density of water to get the mass growth rate of a hail stone:

$$\frac{dm_{rime}}{dt} = \rho_w \int_0^D \frac{\pi}{6} d^3 \frac{\pi}{4} (D+d)^2 \left| u(D) - u(d) \right| E(D,d) \frac{dN}{dd} dd \tag{4.13}$$

where m_{rime} is the mass of rime accreted and again make the assumption that $D+d \cong D$ and $u(D) - u(d) \cong u(D)$ to find that:

$$\frac{dm_{rime}}{dt} = \frac{\pi}{4} D^2 a D^b w_l \tag{4.14}$$

where w_l is either the liquid water mixing ratio or water content and $a = 140$ and $b = 0.5$ in SI units.

4.4.2 Heat balance of a hail /graupel particle

When hail-stones grow by riming, the liquid water that is accreted onto the drops may freeze and as it does it releases the latent heat of fusion, $L_f \cong 3.12 \times 10^5$ J kg^{-1}, thus warming the surface of the particle. This reduces the rate of growth of the particle from the vapour, why?

During heavy riming the surface temperature of the particle can warm to the melting point of water so that the particle will start to melt. Again this changes the rate of growth of the particle from the vapour, why?

We make the assumption that the hail-stone temperature is in steady-state and that the supply of heat to the hail-stone is conducted away (Fourier's law). Therefore:

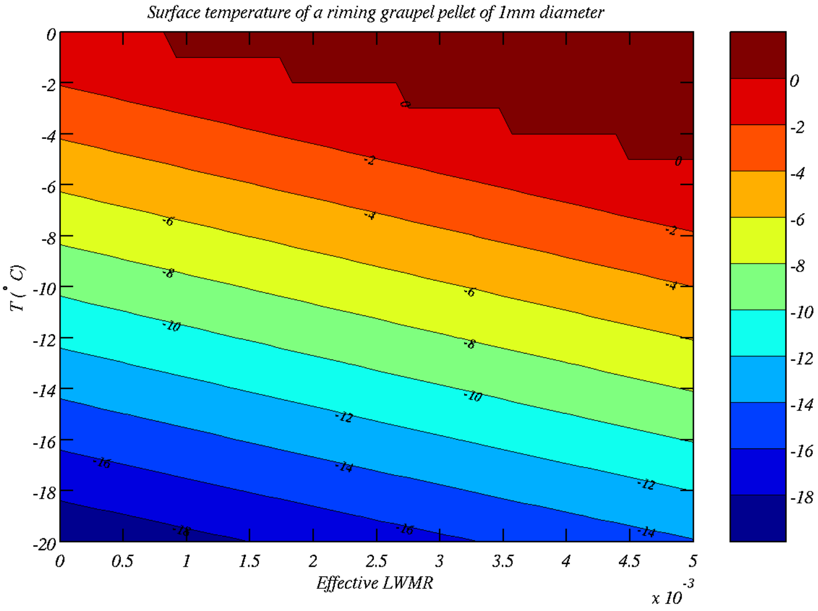

Figure 4.2: Surface temperature of a 1mm graupel pellett growing by riming and vapour diffusion.

$$4\pi ak\left(T_a - T_\infty\right) = L_s\frac{dm_{diff}}{dt} + c_w\left(T_\infty - T_0\right)\frac{dm_{rime}}{dt} + fL_f\frac{dm_{rime}}{dt} + c_i f\left(T_a - T_0\right)\frac{dm_{rime}}{dt} - L_f\frac{dm_{melt}}{dt} \tag{4.15}$$

where $T_0 = 273.15$K and c_w and c_i are the specific heat capacities of water and ice.

At first glance it seems like this equation could be solved for the temperature of the particle, T_a, but the growth rate from the vapour depends on the surface temperature too, $\frac{dm_{diff}}{dt} = \frac{4\pi Da}{R_v}\left(\frac{e}{T_\infty} - \frac{e_s(T_a)}{T_a}\right)$, so the whole thing has to be solved iteratively.

The result is that the surface temperature of a riming hail-stone depends on the liquid water content in cloud. If we assume that the hail-stone is in a mixed-phase cloud then the ambient vapour pressure will be water saturated. Figure 4.2 shows the surface temperature of a hail stone growing by riming.

What do you think will happen under extremely heavy riming in a cloud with ambient temperature lower than the melting point of water? Will the graupel particle melt? Will it sublimate?

4.5 Ice multiplication by rime-splintering

The Hallett–Mossop (H–M) process is the name given to the process that occurs when hail or graupel pellets rime in the temperature region $-7.5 < T < -2.5$°C.

In this temperature regime the freezing pattern of drops on to the hail is such that the drops fracture as they freeze and emit copious ice 'splinters'. The number of splinters produced depends on the temperature, being a maximum at −5°C. The rate of splinter production is roughly 350 mg^{-1} of rime accreted and tails off linearly to zero 2.5 degrees either side of −5°C.

$$f_{H-M}(T) \cong \begin{cases} (2.5 - T)/2.5 & \text{for} - 5.0 < T < -2.5 \\ (7.5 + T)/2.5 & \text{for} - 7.5 < T < -5.0 \\ 0 & \text{otherwise} \end{cases} \tag{4.16}$$

The number of splinters generated as one hail-stone accretes liquid water therefore follows from Equation 4.14:

$$P_{H-M} = 350 \times 10^6 f_{H-M}(T) \frac{dm_{rime}}{dt} \tag{4.17}$$

where P_{H-M} is the production rate of ice due to H–M multiplication.

