Searching for Écalle's analysable functions [1]

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Motivation

Écalle conjectured the existence of a class $\mathcal{E}$ of all germs at $+\infty$ of "analysable functions" such that:

1. $\mathcal{E}$ is a Hardy field closed under taking antiderivatives;
2. the subset of all infinitely increasing germs in $\mathcal{E}$ is a group under composition;
3. $\mathcal{E}$ contains all *tame* functions needed to study problems with analytic data, such as *multisummable functions* and *Dulac functions* (see Rolin's talk), as well as exp and log. In particular, $\mathcal{E}$ should contain the Hardy field $\mathcal{H}_{\text{an,exp}}$ of $\mathbb{R}_{\text{an,exp}}$;
4. each function in $\mathcal{E}$ has a unique transseries asymptotic expansion (*quasianalyticity*);
5. there is an explicit summation procedure to recover the germs from their transseries expansion.

The formal side

Ecalle introduced the field of transseries $\mathbb{T}$ and provided strong evidence that it has (the formal equivalent of) properties (1)–(3).

Aschenbrenner, van den Dries and van der Hoeven [ADH] give an axiomatization T of the first-order theory of $\mathbb{T}$ as an ordered, differential, valued field and find a language extension in which this theory has quantifier elimination.

This axiomatization formulates additional closure properties of $\mathbb{T}$, which guarantee that $\mathbb{T}$ is closed under exponentiation and taking antiderviatives and has the *Intermediate Value Property* for differential polynomials over $\mathbb{T}$.

Rephrasing Écalle's conjecture

Disregarding the search for an explicit summation procedure from now on, we want to find a Hardy field $\mathcal{E}$ of germs at $+\infty$ such that

1. $\mathcal{E} \models T$;
2. $\mathcal{E}$ contains all *tame* functions needed to study problems with analytic data, such as *multisummable functions* and *Dulac functions*;
3. each function in $\mathcal{E}$ has a unique transseries asymptotic expansion (*quasianalyticity*).

Finding a Hardy field model of T is proposed as an open problem in [ADH].

ADH are working on proving that every maximal Hardy field is a model of T.

ADH's strategy

1. (Boshernitzan 1981) Every Hardy field admits a Hardy field extension closed under exp and under taking antiderivatives (Liouville extension).
2. (Conjectured by ADH) Every Hardy field admits a Hardy field extension with the IVP.

A partial result towards (2) is available on the arXiv.

Working with Caulfield and Thomas, we propose to proceed similarly while keeping track of asymptotic transseries expansions, as discussed next.

the series

$\mathcal{C}$ = ring of germs at $+\infty$ of real continuous functions
$\mathcal{M}$ a multiplicative ordered $\mathbb{R}$-vector space (**monomials**), often a set of positive elements of $\mathcal{C}$

Example: $\mathcal{M} = \langle x, \log x \rangle$, the mult. $\mathbb{R}$-vector subspace of $\mathcal{C}$ generated by x and $\log x$

$\mathbb{R}((\mathcal{M}))$ = set of generalized series $F = \sum a_m m$ such that each $a_m \in \mathbb{R}$ and $\text{supp}(F)$ is anti-well ordered

Example: $F = \sum_{\alpha,\beta\in\mathbb{N}} x^{-\alpha}(\log x)^{-\beta}$ has support of order type ω^2

Remark: if $\mathcal{M} \subset \mathcal{C}$, we also consider $\mathcal{F}((\mathcal{M}))$, where $\mathcal{F}$ is a subfield of $\mathcal{C}$ such that every germ in $\mathcal{F}$ grows or decays more slowly than any germ in $\mathcal{M}$.

asymptotic expansions: special case

Fix $\mathcal{M} \subset \mathcal{C}$ and $\mathcal{F} \subset \mathcal{C}$ as on the previous slide.

Let $f \in \mathcal{C}$ and $F = \sum a_m m \in \mathcal{F}((\mathcal{M}))$; we want to define what it means for f to have asymptotic expansion F at $+\infty$.

Aa set $S \subseteq \mathcal{M}$ is **natural** if $S \cap (a, +\infty)_{\mathcal{M}}$ is finite for every $a \in \mathcal{M}$.

If F has natural support, then f has **asymptotic expansion** F at $+\infty$ if, for every $n \in \mathcal{M}$,

$$f(x) - \sum_{m \geq n} a_m(x) m(x) = o(n(x)) \quad \text{as } x \to +\infty.$$

Example: if $F = \sum_{n \in \mathbb{N}} X^{-n}$, then $F(x)$ converges to $f(x) = \frac{x}{x-1}$ for $x > 1$. In particular, there are $A, B > 0$ such that for $n \in \mathbb{N}$,

$$\left| f(x) - \sum_{m \leq n} x^{-m} \right| \leq AB^n |x|^{-n-1} \quad \text{as } x \to +\infty.$$

asymptotic expansions: general case

What makes it work in the special case is that every **truncation** $F_n := \sum_{m \geq n} a_m m$ of F has finite support, so represents the germ of a function.

