PROPERTIES OF ALGEBRAICALLY-δ-CLOSED FIELDS

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Abstract. We introduce the notion of an algebraically-δ-closed field, namely, a differential field of characteristic 0 whose algebraic closure is differentially closed. We prove some basic results around which linear differential equations can be solved in these fields. Furthermore, we show that, by adding the assumption that K^{alg} has a K-basis of constants, these fields are PV-closed.

1. Introduction

There have been many important recent results in the model theory of differential fields. Several authors have realised that algebraic conditions on differential fields, such as being differentially large (cf. [LST20]) or being bounded and PAC-differential (cf. [HLS21]) offer tools to extend the model-theoretic properties of the underlying fields to properties of the differential fields involved. In this way, important properties, such as model-completeness, simplicity, and elimination of imaginaries, have been established for some of these classes of differential fields.

Interestingly, these fields satisfy the following algebraic property: their algebraic closure is differentially closed. Intuitively, this means that all the elements needed to solve the differential equations defined over the field are already algebraic, i.e., they solve an algebraic equation over the field. Thus, the aim of this note is to isolate the class of fields satisfying this property, and study some of its basic algebraic and differential properties, with the hope of providing a basis for further study of their model-theoretic properties.

In order to do this, in Section §2, we define algebraically-δ-closed fields and prove some basic results about the solvability of differential equations in these fields. Then, in Section §3, we introduce a further hypothesis, and prove that algebraically-δ-closed fields with this property are PV-closed.

2. General properties

In this section, we introduce the notion of an algebraically-δ-closed field and we prove several basic results concerning the solvability of some linear differential equations. We assume some basic knowledge of differential algebra; all of the relevant notions can be found in [Kol73, ch. 1]. Throughout, we assume all fields are of characteristic 0.

Definition 2.1. A differential field (K, δ) is **algebraically-δ-closed** if (K^{alg}, δ)[1] is differentially closed.

Example 2.2. (i) All PAC-differential fields are algebraically-δ-closed (cf. [HLS21, Remark 4.7(ii), p. 10]).
(ii) More generally, all differentially large fields are algebraically-δ-closed ([LST20, Corollary 5.12, p. 23]).

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[1] Since there is a unique extension of δ to K^{alg}, we use the same notation for both derivations.

Theorem 2.3. *Let (K,δ) be an algebraically-δ-closed field, and let $b \in K$. Then there exists some $c \in K$ such that $\delta(c) = b$. In other words, the derivation $\delta\colon K \to K$ is surjective.*

Proof. Since (K,δ) is algebraically-δ-closed, (K^{alg},δ) is differentially closed. Thus, there is $c \in K^{\text{alg}}$ such that $\delta(c) = b$. Since c is algebraic over K, it has a minimal polynomial over K, say, $f(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$. By applying δ to $f(c) = 0$ and solving for $\delta(c)$, we have

$$\delta(c) = b = -\frac{\delta(a_{k-1})c^{k-1} + \cdots + \delta(a_1)c + \delta(a_0)}{kc^{k-1} + \cdots + 2a_2c + a_1}.$$

Rearranging, we get

$$(bk + \delta(a_{k-1}))c^{k-1} + \cdots + (2ba_2 + \delta(a_1))c + (ba_1 + \delta(a_0)) = 0. \tag{2.1}$$

Note that the coefficients in (2.1) are all in K. Thus, if not all coefficients are zero, we obtain a polynomial in K with degree less than that of $f(x)$ which vanishes at c. But this contradicts our choice of f. So the coefficients must all vanish. But then we get $\delta(a_{k-1}) = -kb$. Hence, if we let $d = -a_{k-1}/k$, then clearly $d \in K$ and $\delta(d) = b$, as required. □

Corollary 2.4. *Let (K,δ) be an algebraically-δ-closed field, and let $b \in K$ and $n \in \mathbb{N}_0$. Then there exists some $c \in K$ such that $\delta^n(c) = b$.*

Proof. Use induction and Theorem 2.3. □

Theorem 2.5. *Let (K,δ) be an algebraically-δ-closed field, and let $a, b \in K$. Then there exists some $c \in K$ such that $\delta(c) = ac + b$.*

Proof. Since (K,δ) is algebraically-δ-closed, (K^{alg},δ) is differentially closed. Thus, there is $c \in K^{\text{alg}}$ such that $\delta(c) = ac + b$. Since c is algebraic over K, it has a minimal polynomial over K, say, $f(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_1x + a_0$. By applying δ to $f(c) = 0$, we get

$$ac + b = -\frac{\delta(a_{k-1})c^{k-1} + \cdots + \delta(a_1)c + \delta(a_0)}{kc^{k-1} + \cdots + 2a_2c + a_1}.$$

Thus, rearranging, we obtain

$$g(c) := kac^k + (a(k-1)a_{k-1} + kb + \delta(a_{k-1}))c^{k-1} + \cdots + ba_1 = 0. \tag{2.2}$$