The reason for the temperature dependence is thought to be that:

At high T The droplets take longer to freeze, so tend to flatten onto the graupel pellet before freezing.

In the H–M zone The droplets come in contact with the ice surface on one side and there is dendritic ice growth (crystallisation) through the liquid water which eventually contacts the other side of the droplet this causes an ice shell to form around the drop and liquid water to remain trapped in the ice shell. The liquid water in the shell freezes as an ice front propagates through the drop and because ice is less dense than liquid water there is a build up in pressure within the drop, which eventually causes the frozen drop to fracture, ejecting splinters.

At low T The ice front propagates through the drop quicker than the ice shell can form.

4.5.1 Dependence of the surface temperature of hail on the rime-splinter process?

As we have seen the temperature of riming graupel changes with w_l. Figure 4.3 shows this for different size graupel particles. So rime-splintering could occur at many temperatures in a cloud. Why is this?

4.5.2 Rime-splintering as an exponential growth process

Ice crystals ejected by the H–M process can be collected by rain-drops and cause them to freeze. This generates new graupel / hail-stones which can act as new 'rimers' in the H–M zone.

4.5.3 Observations of ice in cumulus clouds

The number of measured ice crystals in cumulus (Cu) frequently exceeds the number of measured IN with an ice nucleus counter by several orders of magnitude. In many of these instances it seems that the H–M process can explain the discrepancy.

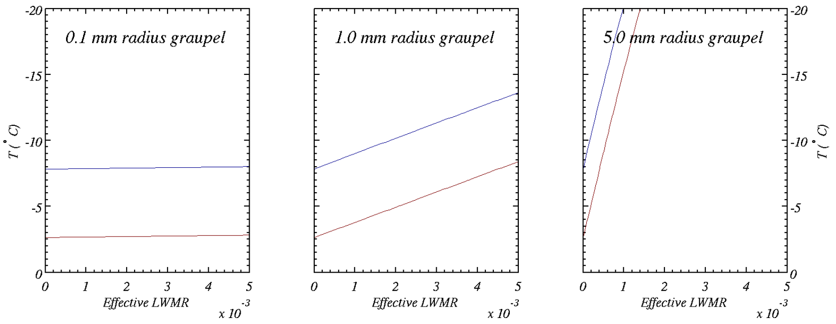

Figure 4.3: Plots showing the bounds of the H–M process for hail-stones of different sizes in T-effective LWMR space.

Example 4.8 *Explain the processes responsible for transferring heat to a hail-stone that is riming.*

Other questions to consider:

- What is the splinter production rate due to the H–M process if a graupel pellet diameter 0.5 mm falls through a supercooled liquid cloud of $w_l = 1 \times 10^{-3}$ kg/kg at −5°C?
- Explain why larger hail-stones have a surface temperature that is elevated more than smaller hail-stones.

THE KEY POINTS TO TAKE HOME HERE ARE:

- Understand and explain the H–M process and associated equations.
- Understand the terms in the heat balance of hail-stones.

Part II

Appendices

Appendix A

Problems and solutions

A.1 Lecture 1 and 2

Example 1.0 A potential temperature perturbation of 1 K is present in the atmosphere, arising through some heterogeneity over land. If the background potential temperature is 300 K what is the initial total acceleration of the air?

Answer Using Equation 1.1 calculate the initial acceleration due to the buoyancy force: $\frac{\delta\theta}{\theta}g = 1/300 \times 9.8 = 0.03$ m s^{-1}.

These examples on adiabatic liquid water mixing ratio and finding cloud base temperature and pressure are not on the exam. They are designed to help you become familiar with common terms in cloud physics and what they mean. You can also use the spreadsheet to help you with the calculation if you like. Obviously you can't use a spreadsheet in the exam but as I am sure you are aware as third year physicists that many problems in Physics do not have readily available analytical solutions. This is more so in research.

Example 1.1 A stratocumulus cloud has $T = 280$K and $P = 900$ hPa at cloud base. Cloud top is at a pressure of 870 hPa. What is the adiabatic liquid water mixing ratio?

Answer To answer this one you can either use the spreadsheet to solve Equation 1.16 iteratively or use the approx. method in the notes. Here we use the approx. method.

- Use Equation 1.19 to calculate the conserved quantity, $\theta_{q,sat}$.
- $\theta_{q,sat}$ requires the saturation vapour pressure, which from Equation 1.23 is $610.7\exp\left(\frac{2.5\times10^6}{461}\left[\frac{T-273.15}{273.15^2}\right]\right) \cong 1004.7$ Pa.
- $\theta_{q,sat} \cong T\left(\frac{1000}{P}\right)^{R/c_p}\left(1+\frac{\epsilon e_s(T)L_v}{PTc_p}\right) \cong 280\left(\frac{1000}{900}\right)^{287/1005}\left(1+\frac{0.622\times1004.7\times2.5\times10^6}{90000\times280\times1005}\right) \cong$ 306.35 K.
- Now we must solve for T_2 in Equation 1.28. If all terms are evaluated we get $\Pi_2 = \left(\frac{1000}{870}\right)^{287/1005} \cong 1.041$ and $T_2 = \frac{\frac{1.7815\times10^7}{87000}+\frac{306.35}{1.041}}{1+\frac{6.868\times10^4}{87000}} \cong 278.98$ K.
- Now use the cloud base and cloud top T and P in Equation 1.18 to find the water content.
- $ALMR = 0.622 \times 610.7 \times \left(\frac{\exp\left(\frac{2.5\times10^6}{461}\left[\frac{280-273.15}{273.15^2}\right]\right)}{90000} - \frac{\exp\left(\frac{2.5\times10^6}{461}\left[\frac{278.98-273.15}{273.15^2}\right]\right)}{87000}\right) \cong$ 0.27×10^{-3} kg/kg.

