Transserial asymptotic expansions, oscillatory integrals and o-minimality

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TRANSSERIES AND PARAMETRIC INTEGRATION

- TRANSSERIES

$\mathbb{K}((x^{\mathbb{R}}))^{LE}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, with $x > \mathbb{R}$

example : $2e^{e^{x^2}} - xe^{x} + \log(\log x) + \sum_{n\in\mathbb{N}} x^{-n} + e^{-x} + e^{-x}\sum_{n\in\mathbb{N}}(\log x)^{-n}$

support : reverse well-ordered, order type: some countable ordinal, bounded logarithmic and exponential depth

but also $\mathbb{A}((x^{\mathbb{R}}))^{LE}$, where $\mathbb{A}$ is a ring of $\mathbb{K}$-valued real functions

example : $\sum_{n\in\mathbb{N}} e^{iP_n(x)} \cdot x^{r_n} \cdot (\log x)^{s_n}$, $P_n \in \mathbb{K}[x]$, $r_n, s_n \in \mathbb{Q}$

TRANSSERIES AND PARAMETRIC INTEGRATION

• PARAMETRIC INTEGRALS

$$X \subseteq \mathbb{R}^m, \quad F : X \times \mathbb{R}^n \longrightarrow \mathbb{K} \qquad (\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C})$$

$\forall \bar{x} \in X$ s.t. $F(\bar{x}, \cdot) \in L^1(\mathbb{R}^n)$, define

$$\mathcal{Y}_F(\bar{x}) := \int_{\mathbb{R}^n} F(\bar{x}, \bar{y})\, d\bar{y}$$

$\bar{x}$ are the <u>parameters</u>, $\bar{y}$ are the <u>integration variables</u>

Examples :

- Primitives : $f(x) = \int_{-\infty}^{x} g(y)\, dy = \int_{\mathbb{R}} \chi_{(-\infty, x)}(y) \cdot g(y)\, dy$
- Fourier transforms : $\hat{f}(x) = \int_{\mathbb{R}} e^{-2\pi i\, x \cdot y} \cdot f(y)\, dy \qquad (\mathbb{K} = \mathbb{C})$
- Families of <u>periods</u> : $a(x) = \int_{\Delta_x} f(x, y)\, dy$, $f : \Delta \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ semi-algebraic / $\mathbb{Q}$

PLAN OF THE TALK

TRANSSERIES + PARAMETRIC INTEGRALS :

1) WHY ?

- O-MINIMALITY : describe primitives of functions definable in a given o-minimal expansion $\mathcal{M}$ of $\overline{\mathbb{R}}$; find concrete examples of functions definable or not definable in $\mathcal{M}$
- FOURIER ANALYSIS, NUMBER THEORY, DYNAMICAL SYSTEMS ...

2) WHAT ?

- Transserial asymptotic expansions
- Oscillatory integrals with subanalytic phase and amplitude
- Some results

3) HOW ?

- How does o-minimality fit in the picture?
- Tools from number theory

O-MINIMAL MOTIVATION (WHY?)

Some history (late '90s, vdDries, MacIntyre, Marker, Speissegger)

Is $\mathbb{R}_{an,exp}$ closed under parametric integration? **NO**

Three examples (1 integration variable + 1 parameter):

1) $F(x) = \int_0^{+\infty} \frac{e^{-y}}{x+y}\, dy \notin \mathbb{R}_{an,exp}$ but $F \in \mathbb{R}_{\mathcal{G},exp}$,

a model-complete o-minimal expansion of $\mathbb{R}_{an,exp}$ ($\Gamma\restriction(0,+\infty) \in \mathbb{R}_{\mathcal{G},exp}$)

Why? Fact: $F \underset{x\to+\infty}{\sim} e^{-x} \sum_{n\geq 0} (-1)^n \cdot n! \cdot x^{-1-n}$ (divergent)

[DMM95] If $f \in \mathbb{R}_{an,exp}$ and $f \underset{x\to+\infty}{\sim} \sum a_n x^{-n}$, then $\sum a_n X^n$ converges in a nbd of 0

2) $G(x) = \int_0^x e^{y^2}\, dy \notin \mathbb{R}_{an,exp}$ et $\notin \mathbb{R}_{\mathcal{G},exp}$, but $\in \mathbb{R}_{Pfaff}$,

an o-minimal expansion of $\mathbb{R}_{exp}$ (model-completeness??)

