A topos-theoretic view of difference algebra

Ivan Tomašić

Queen Mary University of London

Applications of the model theory of fields with operators,
Manchester, 18/06/2019

Outline

Difference categories

The topos of difference sets

Difference homological algebra

Difference algebraic geometry

Applications

Difference categories: Ritt-style

Let $\mathscr{C}$ be a category. Define its associated difference category

$$\boldsymbol{\sigma}\text{-}\mathscr{C}$$

- objects are pairs

  $$(X, \sigma_X),$$

  where $X \in \mathscr{C}$, $\sigma_X \in \mathscr{C}(X, X)$;
- a morphism $f : (X, \sigma_X) \to (Y, \sigma_Y)$ is a commutative diagram in $\mathscr{C}$

  $$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

  i.e., an $f \in \mathscr{C}(X, Y)$ such that

  $$f \circ \sigma_X = \sigma_Y \circ f.$$

Difference categories as functor categories

Let $\boldsymbol{\sigma}$ be the category associated with the monoid $(\mathbb{N}, +)$:

- single object o;
- $\mathrm{Hom}(o, o) \simeq \mathbb{N}$.

Then

$$\boldsymbol{\sigma}\text{-}\mathscr{C} \simeq [\boldsymbol{\sigma}, \mathscr{C}],$$

the functor category:

- objects are functors $\mathcal{X} : \boldsymbol{\sigma} \to \mathscr{C}$
- morphisms are natural transformations.

Translation mechanism: if $\mathcal{X} \in [\boldsymbol{\sigma}, \mathscr{C}]$, then

$$(\mathcal{X}(o), \mathcal{X}(o \xrightarrow{1} o)) \in \boldsymbol{\sigma}\text{-}\mathscr{C}.$$

Difference categories via categorical logic

Let $\mathbb{S}$ be the algebraic theory of a single endomorphism. Then

$$\boldsymbol{\sigma}\text{-}\mathscr{C} = \mathbb{S}(\mathscr{C}),$$

the category of models of $\mathbb{S}$ in $\mathscr{C}$.

Examples

We will consider:

- $\boldsymbol{\sigma}$-$\mathbf{Set}$;
- $\boldsymbol{\sigma}$-$\mathbf{Gr}$;
- $\boldsymbol{\sigma}$-$\mathbf{Ab}$;
- $\boldsymbol{\sigma}$-$\mathbf{Rng}$.

Given $R \in \boldsymbol{\sigma}$-$\mathbf{Rng}$, consider

- R-$\mathbf{Mod}$, the category of difference R-modules.

The topos of difference sets

Note

$$\boldsymbol{\sigma}\text{-}\mathbf{Set} \simeq [\boldsymbol{\sigma}, \mathbf{Set}]$$

is a Grothendieck topos (as the presheaf category on $\boldsymbol{\sigma}^{\mathsf{op}} \simeq \boldsymbol{\sigma}$). The literature also calls it the classifying topos of $\mathbb{N}$, written

$$\mathbf{B}\mathbb{N}.$$

Old adage of topos theory

A topos can serve as a universe for developing mathematics.

A view of difference algebra

Note,

$$\boldsymbol{\sigma}\text{-}\mathbf{Gr} \simeq \mathbf{Gr}(\boldsymbol{\sigma}\text{-}\mathbf{Set})$$
$$\boldsymbol{\sigma}\text{-}\mathbf{Ab} \simeq \mathbf{Ab}(\boldsymbol{\sigma}\text{-}\mathbf{Set})$$
$$\boldsymbol{\sigma}\text{-}\mathbf{Rng} \simeq \mathbf{Rng}(\boldsymbol{\sigma}\text{-}\mathbf{Set}).$$

For $R \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$,

$$R\text{-}\mathbf{Mod} \simeq \mathbf{Mod}(\boldsymbol{\sigma}\text{-}\mathbf{Set}, R)$$

is the category of modules in a ringed topos.

Motto

Difference algebra is the study of algebraic objects internal in the topos $\boldsymbol{\sigma}\text{-}\mathrm{Set}$.