Example 1.1a What percentage of the total water mixing ratio is the adiabatic liquid water mixing ratio?

Answer The total water mixing ratio is the vapour content at cloud base. Why? because at cloud base there is no liquid water, only vapour. The vapour content at cloud base is the saturation vapour mixing ratio. Why? because at cloud base we are at water saturation. Any cooler and we have condensation.

- Use Equations 1.13 and 1.23 to find the saturation vapour mixing ratio:
- $e_{sw} = 610.7 \exp\left(\frac{2.5\times10^6}{461}\left[\frac{280-273.15}{273.15^2}\right]\right) \cong 1004.7$ Pa.
- $w_s(T,P) = \frac{\epsilon e_s(T)}{P} = \frac{0.622\times1004.7}{90000} \cong 6.9\times10^{-3}$ kg/kg.
- So the fraction of liquid water to total water is the answer from Example 1.1 divided by 6.5×10^{-3}, or $0.27/6.9 \cong 0.04$.

Example 1.1b If the number concentration of particles is 200 mg^{-1} of air what is the mean diameter of the drops at cloud top?

Answer We know the total mass of drops in 1 kg of air (0.27×10^{-3}) and the number of drops per kg of air is 200×10^6 so if we assume they are all the same size we can calculate their mean diameter.

- The mass of one drop is total mass divided by total number: $\frac{0.27\times10^{-3}}{200\times10^6} \cong 1.35\times10^{-12}$ kg.
- Equate this to the mass of a sphere, density 1000 kg m^{-3}: $1.35\times10^{-12} = \frac{\pi}{6}\rho_w D^3$.
- Rearrange and solve for D: $D = \left(\frac{1.35\times10^{-12}\times6}{\pi1000}\right)^{1/3} \cong 1.4\times10^{-5}$m or $14\,\mu$m.

Example 1.1c If the liquid water mixing ratio is sub-adiabatic, say 70% of the adiabatic value what is the mean diameter of the drops assuming a drop concentration of 100 mg^{-1} of air?

Answer Very similar arguments to above.

- The mass of one drop is total mass divided by total number: $\frac{0.27\times10^{-3}\times70\%}{100\times10^6} \cong 1.89\times10^{-12}$ kg.
- After equating to the mass of a sphere, rearrange and solve for D: $D = \left(\frac{1.89\times10^{-12}\times6}{\pi1000}\right)^{1/3} \cong 1.53\times10^{-4}$m or $15\,\mu$m.

Example 1.1d If the same initial temperature and pressure are used, but the cloud top is 200m above cloud base what is the adiabatic liquid water mixing ratio? Use a temperature lapse rate of 6.5 K km^{-1}.

Answer This is an approximation because the lapse rate isn't exactly 6.5 K km^{-1}.

- The cloud top temperature is the cloud base temperature minus the cloud thickness multiplied by the lapse rate: $280 - 200\times6.5\times10^{-3} \cong 278.7$ K.
- The adiabatic liquid water mixing ratio (Equation 1.18) requires the cloud base and cloud top temperature and pressure. We have everything except cloud top pressure, so use the hydro-static equation to find cloud top pressure: $\frac{dP}{dz} = -\rho g$, or substituting the ideal gas law $\frac{dP}{dz} = -\frac{P}{R_a}Tg$.

- Solve the hydro-static equation by integrating both sides wrt z: $\int_{P0}^{P1} \frac{1}{P} dP = -\frac{g}{R_a} \int_{z0}^{z0+200} \frac{1}{T} dz$, but we know that $T(z) = 280 - z \times 6.5 \times 10^{-3}$.
- Solving this yields $\ln\left(\frac{P1}{P0}\right) = \frac{g}{R_a \times 6.5 \times 10^{-3}} \ln\left(\frac{280 - 200 \times 6.5 \times 10^{-3}}{280}\right)$
- Or solving for $P1$: $P1 = P0 \exp\left(\frac{g}{R_a \times 6.5 \times 10^{-3}} \ln\left(\frac{280 - 200 \times 6.5 \times 10^{-3}}{280}\right)\right)$.
- $P1 = 900 \exp\left(\frac{9.8}{287 \times 6.5 \times 10^{-3}} \ln\left(\frac{280 - 200 \times 6.5 \times 10^{-3}}{280}\right)\right) \cong 878.3$ hPa.
- Now use the cloud base and cloud top T and P in Equation 1.18 to find the water content.
- $ALMR = 0.622 \times 610.7 \times \left(\frac{\exp\left(\frac{2.5 \times 10^6}{461}\left[\frac{280 - 273.15}{273.15^2}\right]\right)}{90000} - \frac{\exp\left(\frac{2.5 \times 10^6}{461}\left[\frac{278.7 - 273.15}{273.15^2}\right]\right)}{87830}\right) \cong 0.47 \times 10^{-3}$ kg/kg.