Why? Fact: $G \underset{x\to+\infty}{\sim} e^{x^2} \sum_{n\geq 0} b_{2n+1} x^{-1-2n}$, where $b_n = \frac{1\cdot 3 \cdot \ldots \cdot (2n-1)}{2^{n+1}} > 0$

[DS00] If $f \in \mathbb{R}_{\mathcal{G},exp}$ and $f \underset{x\to+\infty}{\sim} \sum a_n x^{-n}$ with $a_n \geq 0$, then $\sum a_n X^n$ converges in a nbd of 0

O-MINIMAL MOTIVATION (WHY?)

3) $H(x) = \int_{\mathbb{R}} \varphi_1(x,y) \cdot \left(\log|\varphi_2(x,y)|\right)^s \cdot e^{\psi(x,y)}\, dy$, where $s \in \mathbb{N}$ and

$\varphi_1, \varphi_2, \psi$ are subanalytic (i.e. $\mathbb{R}_{an}$-definable)

In general, $H \notin \mathbb{R}_{an,exp}$ and $H \notin \mathrm{Pfaff}(\mathbb{R}_{an,exp})$ (H not a primitive)

What can we say about the collection of all such H ? o-minimal ? asymptotics ?

REMARK

3') $\tilde{H}(x) = \int_{\mathbb{R}} \varphi_1(x,y) \cdot \left(\log|\varphi_2(x,y)|\right)^s dy \in \mathbb{R}_{an,exp}$

[CLR; CM] $\tilde{H}(x) = \sum_{j=1}^{N} f_j(x) \prod_{k=1}^{K} \log(g_{j,k}(x))$ $\quad f_j, g_{j,k} \in \mathbb{R}_{an}$

Lion-Rolin preparation: $\varphi_i(x,y) = a_i(x) \cdot y^{r_i} \cdot \left(1 + \varepsilon_i(x \cdot y^{-1/d_i})\right)$ piecewise $\left(\begin{matrix} r_i \in \mathbb{Q}, d_i \in \mathbb{N} \\ a_i, \varepsilon_i \text{ analytic} \end{matrix}\right)$

$\Rightarrow \varphi_1 \cdot (\log \varphi_2)^s = A(x) \cdot y^r \cdot (\log y)^s + B(x,y)$ $\quad \left(\log(1+\varepsilon(x,y)) \in \mathbb{R}_{an}, \sim \sum a_i(x) \cdot y^{-i/d}\right)$

$\int_{\mathbb{R}} y^r \cdot (\log y)^s dy = y^q \cdot P(\log y)$, $P \in \mathbb{Q}[Y]$, $q \in \mathbb{Q}$

SOME MORE MOTIVATION (WHY?)

PROBLEM: Given a family $\mathcal{F}$ of "civilized" functions, describe the smallest collection $\hat{\mathcal{F}}$ containing $\mathcal{F}$ and closed under parametric integration.

EXAMPLE: $\mathcal{F} = \mathcal{S} := \mathrm{Def}(\mathbb{R}_{an})$ subanalytic functions $\Rightarrow \int_1^x \frac{1}{y}\, dy = \log x \in \hat{\mathcal{F}}$

[CM] $\hat{\mathcal{F}} = \mathcal{C} := \sum \mathcal{S} \prod \log \mathcal{S}$ constructible functions

but if $\mathcal{F} = \mathrm{Def}(\mathbb{R}_{an,exp})$ then $\hat{\mathcal{F}} = ???$

IN THIS TALK: we describe $\hat{\mathcal{F}}$, where $\mathcal{F} = \mathcal{S} \cup e^{i\mathcal{S}}$

$\hat{\mathcal{F}} \ni \int_{\mathbb{R}^n} \psi(\bar{x},\bar{y})\, e^{i\varphi(\bar{x},\bar{y})}\, d\bar{y} \quad \varphi, \psi \in \mathcal{S}$ (OSCILLATORY INTEGRALS WITH PHASE AND AMPLITUDE IN $\mathcal{S}$)

FOURIER TRANSFORMS OF $\mathcal{S}$: $f \in \mathcal{S} \Rightarrow \hat{f}(\bar{x}) = \int_{\mathbb{R}^n} e^{-2\pi i\, \bar{x}\cdot\bar{y}} f(\bar{y})\, d\bar{y} \in \hat{\mathcal{F}}$

In particular, $e^{-|x|} = \mathrm{Fourier}\left(\frac{1}{1+4\pi^2x^2}\right) \in \hat{\mathcal{F}}$

So $H = \int \varphi_1 \cdot (\log \varphi_2)^s \cdot e^{\psi}\, dy \in \hat{\mathcal{F}}$ when $\psi \leq \mathrm{cost}$. (example 3)

Families of exponential periods [KZ01] $\int_{\Delta_x} f(x,y)\, e^{g(x,y)}\, dy$, $f, g : \Delta \to \mathbb{R}$ semialg. $/\mathbb{Q}$

TRANSSERIAL ASYMPTOTIC EXPANSIONS (WHAT?)