Note that all the coefficients in (2.2) are in K. Thus, we obtain a polynomial in K of degree k that vanishes at c. This can happen iff all coefficients vanish, or $g(x) = kaf(x)$. If the latter holds, then, comparing the coefficients of c^{k-1}, we obtain

$$a(k-1)a_{k-1} + kb + \delta(a_{k-1}) = kaa_{k-1}$$
$$\implies \delta(a_{k-1}) = aa_{k-1} - kb.$$

Thus, if we let $d := -a_{k-1}/k$, then we get

$$\delta(d) = -\frac{1}{k}\delta(a_{k-1}) = -\frac{1}{k}(aa_{k-1} - kb) = ad + b.$$

Since $d \in K$, this gives us our result. On the other hand, if all coefficients of $g(x)$ vanish, then we get $ka = 0$, and thus, $a = 0$. So the original equation is $\delta(x) = b$, which has a solution in K by Theorem 2.3. Thus, either way, solutions in K to $\delta(x) = ax + b$ exist, as required. □

General results about the solvability of higher order linear differential equations have not been obtained yet. Nonetheless, we conjecture the following:

Conjecture 2.6. *Let (K,δ) be an algebraically-δ-closed field. For any $a_0, \dots, a_{n-1}, b \in K$, there exists some $c \in K$ such that $\delta^n(c) + a_{n-1}\delta^{n-1}(c) + \cdots + a_1\delta(c) + a_0c = b$.*

In the following section, we give a positive solution to the conjecture in the case when K^{alg} admits a K-basis of constants.

3. Assuming K^{alg} has a K-basis of constants

In this section, viewing K^{alg} as a vector space over K, we add an extra assumption to our algebraically-δ-closed fields, namely, that K^{alg} has a K-basis of constants, i.e., a basis $\{1, \alpha_1, \alpha_2, \dots\}$ with $\alpha_i \in K^{\text{alg}} \setminus K$ and $\delta(\alpha_i) = 0$ for all $i \geq 1$. We then show that these fields solve every linear differential equation defined over them, and we conclude by proving that these fields are PV-closed, i.e., any algebraically-δ-closed field K is already a PV-extension for any linear homogeneous system over K. All the relevant notions from differential Galois theory employed in this section can be found in [Mag94, ch. 2 & 3].

Theorem 3.1. *Let (K, δ) be an algebraically-δ-closed field, and let $a_0, \dots, a_{n-1}, b \in K$, with $n \in \mathbb{N}_0$. Suppose further that K^{alg} has a K-basis of constants. Then there exists some $c \in K$ such that $\delta^n(c) + a_{n-1}\delta^{n-1}(c) + \dots + a_1\delta(c) + a_0 c = b$.*

Proof. Since K^{alg} is differentially closed, there is $d \in K^{\text{alg}}$ such that

$$\delta^n(d) + a_{n-1}\delta^{n-1}(d) + \dots + a_1\delta(d) + a_0 d = b. \tag{3.1}$$

Since K^{alg} has a K-basis of constants, there are $d_0, d_1, \dots, d_k \in K$ such that

$$d = d_0 + d_1\alpha_1 + \dots + d_k\alpha_k. \tag{3.2}$$

Thus, substituting this into (3.1), we get (setting $a_n = \alpha_0 = 1$)

$$\sum_{i=0}^{n}\sum_{j=0}^{k} a_i\delta^i(d_j)\alpha_j = b.$$

Since $b = b + 0\alpha_1 + 0\alpha_2 + \dots$, comparing the coefficients of $\alpha_0 = 1$ on both sides we get

$$\delta^n(d_0) + a_{n-1}\delta^{n-1}(d_0) + \dots + a_1\delta(d_0) + a_0 d_0 = b,$$

and so d_0 is a solution to (3.1) in K. □

Our next goal is to show that algebraically-δ-closed fields K with K^{alg} having a K-basis of constants are PV-closed (compare to the result from [HLS21, Lemma 5.8, p. 21]). More precisely, we want to show the following stronger result:

Theorem 3.2 (cf. [HLS21, Proposition 5.9, p. 22])**.** *Let (K, δ) be an algebraically-δ-closed field, and let $a_0, \dots, a_{n-1} \in K$, with $n \in \mathbb{N}$. Suppose further that K^{alg} has a K-basis of constants. Let $g \in K\{x\} \setminus \{0\}$ be such that $\operatorname{ord}(g) < n$. Then there exists some $d \in K$ which solves the following system:*

$$\begin{cases} \delta^n(x) + a_{n-1}\delta^{n-1}(x) + \dots + a_1\delta(x) + a_0 x = 0, \\ g(x) \neq 0. \end{cases}$$

We will use the following lemma:

Lemma 3.3 ([Lam21, Proof of Proposition 4.3, p. 15])**.** *Let $g \in K\{x\}$ be a non-zero differential polynomial with $\operatorname{ord}(g) = k$. If there exist $d_0, \dots, d_k \in K$ linearly independent over C_K (the field of constants of K), then there exist $c_0, \dots, c_k \in C_K$ such that $g(c_0 d_0 + \dots + c_k d_k) \neq 0$.*

Proof of Theorem 3.2. We construct inductively solutions $d_0, d_1, \dots, d_{n-1} \in K$ to the homogeneous linear equation $0 = l(x) := \delta^n(x) + a_{n-1}\delta^{n-1}(x) + \dots + a_0 x = 0$ which are linearly independent over C_K.