Topoi

A category $\mathscr{E}$ is an elementary topos if

1. $\mathscr{E}$ has finite limits (all pullbacks and a terminal object e);
2. $\mathscr{E}$ is cartesian closed; for each $X \in \mathscr{E}$, the functor $- \times X$ has a right adjoint
$$[X, -] : \mathscr{E} \to \mathscr{E};$$
3. $\mathscr{E}$ has a subobject classifier, i.e., an object Ω and a morphism $e \xrightarrow{t} \Omega$ such that, for each monomorphism $Y \xrightarrow{u} X$ in $\mathscr{E}$, there is a unique morphism $\chi_u : X \to \Omega$ making
$$\begin{array}{ccc} Y & \longrightarrow & e \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{\chi_u} & \Omega \end{array}$$

a pullback diagram.

Monoidal closed categories

- A symmetric monoidal category $\mathscr{V}$ is closed when we have internal hom objects

$$[B, C] \in \mathscr{V}$$

  so that

$$\mathscr{V}(A \otimes B, C) \simeq \mathscr{V}(A, [B, C]),$$

  for all $A, B, C \in \mathscr{V}$.
- $\mathscr{V}$ is cartesian closed when monoidal closed for $\otimes = \times$.

Internal homs for difference sets

Consider $N = (\mathbb{N}, i \mapsto i+1) \in \boldsymbol{\sigma}\text{-}\mathbf{Set}$.

Internal homs for $\boldsymbol{\sigma}$-Set

- $$\begin{aligned} [X, Y] &= \boldsymbol{\sigma}\text{-}\mathbf{Set}(N \times X, Y) \\ &\simeq \{(f_i) \in \mathbf{Set}(X, Y)^{\mathbb{N}} : f_{i+1} \circ \sigma_X = \sigma_Y \circ f_i\}. \end{aligned}$$

$$\begin{array}{ccc} X & \xrightarrow{f_0} & Y \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\ X & \xrightarrow{f_1} & Y \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\ X & \xrightarrow{f_2} & Y \\ \vdots & & \vdots \end{array}$$

- shift $s : [X, Y] \to [X, Y]$, $s(f_0, f_1, \ldots) = (f_1, f_2, \ldots)$.

Internal homs vs homs

Note

$$\Gamma([X, Y]) = \mathrm{Fix}[X, Y] = \boldsymbol{\sigma}\text{-}\mathbf{Set}(X, Y),$$

so the Ritt-style difference algebra is the underlying category side of the enriched framework; it only sees the tip of an iceberg.

Subobject classifier in difference sets

The subobject classifier in $\boldsymbol{\sigma}$-$\mathbf{Set}$ is

$$\Omega = \mathbb{N} \cup \{\infty\}, \quad \sigma_\Omega : 0 \mapsto 0, \infty \mapsto \infty, i+1 \mapsto i \ (i \in \mathbb{N}).$$

For a monomorphism $Y \xrightarrow{u} X$, the classifying map is

$$\chi_u : X \to \Omega, \ \ \chi_u(x) = \min\{n : \sigma_X^n(x) \in Y\},$$

and

$$Y = \chi_u^{-1}(\{0\}).$$

Logic of difference sets

$\Omega = \mathbb{N} \cup \{\infty\}$ is a Heyting algebra with:

- $\text{true} = 0$, $\text{false} = \infty$;
- $\wedge(i,j) = \max\{i,j\}$;
- $\vee(i,j) = \min\{i,j\}$;
- $\neg(i) = \begin{cases} 0, & i = \infty; \\ \infty, & i \in \mathbb{N}. \end{cases}$
- $\Rightarrow(i,j) = \begin{cases} 0, & i \geq j; \\ j, & i < j. \end{cases}$

Warning:

$\neg\neg \neq \mathrm{id}_\Omega$ so $\boldsymbol{\sigma}$-$\mathbf{Set}$ is not a Boolean topos.