$f : (a, +\infty) \longrightarrow \mathbb{k} \ (= \mathbb{R} \text{ or } \mathbb{C})$, $T \in A((x))^{LE}$ (A $\mathbb{k}$-algebra)

<u>ASYMPTOTIC SCALE</u>: Set $\mathcal{G}$ of germs which is totally ordered asymptotically:

$$\mathcal{G} = (g_\alpha)_{\alpha \in (I;<)} \text{ s.t. } \alpha < \beta \Longrightarrow g_\beta = o_{+\infty}(g_\alpha) \quad \left(\lim_{x \to +\infty} \frac{g_\beta(x)}{g_\alpha(x)} = 0 \right)$$

Example: $\mathcal{G}$ = log-exp monomial germs

1) $\mathcal{G}$-ASYMPTOTIC EXPANSION (BABY VERSION - Poincaré): $f \sim_{+\infty} T$ if $\exists$ sequence $(g_{\alpha_n})_{n \in \mathbb{N}}$, $\exists \ (c_n)_{n \in \mathbb{N}} \subseteq \mathbb{k}$ s.t.

$$\forall n \in \mathbb{N} \quad \left| f - \sum_{i=0}^{n} c_i \, g_{\alpha_i} \right| = o_{+\infty}(g_{\alpha_n})$$

$T := \sum_{n \in \mathbb{N}} c_n \, g_{\alpha_n} \in \mathbb{k}((x))^{LE}$ and $c_n = \lim_{x \to +\infty} \left(f(x) - \sum_{i=0}^{n-1} c_i \, g_{\alpha_i} \right) \cdot g_{\alpha_n}^{-1}$ is uniquely determined

Example: $$\int_1^x \frac{e^t}{t} \, dt \sim_{+\infty} e^x \sum_{n \in \mathbb{N}} n! \, x^{-1-n} \quad \nearrow_{+\infty}$$

TRANSSERIAL ASYMPTOTIC EXPANSIONS (WHAT?)

$f:(a,+\infty)\longrightarrow \mathbb{k}$ $(=\mathbb{R}$ or $\mathbb{C})$, $T\in \mathbb{A}((x))^{LE}$ ($\mathbb{A}$ $\mathbb{k}$-algebra)

ASYMPTOTIC SCALE: Set $\mathcal{G}$ of germs which is totally ordered asymptotically:

$$\mathcal{G}=(g_\alpha)_{\alpha\in(I;<)} \text{ s.t. } \alpha<\beta \Longrightarrow g_\beta=o_{+\infty}(g_\alpha) \quad \left(\lim_{x\to+\infty}\frac{g_\beta(x)}{g_\alpha(x)}=0\right)$$

2) $\mathcal{G}$-ASYMPTOTIC EXPANSION (GROWN-UP VERSION - Erdélyi): $f\sim_{+\infty} T$ if $\exists$ sequence $(g_{\alpha_n})_{n\in\mathbb{N}}$, $\exists\,(h_n(x))_{n\in\mathbb{N}}\subseteq\mathbb{A}$ (functions!) s.t.

$$\forall n\in\mathbb{N} \quad \left| f-\sum_{i=0}^{n}h_i(x)\,g_{\alpha_i}\right| = o_{+\infty}(g_{\alpha_n})$$

$\mathcal{G}$ = log-exp monomial germs $\Rightarrow$ $T:=\sum_{n\in\mathbb{N}}h_n(x)g_{\alpha_n}\in\mathbb{A}((x))^{LE}$

REMARK. The (h_n) are not, in general, uniquely determined!

Example: $\mathcal{G}=(x^{-n})_{n\in\mathbb{N}}$, $\mathbb{A}=\langle e^{-\frac{1}{x^2}}\rangle$

$$e^{-\frac{1}{x^2}}\sim_{+\infty}0 \quad \text{but also} \quad e^{-\frac{1}{x^2}}\sim_{+\infty}e^{-\frac{1}{x^2}}\cdot x^0$$

TRANSSERIAL ASYMPTOTIC EXPANSIONS (WHAT?)