For the base case, we need to find a solution $d_0 \in K$ of $l(x) = 0$ such that $\{c\}$ is linearly independent over C_K, i.e., $c \neq 0$. Since K^{alg} is differentially closed, the system

$$\begin{cases} l(x) = 0, \\ x \neq 0 \end{cases}$$

has a solution, say $\beta \in K^{\mathrm{alg}}$. Since K^{alg} has a K-basis of constants, there exist $e_0, e_1, \ldots, e_k \in K$ such that $\beta = e_0 + e_1\alpha_1 + \cdots + e_k\alpha_k$. Substituting this into $l(\beta) = 0$, we can see, by homogeneity, that each e_i is a solution to $l(x) = 0$. Furthermore, since $\beta \neq 0$, there exists some $0 \leq i \leq k$ such that $e_i \neq 0$. Therefore, let $d_0 := e_i$.

For the inductive step, suppose we already have linearly independent solutions $d_0, d_1, \ldots, d_i \in K$ over C_K of $l(x) = 0$, for some $i < n-1$. Since K^{alg} is differentially closed, there exists some $\beta \in K^{\mathrm{alg}}$ such that $l(\beta) = 0$ and (using the characterization of linear independence in terms of the Wronskian)

$$\begin{vmatrix} d_0 & d_1 & \ldots & d_i & \beta \\ \delta(d_0) & \delta(d_1) & \ldots & \delta(d_i) & \delta(\beta) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta^{i+1}(d_0) & \delta^{i+1}(d_1) & \ldots & \delta^{i+1}(d_i) & \delta^{i+1}(\beta) \end{vmatrix} \neq 0.$$

Thus, we have

$$M_{1,i+2}\beta + M_{2,i+2}\delta(\beta) + \cdots + M_{i+1,i+1}\delta^{i+1}(\beta) \neq 0, \tag{3.3}$$

where $M_{i,j}$ denotes the minor of the above matrix obtained by removing the ith row and jth column. Note that $M_{j,i+2} \in K$ for all $1 \leq j \leq i+2$.

Now, since K^{alg} has a K-basis of constants, we can write $\beta = e_0 + e_1\alpha_1 + \cdots + e_k\alpha_k$, where $e_l \in K$ for all $0 \leq l \leq k$. Note that $l(e_l) = 0$ for all l by the homogeneity of l. Furthermore, substituting this into (3.3), it follows that there is $j \in \{0, \ldots, k\}$ such that $M_{1,i+2}e_j + M_{2,i+2}\delta(e_j) + \cdots + M_{i+1,i+1}\delta^{i+1}(e_j) \neq 0$, or equivalently,

$$\begin{vmatrix} d_0 & d_1 & \ldots & d_i & e_j \\ \delta(d_0) & \delta(d_1) & \ldots & \delta(d_i) & \delta(e_j) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta^{i+1}(d_0) & \delta^{i+1}(d_1) & \ldots & \delta^{i+1}(d_i) & \delta^{i+1}(e_j) \end{vmatrix} \neq 0.$$

Therefore, e_j is linearly independent from $d_0, \ldots, d_i$ over C_K, so we set $d_{i+1} = e_j$.

Having $d_0, \ldots, d_{n-1}$, we apply Lemma 3.3 to get $c_0, \ldots, c_{n-1} \in C_K$ such that $g(c_0d_0 + \cdots + c_{n-1}d_{n-1}) \neq 0$. Furthermore, by homogeneity, $l(c_0d_0 + \cdots + c_{n-1}d_{n-1}) = 0$. Since $c_0d_0 + \cdots + c_{n-1}d_{n-1} \in K$, we are done. □

Corollary 3.4. *Let (K, δ) be an algebraically-δ-closed field such that K^{alg} has a K-basis of constants. Then (K, δ) is PV-closed. In other words, K is already a PV-extension for any linear homogeneous system over K.*

References

[HLS21] D.M. Hoffmann and O. León Sánchez, *Model theory of differential fields with finite group actions*, 2021, available on https://arxiv.org/abs/2012.14376.

[Kol73] E. R. Kolchin, *Differential algebra and algebraic groups*, Pure and Applied Mathematics, Vol. 54, Academic Press, New York-London, 1973.

[Lam21] K. Lamont, *The differential primitive element theorem*, Master's thesis, Department of Mathematics, The University of Manchester, 2021.

[LST20] O. León Sánchez and M. Tressl, *Differentially large fields*, 2020, available on https://arxiv.org/abs/2005.00888.

[Mag94] A. R. Magid, *Lectures on differential Galois theory*, University Lecture Series, vol. 7, American Mathematical Society, Providence, RI, 1994.