A.2 Lectures 2 and 3

This question on seeding stratocumulus clouds can be calculated analytically, meaning I could ask you about it on the exam. If so you will be given Equation 1.33 and 1.37 There is a spreadsheet to do the calculations if you're struggling.

Example 1.2 A stratocumulus cloud, 100m thick, has a liquid water mixing ratio of 0.5 g m^{-3} and a number concentration of 50 drops cm^{-3} of air. Over a period of time you are able to increase the concentration of salt particles entering the cloud base to 500 drops cm^{-3} of air. In both cases the size distribution can be considered to be exponentially distributed. What is the change in the cloud albedo?

Answer You should be able to derive (and demonstrate the derivation in the exam) Equation 1.38 in the notes. Once you get to here it is quite straight forward.

- The third moment, M_3 can be calculated by dividing the liquid water mixing ratio by $\frac{\pi\rho_w}{6}$ so that $M_3 = \frac{0.5\times10^{-3}\times 6}{\pi\times1000} \cong 9.55\times10^{-7}$ m^3 m^{-3}.
- The zeroth moment, $M_0 = 50\times10^6$ m^{-3} so the second moment from Equation 1.38 is $0.6\left(\times50\times10^6\right)^{1/3}\times\left(9.55\times10^{-7}\right)^{2/3}\cong 0.0214$.
- The albedo from Equation 1.37 is $A_c = \frac{\pi\times0.0214\times100}{\pi\times0.0214\times100+15.4} = 0.30$.
- After addition of seed aerosol, $M_0 = 500\times10^6$ m^{-3} so the second moment from Equation 1.38 is $0.6\left(\times500\times10^6\right)^{1/3}\times\left(9.55\times10^{-7}\right)^{2/3}\cong$ 0.0462.
- The albedo from Equation 1.37 is $A_c = \frac{\pi\times0.0462\times100}{\pi\times0.0462\times100+15.4} = 0.49$.
- So $\Delta A_c = 0.49 - 0.30 = 0.19$.

A.3 Lecture 4

Some of the examples on CCN activation could in principle be used in the exam although you would be given Equations 2.21 and 2.22. Also see the spreadsheet which does this calculation:

Example 2.3 A CCN counter is used to measure the number of active CCN verses supersaturation at the ground. From analysis of the data a power-law (Equation 2.21) is fitted to the data, and the parameters of the fit are $C = 200$ cm^{-3} and $k = 0.50$. Calculate the number of cloud drops activated at cloud base for a wind speed of $w = 0.2, 0.5, 1.0$ and 5.0 ms^{-1}.

Answer Using Equation 2.22, $N_c \cong 0.88{\times}200^{2/(0.50+2)}\left[70 \times 0.2^{3/2}\right]^{0.50/(0.50+2)} \cong 88.03$mg^{-1}.

A.4 Lecture 5

The questions on droplet growth, or ice crystal growth can be solved analytically and so could be used in the exam. You would be given any of Equation 2.25, 2.27, 2.36 or 2.37 and be expected to be able to use them. The purpose of the spreadsheet is to show that the analytical solution is a very good approximation to the exact solution (which can only be solved iteratively).

Example 2.4 How long does it take for a drop of diameter 10μm to grow to a precipitation-sized drop of diameter 2 mm given an in-cloud supersaturation of 2%? Assume the temperature is 290k, pressure is 900hPa and calculate the diffusivity and thermal conductivity from the expressions for D_v and k in your notes. Use Equation 1.11 for e_s.

Answer For this use Equation 2.37 and recognise that all terms on the right hand side are a constant,

- So that after integration the solution is $a(t) = \sqrt{2At + a_0^2}$, where $A \cong 2.3 \times 10^{-12}$ m^{2}s^{-1} is everything on the right hand side of Equation 2.37.
- Rearrange for t: $t = \frac{a(t)^2 - a_0^2}{2A}$.
- Substitute $a(t) = \frac{a(t)^2 - a_0^2}{2A} \cong \frac{\left(1\times10^{-3}\right)^2 - \left(5\times10^{-6}\right)^2}{2\times2.3\times10^{-12}} \cong 2.2 \times 10^5$ s or 2.5 days!

Example 2.4a For the problem above what is the drop growth rate after 100 s?

Answer Could do this a number of ways, try differentiating $a(t) = \sqrt{2At + a_0^2}$: $\frac{da}{dt} = \frac{A}{\left(2At + a_0^2\right)^{1/2}}$ and substitute $t = 100$s so $\frac{da}{dt} \cong 1 \times 10^{-7}$ ms^{-1} or 0.1μm s^{-1}. The size $a(t = 100s) \cong 21\mu$m.

Example 2.4b With reference to the previous questions on drop growth what is the temperature of the drop in the above question after 100s?

Answer This requires the balance of Fourier's law with the latent heat released by the mass growth rate by diffusion (Equation 2.29).

- The mass growth rate can be written in terms of the radial growth rate: $\frac{dm}{dt} = 4\pi a^2 \rho_w \frac{da}{dt}$.
- So using the numbers for a and $\frac{da}{dt}$ in the previous question we have $\frac{dm}{dt} \cong 4\pi \times \left(21 \times 10^{-6}\right)^2 \times 1000 \times 1 \times 10^{-7} \cong 6.3 \times 10^{-13}$kg s^{-1}.
- Multiply the mass growth rate by the latent heat of vapourisation and equate to Fourier's law: $6.3 \times 10^{-13} \times 2.5 \times 10^6 \cong 4\pi \times ak\,(T_a - T_\infty)$.
- Rearranging for T_a: $T_a - T_\infty \cong 0.2295$. So the drop temperature is elevated above 290K by 0.2 K. Incidentally you do not need to include the time or size of the drop in this calculation as they cancel so at all times and all drops sizes the difference in drop and ambient temperature will be the same. Can you proove this?