Difference subsets

For $X \in \boldsymbol{\sigma}\text{-}\mathbf{Set}$,

$$\mathrm{Sub}(X)$$

is a Heyting algebra; for $U \rightarrowtail X$, $V \rightarrowtail X$,

$$U \wedge V = U \cap V, \quad U \vee V = U \cup V.$$

However

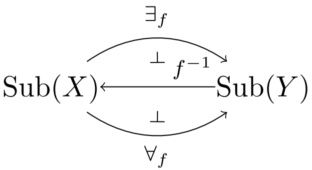
$$U \Rightarrow V = \{x \in X : \text{for all } n \in \mathbb{N}, \sigma_X^n x \in U \text{ implies } \sigma_X^n x \in V\},$$

and

$$\neg U = \{x \in X : \text{for all } n \in \mathbb{N}, \sigma_X^n x \notin U\}.$$

Quantifiers in difference sets

Given $f : X \to Y$ in $\boldsymbol{\sigma}$-$\mathbf{Set}$, we have functors



where

$$f^{-1}(V \rightarrowtail Y) = f^{-1}(V) \rightarrowtail X$$

is the usual set-theoretic preimage,

$$\exists_f(U \rightarrowtail X) = \mathrm{Im}(U \rightarrowtail X \xrightarrow{f} Y) = f(U) \rightarrowtail Y$$

is the usual set-theoretic image of U along f, and

$$\forall_f(U \rightarrowtail X) = \{y \in Y : \text{for all } n \in \mathbb{N}, U_{\sigma^n y} = X_{\sigma^n y}\} \rightarrowtail Y.$$

In search of difference cohomology

Goals

- Homological algebra of $\boldsymbol{\sigma}\text{-}\mathbf{Ab}$ and $R\text{-}\mathbf{Mod}$, for $R \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$.
- Solid foundation for difference algebraic geometry.

Difference modules are monoidal closed

Let $R \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$.

Internal homs for R-$\mathbf{Mod}$

Given $A, B \in R\text{-}\mathbf{Mod}$,

$$f = (f_i) \in [A, B]_R \in R\text{-}\mathbf{Mod}$$

is a 'ladder' with

$$f_i \in \lfloor R \rfloor\text{-}\mathbf{Mod}(\lfloor A \rfloor, \lfloor B \rfloor).$$

Hom-tensor duality for difference modules

$$\mathrm{Hom}_R(A \otimes B, C) \simeq \mathrm{Hom}_R(A, [B, C]_R).$$

Difference homological algebra

Let $R \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$.

Fact

$R\text{-}\mathbf{Mod} = \mathbf{Mod}(\boldsymbol{\sigma}\text{-}\mathbf{Set}, R)$ is abelian with enough injectives and enough internal injectives.

Difference cohomology is an instance of topos cohomology:

- $\mathrm{Ext}^i_R(M, N)$;
- $\mathrm{Ext}^i_R[M, N]$.

Difference algebraic geometry

Recall: topos theory philosophy

The universe of sets can be replaced by an arbitrary base topos, and one can develop mathematics over it.

Motto

Difference algebraic geometry is algebraic geometry over the base topos $\boldsymbol{\sigma}$-$\mathbf{Set}$.

Difference schemes

Hakim-Cole Zariski spectrum

For a ringed topos $(\mathscr{E}, A)$, $\mathrm{Spec.Zar}(\mathscr{E}, A)$ is the locally ringed topos equipped with a morphism of ringed topoi

$$\mathrm{Spec.Zar}(\mathscr{E}, A) \to (\mathscr{E}, A)$$

which solves a certain 2-universal problem.

Definition

The affine difference scheme associated to a difference ring A is the locally ringed topos

$$(X, \mathscr{O}_X) = \mathrm{Spec.Zar}(\boldsymbol{\sigma}\text{-}\mathbf{Set}, A)$$

General relative schemes can be treated using stacks.

Difference étale topos

Hakim-Cole étale spectrum

For a locally ringed topos $(\mathscr{E}, A)$, $\mathrm{Spec.\acute{E}t}(\mathscr{E}, A)$ is a strictly locally ringed topos equipped with a morphism of locally ringed topoi

$$\mathrm{Spec.\acute{E}t}(\mathscr{E}, A) \to (\mathscr{E}, A)$$

which solves a certain 2-universal problem.