In general, we call a set $\mathcal{S} \subseteq \mathcal{F}((\mathcal{M}))$ **truncation closed** if, for every $F \in \mathcal{S}$ and every $n \in \mathcal{M}$, the truncation F_n also belongs to $\mathcal{S}$.

Definition

A triple $(\mathcal{K}, \mathcal{M}, T)$ is a **quasianalytic asymptotic (qaa) algebra** if $\mathcal{K}$ is a subalgebra of $\mathcal{C}$ such that

- $T : \mathcal{K} \longrightarrow \mathbb{R}((\mathcal{M}))$ is an injective algebra homomorphism (quasianalyticity);
- $T(\mathcal{K})$ is truncation closed;
- for $f \in \mathcal{K}$ and $n \in \mathcal{M}$, we have

$$f(x) - T^{-1}((Tf)_n)(x) = o(n(x)) \quad \text{as } x \to +\infty.$$

Example: the Hardy field of $\mathbb{R}_{\text{an,exp}}$

The set $\mathbb{T}$ of *transseries* [Ecalle, ADH] is a set of generalized series in some $\mathbb{R}((\mathcal{M}))$ closed under exp and log. The set $\mathcal{M}$ of *transmonomials* is constructed in parallel with $\mathbb{T}$ and has the property that every Archimedean class of $\mathbb{T}$ has a unique representative in $\mathcal{M}$.

We identify $\mathcal{H}_{\text{an,exp}}$ with the subset of $\mathbb{T}$ consisting of all *convergent* transseries and let

$$\mathcal{L} := \mathcal{H}_{\text{an,exp}} \cap \mathcal{M}$$

be the set of all convergent transmonomials.

Proposition (van den Dries, Macintyre, Marker; Galal, Kaiser, S)

Under this identification,

1. *every $h \in \mathcal{H}_{\text{an,exp}}$ is the sum of a convergent transseries $S(h) \in \mathbb{R}((\mathcal{L}))$;*
2. *the triple $(\mathcal{H}_{\text{an,exp}}, \mathcal{L}, S)$ is a qaa field.*

Rephrasing Écalle's conjecture, again

Find a qaa field $(\mathcal{E}, \mathcal{M}, T)$, with $\mathcal{M}$ the set of transmonomials, such that

1. $\mathcal{E} \models T$;
2. $\mathcal{E}$ contains all *tame* functions needed to study problems with analytic data, such as *multisummable functions* and *Dulac functions*.

To achieve (1), we (with Caulfield and Thomas) hope to adapt ADH's strategy to show that every qaa Hardy field has a qaa field extension that is a model of T.

To achieve (2), we suggest using the recent construction of Ilyashenko fields (done with Galal and Kaiser), a particular kind of qaa Hardy field containing Dulac functions, as discussed next.

complex analysis needed to obtain quasianalyticity

A **standard power domain** is a complex domain

$$U_C^\epsilon := \{z + C(1+z)^\epsilon : \operatorname{re} z > 0\},$$

where $C > 0$ and $0 < \epsilon < 1$.

$\exp^{-1} = o(1)$ as $|z| \to \infty$ uniformly on standard power domains (but not on half-planes).

Uniqueness Principle (Ilyashenko)

Let U be a standard power domain and $\mathbf{h} : U \longrightarrow \mathbb{C}$ be bounded and holomorphic. If

$$\mathbf{h}(x) = o\left(e^{-nx}\right) \quad \text{as } x \to +\infty, \quad \text{for all } n \in \mathbb{N},$$

then $\mathbf{h} = 0$.

This UP does not hold on sectors of opening strictly less than π.

strong asymptotic expansions

$L_0 := \langle \exp \rangle$ is a **scale** on standard power domains, that is, $m \in L_0$ and $m = o(1)$ imply $m = o(1)$ on standard power domains.

Definition

$f : (a, +\infty) \longrightarrow \mathbb{R}$ has **strong asymptotic expansion** $F = \sum a_m m \in \mathbb{R}(\!(L_0)\!)$ if supp(F) is natural and there is a standard power domain U such that

1. f has a holomorphic extension $\mathbf{f} : U \longrightarrow \mathbb{C}$;
2. for $n \in L_0$ we have

$$\mathbf{f}(z) - \sum_{m \geq n} a_m m(z) = o(n(z)) \quad \text{as } |z| \to \infty.$$

Stage 0 of the construction

Let $\mathcal{F}_0$ be the set of all germs f at $+\infty$ with a strong asymptotic expansion $T_0 f \in \mathbb{R}(\!(L_0)\!)$.