Example 2.4c Formation of rain requires particles of different sizes, which have different terminal fall-speeds. Given this can you say anything about why the growth of drops by vapour diffusion is not able to explain the formation of rain in real clouds?

Answer Use $a(t) = \sqrt{2At + a_0^2}$. If we start with a small drop radius a_0 and a slightly larger drop with radius a_1 then the difference between the squares of these two drops with time is $2At + a_1^2 - \left(2At + a_0^2\right) = a_1^2 - a_0^2$. If the difference of the squares of the drops is constant with time this means that they get closer together in size. After an infinite amount of time the drop size distribution will be infinitesimally narrow and so if they were precipitation sized the whole cloud would fall as rain.

A.5 Lecture 6

Example 2.5 Derive the ‘radial’ growth vs time of an ice crystal with circular disk morphology, with initial starting radius a_0.

Answer Use the capacitance $C_0 = \frac{2a}{\pi}$. After this an analogy can be made with the droplet growth equation: $\frac{2a(t)}{\pi} = \sqrt{2At + \left(\frac{2a_0}{\pi}\right)^2}$ or $a(t) = \sqrt{\frac{At\pi^2}{2} + a_0^2}$. A is analogous to the right hand side of Equation 2.37, but uses the latent heat of sublimation and the saturation vapour pressure over an ice surface instead of liquid.

Example 2.5a For the problem above what is the radial growth rate after 100 seconds? and therefore what is the temperature of the ice crystal? (assume the saturation ratio $s_i = 1.10$ and $T_\infty = -15$°C, $P = 900$hPa and the intial size is $a_0 = 5\mu$m).

Answer Differentiate $a(t)$ wrt time: $\frac{da}{dt} = \frac{A\pi^2}{2\left(\frac{At\pi^2}{2}+a_0^2\right)^{1/2}}$.

- For ice crystals at these ambient conditions you should find that $A = \frac{s_i-1}{\frac{\rho_i R_v T_\infty}{e_{si}(T_\infty)D_v}+\rho_i \frac{L_s}{T_\infty k}\left(\frac{L_s}{R_v T_\infty}-1\right)}$ is equal to $\cong 2.53 \times 10^{-12}$ m^2 s^{-1}. Note that $\rho_i = 910$ kg m^{-3} and $L_s = 2.8 \times 10^6$Jkg^{-1}.
- So the growth rate is $\frac{da}{dt} = \frac{2.5\times10^{-12}\pi^2}{2\left(\frac{2.5\times10^{-12}\times100\pi^2}{2}+\left(5\times10^{-6}\right)^2\right)^{1/2}} \cong 3.5 \times 10^{-7}$ ms^{-1}.

  The ‘radius’ is $a(t) = \sqrt{\frac{At\pi^2}{2} + a_0^2} \cong 36\mu$m.
- For the initial temperature of the crystal find the mass growth rate and equate the latent heat of sublimation to the heat lost by Fourier’s law.
- Approximate the crystal as a sphere (which it isn’t): $\frac{dm}{dt} = 4\pi a^2 \rho_i \frac{da}{dt} \cong 4\pi\left(36 \times 10^{-6}\right)^2 \times 910 \times 3.5 \times 10^{-7} \cong 5.2 \times 10^{-12}$ kg s^{-1}.
- Multiply the mass growth rate by the latent heat of sublimation and equate to Fourier’s law: $5.2\times10^{-12}\times2.82\times10^6 \cong 4\pi\times2\times a/\pi k\,(T_a - T_\infty)$.
- Rearranging for T_a: $T_a - T_\infty \cong 2.26$K. So the ice crystal temperature is elevated above 258.15K by 2.26 K.

Example 2.5b What is the radial growth rate of a drop at the same conditions?

Answer $\frac{da}{dt} = \frac{A}{\left(2At+a_0^2\right)^{1/2}}$, under these conditions, A is the right hand side of Equation 2.37, but the supersaturation is not 0.10 as that is the supersaturation over an ice surface.

- The saturation ratio over ice (1.10 in this case) can be converted to vapour pressure, e by multiplying by the saturation vapour pressure e_{si} from the saturation vapour pressure (Equation 1.12).
- This gives $e = 1.10 \times 610.7 \exp\left(\frac{2.82\times10^6}{461}\left[\frac{1}{273.15} - \frac{1}{258.15}\right]\right) \cong 182$ Pa.
- To get the saturation over liquid water divide the vapour pressure by the saturation vapour pressure over a liquid water surface: $\frac{182}{\exp\left(\frac{2.5\times10^6}{461}\left[\frac{1}{273.15}-\frac{1}{258.15}\right]\right)} \cong 0.95$. So the supersaturation is $0.95 - 1 \cong -0.05$. Meaning the drop will

evaporate. The importance of this will become apparent when we cover supersaturation in clouds (Bergeron-Findeison process).

- A in this case is therfore $\cong -8.5\times10^{-15}$ ms^{-1}.
- So the growth rate is $\frac{A}{\left(2At+a_0^2\right)^{1/2}} \cong \frac{-8.5\times10^{-15}}{\left(2\times-8.5\times10^{-15}\times100+\left(5\times10^{-6}\right)^2\right)^{1/2}} \cong -1.76\times 10^{-9}$ ms^{-1}.