Definition

Let $(X, \mathscr{O}_X)$ be a difference scheme as before. Its étale topos is the strictly locally ringed topos

$$(X_{\text{ét}}, \mathscr{O}_{X_{\text{ét}}}) = \mathrm{Spec.\acute{E}t}(X, \mathscr{O}_X)$$

Étale fundamental group of a difference scheme

Definition

Let $(X, \mathscr{O}_X)$ be a difference scheme, and $\bar{x} : \boldsymbol{\sigma}\text{-}\mathbf{Set} \to X_{\text{ét}}$ a point. Then

$$\pi_1^{\text{ét}}(X, \bar{x}) = \pi_1(X_{\text{ét}}, \bar{x}),$$

the Bunge-Moerdijk pro-$(\boldsymbol{\sigma}\text{-}\mathbf{Set})$-localic fundamental group associated to the geometric morphism $X_{\text{ét}} \to \boldsymbol{\sigma}\text{-}\mathbf{Set}$.

A special case: Galois theory of difference field extensions

Let L/k be a difference field extension of finite σ-type with $\lfloor L \rfloor / \lfloor k \rfloor$ Galois (it can happen to be an infinite extension with no finite σ_L-stable subextensions).
The Galois group is the difference group

$$G = \underline{\mathrm{Aut}}(L/k) \rightarrowtail [L, L]_{k\text{-}\mathbf{Alg}},$$

topologised by basic opens

$$\langle a, b, n \rangle = \{ f = (f_i) \in \underline{\mathrm{Aut}}(L/k) : f_n(a) = b \},$$

for $a, b \in \lfloor L \rfloor$.
Details on finite Galois theory: talk by Rachael.

Difference étale cohomology

Definition

Let $(X, \mathscr{O}_X)$ be a difference scheme with structure geometric morphism $\gamma : X \to \boldsymbol{\sigma}\text{-}\mathbf{Set}$, and let M be a $\mathscr{O}_{X_{\text{ét}}}$-module. Then

$$H^n_{\text{ét}}(X, M) = R^i \gamma_*(M),$$

the abelian difference groups obtained through relative (enriched) topos cohomology.

Difference-differential algebra

Keigher: differential algebra in a topos $\mathscr{E}$

we have the category of differential rings

$$\mathbf{DRng}(\mathscr{E}).$$

Difference-differential rings

Note

$$\delta\text{-}\sigma\text{-}\mathbf{Rng} \simeq \mathbf{DRng}(\boldsymbol{\sigma}\text{-}\mathbf{Set}).$$

Work in progress by Antonino.

Some calculations

- (with M. Wibmer) Cohomology of difference algebraic groups. Explicit calculations for twisted groups of Lie Type as difference group schemes;
- Ext of modules over skew-polynomial rings.

Cohomology of the Suzuki difference group scheme

Let $\theta : \mathrm{Sp}_4 \to \mathrm{Sp}_4$ be the algebraic endomorphism satisfying

$$\theta^2 = F_2.$$

The Suzuki difference group scheme $\mathbf{G}$:

$$\mathbf{G}(R, \sigma) = \{X \in \mathrm{Sp}_4(R) : F_2 \circ \sigma(X) = \theta(X)\}.$$

naturally acts on the module

$$\mathbf{F}(R, \sigma) = \{(x_1, x_2, x_3, x_4)^T \in R^4 : \sigma^2 x_i^2 = x_i\}.$$

Note

$$\mathbf{G}(\bar{\mathbb{F}}_2, F_q) = {}^2B_2(2q^2),$$

the (familiar) finite Suzuki group.
We have

$$\mathrm{H}^1(\mathbf{G}, \mathbf{F}) \text{ is 1-dimensional.}$$

Extensions of modules over skew-polynomial rings

For $k \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$, have the skew-polynomial ring

$$R = k[T; \sigma_k].$$

Equivalence of categories:

$$k\text{-}\mathbf{Mod} \simeq R\text{-}\mathbf{Mod}.$$

If F is an étale k-module, then

$$\mathrm{Ext}^i_{R\text{-}\mathbf{Mod}}(F, F') = \begin{cases} [F, F']_s, & i = 1, \\ 0, & i > 1, \end{cases}$$

where $[F, F']_s = [F, F'] / \operatorname{Im}(s - \mathrm{id})$ is the module of s-coinvariants of $[F, F']$.
In particular, if k is linearly difference closed and F, F' are finite étale, then, for $i > 0$,

$$\mathrm{Ext}^i(F, F') = 0.$$

Studying Elephant