Proposition

$\mathcal{F}_0$ is a field and $T_0 : \mathcal{F}_0 \longrightarrow \mathbb{R}(\!(L_0)\!)$ is a field homomorphism.

Next, we right-shift $\mathcal{F}_0$ by log: set $L_1' := L_0 \circ \log$, $\mathcal{F}_1' := \mathcal{F}_0 \circ \log$ and $T_1' := T_0 \circ \log$.

Corollary

$\mathcal{F}_1'$ is a field and $T_1' : \mathcal{F}_1' \longrightarrow \mathbb{R}(\!(L_1')\!)$ is a field homomorphism. Every $f \in \mathcal{F}_1'$ is polynomially bounded.

To iterate, we use $\mathcal{F}_1'$ as coefficients in strong asymptotic expansions.

Stage 1 of the construction

Let $\mathcal{F}_1$ be the set of all germs f at $+\infty$ with a strong asymptotic expansion $\tau_1 f \in \mathcal{F}_1' \left(\!\left(L_0\right)\!\right)$.

However, the series we want are in the monomials $L_1 := \langle \exp, x \rangle^\times$, a scale on standard power domains. To get there, let $\sigma_1 : \mathcal{F}_1' \left(\!\left(L_0\right)\!\right) \longrightarrow \mathbb{R} \left(\!\left(L_1\right)\!\right)$ be the field homomorphism

$$\sigma_1 \left(\sum a_m m \right) := \sum T_1'(a_m) m,$$

and set $T_1 := \sigma_1 \circ \tau_1$.

Proposition

$\mathcal{F}_1$ is a field and τ_1 and T_1 are field homomorphisms.

Again set $L_2' := L_1 \circ \log$, $\mathcal{F}_2' := \mathcal{F}_1 \circ \log$ and $T_2' := T_1 \circ \log$.

Corollary

$T_2' : \mathcal{F}_2' \longrightarrow \mathbb{R} \left(\!\left(L_2'\right)\!\right)$ is a field homomorphism, and every germ in $\mathcal{F}_2'$ is polynomially bounded.

Stage ω of the construction

Iterating this construction, we obtain fields $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots$ with field homomorphisms $T_i : \mathcal{F}_i \longrightarrow \mathbb{R}(\!(L_i)\!)$ such that T_{i+1} extends T_i for each i, where $L_i := \langle \exp, \dots, \log_{i-1} \rangle^\times$.
So we set $\mathcal{F} := \bigcup_{i \in \mathbb{N}} \mathcal{F}_i$ and let $T : \mathcal{F} \longrightarrow \mathbb{R}(\!(L)\!)$ be the common extension of all T_i, where $L := \langle \exp, x, \log, \dots \rangle^\times$.

Theorem 1 (S)

1. $(\mathcal{F}, L, T)$ is a qaa field.
2. *$\mathcal{F}$ is closed under differentiation and* log*-composition.*

This qaa field was used by Belotto da Silva, Figalli, Parusinski and Rifford in their recent solution of the strong Sard conjecture in dimension 3.

Problem: $(\mathcal{F}, L, T)$ does not extend $(\mathcal{H}_{\text{an,exp}}, \mathcal{L}, S)$, so $\mathcal{F}$ is not big enough for the construction of Écalle's class of analysable functions.

The construction for convergent transmonomials

Let $\overline{m} \in \mathcal{L}^k$ for some $k \in \mathbb{N}$, and set $L_{\overline{m}} := \langle \overline{m} \rangle$.

Theorem 2 (Galal, Kaiser and S)

There exists a qaa field $(\mathcal{K}_{\overline{m}}, L_{\overline{m}}, T_{\overline{m}})$. Moreover, if $\overline{n}$ is a subtuple of $\overline{m}$, then $(\mathcal{K}_{\overline{m}}, L_{\overline{m}}, T_{\overline{m}})$ extends $(\mathcal{K}_{\overline{n}}, L_{\overline{n}}, T_{\overline{n}})$.

Let $\mathcal{K}$ and T be the direct limits of the $\mathcal{K}_{\overline{m}}$ and $T_{\overline{m}}$.

Theorem 3 (Galal, Kaiser and S)

1. *$(K, \mathcal{L}, T)$ is a qaa field that extends each $(\mathcal{K}_{\overline{m}}, L_{\overline{m}}, T_{\overline{m}})$.*
2. *$(\mathcal{K}, \mathcal{L}, T)$ extends both $(\mathcal{F}, L, T)$ and $(\mathcal{H}_{\text{an,exp}}, \mathcal{L}, S)$, and $\mathcal{K}$ is a Hardy field.*

Theorem 3 is a possible starting point for the construction of Écalle's class of analysable functions, but it is still not enough, as it does most likely not contain the multisummable functions. Work in progress...