Example 2.5c If there are 100 mg^{-1} of drops radius 10μ m at -36°C how long in seconds would it take to freeze halve of them by homogeneous freezing?

Answer Use Equation 2.42 which defines a stochastic process (like radioactive decay) in which drops freeze to form ice crystals.

- It can be recognised that the rate of change of ice crystals is minus the rate of change of drops so Equation 2.42 can be written: $\frac{dN_{drops}}{dt} = N_{drops}V_{drop}J(T)$, which has a solution similar to radioactive decay: $t_{1/2} = \frac{1}{VJ(t)}\ln\left(\frac{N_I}{N_F}\right) = \frac{1}{VJ(t)}\ln(2)$
- The nucleation rate, J, (Equation 2.41) has a value of 3.15×10^{12} at -36°C.
- The volume of a 10μ m drop is $\frac{\pi}{6}\left(10\times10^{-6}\right)^3 \cong 5.23\times10^{-16}$ m^3.
- The half-life is therefore: $\frac{1}{VJ(t)}\ln(2) \cong \frac{1}{5.23\times10^{-16}\times3.15\times10^{12}}\times0.693 \cong 420$s

A.6 Lecture 7 and 8

The topic on supersaturation in clouds brings together the droplet and ice crystal growth equations, the hydrostatic relation, and the first law of thermodynamics (which is used to derive θ and $\theta_{q,sat}$). It therefore brings together everything done so far and is a very important concept in cloud physics. It is also a difficult concept to grasp and the Matlab function m-file was written so that you may play with it to sharpen your intuition about supersaturation in single and mixed phase clouds. I obviously can't ask you to solve the ode describing supersaturation in the exam, but I can ask you to explain what determines the supersaturation in a cloud. Reading through and answering the descriptive examples in the notes will help you for the exam.

Example 3.7 Explain what all the terms in Equation 3.7 mean.

Answer see notes.

Example 3.7a For an ascent rate of 2 m s^{-1}, what is the steady-state supersaturation in a liquid only cloud containing drops of concentration 1000 mg^{-1} of air, radius 10 μm at a pressure of 900 hPa and temperature of 273 K?

Answer Use $S_l = \frac{a_0 w}{b_l N_l a_l}$ where $a_0 = \frac{g}{c_p} - \frac{g}{R_a T} \sim 9.63 \times 10^{-3}$, $N_l = 1000 \times 10^6$ kg^{-1}, $a_l =$ 10 μm and b_l is the thermodynamic factor $4\pi \frac{\left(\frac{1}{w_v} + \frac{L_v^2}{R_v T^2 c_p}\right)}{\frac{R_v T}{e_{s,liq}(T) D_v} + \frac{L_v}{Tk}\left(\frac{L_v}{R_v T} - 1\right)} \sim 3.1 \times 10^{-4}$. The water vapour mixing ratio w_v can be assumed to be at water saturation so $w_v \cong \frac{\epsilon e_{s,liq}(T)}{P} \sim 4.18 \times 10^{-3}$ kg/kg and $e_{s,liq}(T) \sim 605$ Pa is defined from the saturation vapour pressure equation, D_v and k were defined in previous lectures $D_v \sim 2.11 \times 10^{-5}$ and $k \sim 0.025$. Taking all this together gives:

$$S_l \cong \frac{9.63 \times 10^{-3} \times 2}{2.45 \times 10^{-5} \times 1000 \times 10^6 \times 10^{-6}} \cong 0.006$$

Example 3.7b In a mixed phase cloud, what is the ascent rate required to maintain supercooled liquid water at −15°C and 700 hPa if the ice crystal concentration is 1 mg^{-1} of air and the ice crystal average radii are 100 μm?

Answer Use $w = \frac{b_{1,i} N_i a_i}{a_0}$ where $a_0 = \frac{g}{c_p} - \frac{g}{R_a T} \sim 9.62 \times 10^{-3}$, $N_i = 1 \times 10^6$ kg^{-1}, $a_i =$ 100 μm and $b_{1,i}$ is the thermodynamic factor $4\pi \left(\frac{e_{sat,l}}{e_{sat,i}} - 1\right) \frac{\left(\frac{1}{w_v} + \frac{L_v L_s}{R_v T^2 c_p}\right)}{\frac{R_v T}{e_{s,ice}(T) D_v} + \frac{L_s}{Tk}\left(\frac{L_s}{R_v T} - 1\right)} \sim 4.45 \times 10^{-5}$. The water vapour mixing ratio w_v can be assumed to be at water saturation so $w_v \cong \frac{\epsilon e_{s,liq}(T)}{P} \sim 1.3 \times 10^{-3}$ kg/kg and $e_{s,liq}(T) \sim 189$ Pa and $e_{s,ice}(T) \sim 163$ are defined from the saturation vapour pressure equations, D_v and k were defined in previous lectures $D_v \sim 2.11 \times 10^{-5}$ and $k \sim 0.025$. Taking all this together gives:

$$w \cong \frac{4.45 \times 10^{-5} \times 1 \times 10^6 \times 100^{-6}}{9.63 \times 10^{-3}} \cong 0.45\,\mathrm{m\,s^{-1}}$$

A.7 Lecture 9 and 10

The continuous growth model again can be solved analytically so could in principle show up on the exam. You would be given Equation 4.8 and asked to worked from there.