$f:(a,+\infty)\longrightarrow \mathbb{k}$ $(=\mathbb{R}$ or $\mathbb{C})$, $T\in A((x))^{LE}$ (A $\mathbb{k}$-algebra)

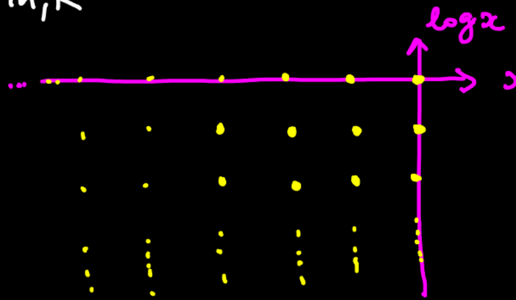
<u>ASYMPTOTIC SCALE</u>: Set $\mathcal{G}$ of germs which is totally ordered asymptotically:

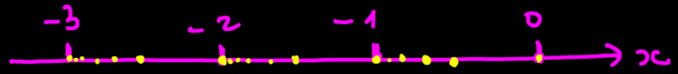
$$\mathcal{G}=(g_\alpha)_{\alpha\in(I;<)} \text{ s.t. } \alpha<\beta \Longrightarrow g_\beta = o_{+\infty}(g_\alpha) \quad \left(\lim_{x\to+\infty} \frac{g_\beta(x)}{g_\alpha(x)}=0\right)$$

3) $\mathcal{G}$-ASYMPTOTIC EXPANSION (SUPER-STAR VERSION - ...):

Let $T=\sum_{m,k} c_{m,k}\, x^{-m}(\log x)^{-k}$ or $T=\sum_{m,k\geq 1} c_{m,k}\, x^{-m+\frac{1}{k}}$ $\in \mathbb{R}((x))^{LE}$

Support:





$\omega\cdot\omega$

What does it mean that $f\sim_{+\infty} T$?

$$\left|f-\sum^{N} c_{0,k}(\log x)^{-k}\right| = o\left((\log x)^{-N}\right) \text{ but } \sum_{k=0}^{\infty} c_{0,k}(\log x)^{-k} \text{ might diverge,}$$

so the sentence "$\left|f-\sum_{k=0}^{\infty} c_{0,k}(\log x)^{-k}\right| = O(x^{-1})$" doesn't make sense!

TRANSSERIAL ASYMPTOTIC EXPANSIONS OF OSCILLATORY INTEGRALS (WHAT?)

$\mathcal{S} = \mathcal{D}ef(\mathbb{R}_{an})$, $\mathcal{C} = \Sigma\, \mathcal{S}\, \Pi \log \mathcal{S}$, $\mathcal{J} = \mathcal{S} \cup e^{i\mathcal{S}}$

AIM : describe $\hat{\mathcal{J}} :=$ closure of $\mathcal{J}$ under parametric integration

NATURAL CANDIDATE : $\mathcal{D} := \Sigma\, \mathcal{C} \cdot e^{i\mathcal{S}}$

THM 1 (Cluckers-Comte-Miller-Rolin-S., 2017)

Every $f \in \mathcal{D}$ has a $\mathcal{G}$-asymptotic expansion (version 2), where $\mathcal{G} = \left(x^q \cdot (\log x)^m\right)_{m \in \mathbb{N}}^{q \in \mathbb{Q}}$ and the space of coefficients is $A = \left(\{E_k(x) := \sum_{j=1}^{N} c_{j,k} e^{i p_j(x^{1/d})} : N, d \in \mathbb{N},\ c_{j,k} \in \mathbb{C},\ p_j \in \mathbb{C}[z]\}\right)$.

Moreover, such an asymptotic expansion $T(x) = \sum_{k \in \mathbb{N}} E_k(x) \cdot x^{q_k} \cdot (\log x)^{m_k}$ is

UNIQUE and CONVERGENT : the series $f_j(x) := \sum_{k \in \mathbb{N}} c_{j,k} \cdot x^{q_k} \cdot (\log x)^{m_k}$ converge and $f(x) = T(x)$.

COROLLARY. $\hat{\mathcal{J}} \neq \mathcal{D}$.