Example 4.7 Explain why with reference to the solutions of the relevant growth equations why growth by vapour diffusion leads to a narrowing of the distribution, whereas growth by collision and coalescence leads to a broadening.

Answer See example 2.4c to explain why growth from the vapour leads to narrowing of the distribution. For collision-coalescence:

- The solution to the growth equation is (Equation 4.12): $D(t) = \left(\frac{\bar{E}w_l a(1-b)}{2\rho_w}t + D_0^{1-b}\right)^{\frac{1}{1-b}}$.
- Or rearranging with $b = 2$: $D(t)^{-1} = \frac{\bar{E}w_l a(1-b)}{2\rho_w}t + D_0^{-1}$.
- So if we consider two drops starting at different initial sizes, D_1 and D_0, the difference in the reciprical of their sizes with time is: $D_1(t)^{-1} - D_0(t)^{-1} = D_1^{-1} + D_0^{-1}$. If the difference between the reciprical of their sizes is a constant with time then the difference in size gets larger with time (if they are growing).

Example 4.7a Assuming that the collision efficiency is equal to 1, the liquid water mixing ratio is 1×10^{-3}kg kg^{-1} and the intial diameter is $D_0 = 10\,\mu$m what will the diameter be after 6350 seconds?

Answer Use Equation 4.12: $D(t) = \left(\frac{\bar{E}w_l a(1-b)}{2\rho_w}t + D_0^{1-b}\right)^{\frac{1}{1-b}}$ with $b = 2$ and $a = 3.129 \times 10^7$, to get that $D(t = 1000) \cong 1.5$ mm. Growth at 2% supersaturation at 900 hPa gives 117 μm radius, or 234 μm diameter (see Example 2.4) for details on how to do the calculation.

Example 4.7b The rain-drop size distribution is exponentially distributed and has an intercept of n_0 and a slope parameter of λ_0. If the terminal velocity of the rain drops can be described as a power law: $v_t = aD^b$ derive an expression for the mass flux of precipitation.

Answer The flux of precipitation is the integral of the product of the mass and fall speed: $\chi_f = \int aD^b \frac{\pi\rho_w}{6} D^3 n_0 \exp\left(-\lambda_0 D\right) dD$. By noting the integrals of modified gamma functions (previous lecture) we can write this as:

$$\chi_f = \frac{a\pi\rho_w n_0 \Gamma\left(4+b\right)}{6\lambda_0^{4+b}}$$

The heat budget of a hail-stone can have important consequences in cloud microphysics and crops up in several areas (e.g. thunderstorm electrification, rime-splintering). I won't ask you to solve the heat budget equation on the exam, but I could ask you to explain what determines the temperature of a riming hail-stone. Make sure you can do this.

Example 4.8 Explain the processes responsible for transferring heat to a hail-stone that is riming.

Answer With reference to Equation 4.15:

- The accreted water at the ambient cloud temperature $T > 0°\mathrm{C}$ should approach the triple point before it can freeze. This requires heat to be transfered to / from the liquid water.
- Once frozen the rime should approach the temperature of the hail-stone, which requires heat to be transfered away from the rime. Ice has a difference heat capacity than liquid water though.
- Note if the rime water is already colder than 0°C then it doesn't have to freeze at the triple point, but the amount of heat required for the ice to approach the temperature of the hail stone is equivalent to the specific heat of liquid water approaching the triple point and then ice water approaching the hail temperature (by conservation of energy).
- The latent heat of fusion of the accreted water is released once it freezes, heating the hail stone.
- The latent heat of sublimation is released as the hail-stone grows from the vapour.

Example 4.8a What is the splinter production rate due to the H–M process if a graupel pellet diameter 0.5 mm falls through a supercooled liquid cloud of $w_l = 1 \times 10^{-3}$ kg/kg at −5°C?

Answer Use Equations 4.14 and 4.17: $\frac{\pi}{4}D^2aD^bw_l \times 350 \times 10^6$, with $a = 140$ and $b = 0.5$. Thus $\frac{\pi}{4}\left(0.5 \times 10^{-3}\right)^2 \times 140\left(0.5 \times 10^{-3}\right)^{0.5} \times 1 \times 10^{-3} \times 350 \times 10^6 \cong 0.2$ particles s^{-1}.

Example 4.8b Explain why larger hail-stones have a surface temperature that is elevated more than smaller hail-stones.

Answer Because they sweep out a larger volume of air per second (due to their larger size and larger fall speed). Hence they collect more liquid water per second and because this water freezes it releases more heat by the latent heat of fusion.

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List of Symbols

$\bar{\mathbf{j}}$	Current density of vapour mass, page 20
$\bar{\mathbf{j}}_h$	Current density of heat, page 21
$\ddot{z}$	Acceleration of drop, page 45
$\delta\rho$	Air density perturbation, page 7
δP	Pressure perturbation, page 7
ϵ	Ratio of gas constants for dry and moist air = 0.622, page 9
η	Viscosity of air., page 45
$\hat{\mathbf{k}}$	Unit vector in vertical direction., page 7
λ_0	Slope parameter for exponential fit (m^{-1}), page 14
ρ_0	Reference air density, page 7
ρ_v	Mass of water vapour per unit volume of air, page 20
$\rho_{v,\infty}$	Vapour density at infinite distance from drop, page 27
$\rho_{v,a}$	Vapour density on drop surface, page 27
τ	Residence time of salt particles in atmosphere, page 19
$\mathbf{v}$	Wind velocity vector field, page 6
$\theta_{q,sat}$	Wet equivalent potential temperature (K), page 10
a	Drop radius, page 27
a	Radius of salt water drop, page 19
A_c	Albedo of a liquid layer cloud (fraction), page 16
A_c	Albedo of a liquid layer cloud (fraction), page 18
A_E	Area of Earth's surface, page 19

a_i	Average radius of ice crystals, page 40
a_l	Average radius of liquid drops, page 40
A_p	Planetary albedo, page 18
$ALMR$	Adiabatic liquid water mixing ratio kg/kg, page 10
C	Capacitance of an ice crystal, page 33
C_0	Capacitance of a thin circular disk: $\frac{2a}{\pi}$, page 33
C_D	Drag coefficient of sphere falling in air, page 45
c_i	Specific heat capacity of ice, page 51
c_p	Heat capacity of dry air = $1005\mathrm{Jkg}^{-1}\mathrm{K}^{-1}$, page 10
c_p	Specific heat capacity of air, page 38
c_w	Heat capacity of liquid water = $4187\mathrm{Jkg}^{-1}\mathrm{K}^{-1}$, page 10
c_w	Specific heat capacity of liquid water, page 51
D	Particle diameter, page 14
D_v	Diffusivity of water vapour in air, page 20
e	Vapour pressure, page 37
$E(D,d)$	Collision efficiency between drop of diameter D and d, page 46
e_∞	vapour pressure at infinite distance from drop, page 27
e_v	Vapour pressure at surface of drop, page 23
$e_{s,drop}$	Vapour pressure at surface of drop, page 23
$e_{sat,l}$	Saturation vapour pressure over liquid water, page 37
e_{si}	Saturation vapour pressure over flat liquid water surface, page 9
e_{sw}	Saturation vapour pressure over flat ice water surface, page 9
F	Average solar irradiance, page 18
F_0	Solar flux at top of the atmosphere, page 18
f_1	Fraction of Earth's surface covered by ocean, page 18
f_2	Fraction of Earth's oceans covered by marine clouds, page 18

f_3	Fraction of marine clouds that are seeded, page 18
$F_{diabatic}$	Processes changing θ that are not adiabatic., page 7
f_{H-M}	Temperature dependent function for H-M multiplication., page 52
g	Gravitational field strength, page 6
g	Gravitational field strength, page 38
H	Depth of boundary layer, page 19
J	Homogeneous nucleation rate of ice in supercooled water, page 32
k	Thermal conductivity of air, page 21
L_f	Latent heat of fusion, page 50
L_s	Latent heat of sublimation for water, $\sim 2.82 \times 10^6$ J kg^{-1}, page 38
L_v	latent heat of vapourisation for water $\cong 2.5 \times 10^6$J kg^{-1}, page 38
L_v	latent heat of vapourisation for water $\cong 2.5 \times 10^6$Jkg^{-1}, page 10
M_0	Zeroth moment of a size distribution, page 15
M_2	Second moment of a size distribution, page 15
M_3	Third moment of a size distribution, page 15
m_{rime}	The mass of rime accreted, page 50
N	Concentration of cloud drops in seeded cloud, page 18
N	Particle number density, page 14
N_0	Concentration of cloud drops in un-seeded cloud, page 18
n_0	Intercept parameter for exponential fit (m^{-4}), page 14
N_c	Number concentration of activated CCN, page 25
N_i	Number concentration of ice crystals, page 40
N_l	Number concentration of liquid drops, page 40
N_{IN}	Number concentration of ice nuclei, page 32
n_{spray}	Number of drops sprayed, page 19
P	Pressure, page 37

P	Total pressure, page 6
P_0	Reference pressure, page 7
P_1	Temperature of air at cloud base, K, page 10
P_2	Pressure of air above cloud base, Pa, page 10
P_c	Pressure difference over drop interface, page 23
P_{H-M}	Production rate of ice due to H-M multiplication., page 52
Q	Total water mixing ratio (vapour plus liquid) kgkg^{-1}, page 10
R'	Gas constant for dry air = $287\mathrm{Jkg}^{-1}\mathrm{K}^{-1}$, page 10
R_a	Specific gas constant for air, page 38
R_v	Specific gas constant for water vapour, page 38
Re	Reynolds number $Re = \frac{ud\rho_a}{\eta}$, page 45
S_i	Supersaturation over ice, page 9
s_i	saturation over ice, page 9
S_l	Supersaturation over liquid water, page 9
S_l	Supersaturation over liquid water, page 37
s_l	saturation over liquid water, page 9
T	Temperature, page 38
T_1	Temperature of air at cloud base, K, page 10
T_2	Temperature of air above cloud base, Pa, page 10
T_∞	Temperature at infinite distance from drop, page 27
T_a	Temperature at the surface of drop, page 27
T_c	Temperature in degrees celcius., page 32
u	Velocity of sphere falling in air, page 45
V_{spray}	Total volume of sea spray, page 19
w	Ascent velocity of parcel, page 38
w	Vertical wind speed, page 25

w_i	Ice water mixing ratio, page 38
w_l	Liquid water mixing ratio., page 47
w_l	Liquid water mixing ratio, page 38
w_s	Saturated vapour mixing ratio kgkg^{-1}, page 10
w_v	Water vapour mixing ratio, page 37
y	Maximum displacement between drop centres which may result in collision, page 46