$Si(x) := \int_0^x \frac{e^{iy} - e^{-iy}}{2iy}\, dy \in \hat{\mathcal{J}} \setminus \mathcal{D}$ because $Si \sim_{+\infty} \Sigma\, c_m \cdot e^{(-1)^m i x} \cdot x^{-m}$, where $|c_m| \sim m!$

$\uparrow +\infty$

(Analogy with $\int_0^{+\infty} \frac{e^{-y}}{x+y}\, dy \notin \mathbb{R}_{an,\exp}$)

OSCILLATORY INTEGRALS WITH SUBANAYTIC PHASE AND AMPLITUDE (WHAT ?)

$\mathcal{S} = \mathcal{D}ef\,(\mathbb{R}_{an})$, $\mathcal{C} = \sum \mathcal{S}\, \Pi \log \mathcal{S}$, $\mathcal{Y} = \mathcal{S} \cup e^{i\mathcal{S}}$, $\mathcal{D} = \sum \mathcal{C} \cdot e^{i\mathcal{S}}$

AIM: describe $\widehat{\mathcal{Y}} :=$ closure of $\mathcal{Y}$ under parametric integration

THM 2 (Cluckers - Comte - Miller - Rolin - S., 2017)

All $f \in \widehat{\mathcal{Y}} \setminus \mathcal{D}$ can be written in terms of 1-dimensional integrals of a simple form:

Let $\Gamma := \left\{ \gamma_{h,s}(\bar{x}) = \int_{\mathbb{R}} h(\bar{x},t) \cdot (\log|t|)^s \cdot e^{it}\, dt \ : \ h \in \mathcal{S},\ h(\bar{x},\cdot) \in L^1(\mathbb{R}),\ s \in \mathbb{N} \right\}$.

Then $\boxed{\widehat{\mathcal{Y}} = \sum \mathcal{D} \cdot \Gamma}$ (module generated by Γ over $\mathcal{D}$).

Moreover, if $f(x) \in \widehat{\mathcal{Y}}$ then $\exists\, g(x) \in \mathcal{D}$ s.t. $\boxed{f(x) = g(x) + o(x^{-1})}$.

COROLLARY

1) $\widehat{\mathcal{Y}}$ is a $\mathbb{C}$-algebra: $\gamma_1 \cdot \gamma_2 \overset{\text{FUBINI}}{=} \sum \iint_{\mathbb{R}^2} \mathcal{C} \cdot e^{i\mathcal{S}}\, dt\, du \overset{\text{THM 2}}{=} \sum \mathcal{D} \cdot \Gamma$

2) Let $f(x) \in \widehat{\mathcal{Y}}$ s.t. $\lim\limits_{x \to +\infty} f(x) = +\infty$. Then $f \underset{\text{THM 2}}{\sim_{+\infty}} g \in \mathcal{D} \underset{\text{THM 1}}{\sim_{+\infty}} E(x) \cdot x^q \cdot (\log x)^s$

bounded

In particular, $e^x \notin \widehat{\mathcal{Y}}$.

PROOF TOOLS (HOW?)

$\mathcal{S} = \mathrm{Def}(\mathbb{R}_{an})$, $\mathcal{C} = \sum \mathcal{S} \prod \log \mathcal{S}$, $\mathcal{F} = \mathcal{S} \cup e^{i\mathcal{S}}$, $\mathcal{D} = \sum \mathcal{C} \cdot e^{i\mathcal{S}}$

<u>THM 1</u> Every $f \in \mathcal{D}$ has a **UNIQUE** and **CONVERGENT** asymptotic expansion

$$T(x) = \sum_{k \in \mathbb{N}} E_k(x) \cdot x^{q_k} \cdot (\log x)^{n_k}, \text{ where } E_k(x) := \sum_{j=1}^{N} c_{j,k} e^{i p_j (x^{1/d})} \quad (q_k \in \mathbb{Q},\ n_k, d \in \mathbb{N},\ p_j \in \mathbb{C}[t])$$

PROOF. $f = \sum g \cdot e^{i\varphi}$ $\quad (g \in \mathcal{C},\ \varphi \in \mathcal{S})$

- EXISTENCE AND CONVERGENCE OF T : O-MINIMAL CELL. DEC. AND PREPARATION OF g AND φ

$x^q \cdot (\log x)^n \in \mathrm{Supp}(T) \Rightarrow q$ is an integer multiple of $\frac{1}{d}$ (Puiseux) $\rightsquigarrow$ no accumulation, and $\forall q\ \exists^{<\infty} n$ s.t. $x^q \cdot (\log x)^n \in \mathrm{Supp}(T)$ $\rightsquigarrow$ no need for super-star expansions

- UNIQUENESS: C.U.D. MOD 1 MAPS (WEYL et al.)

$$\boxed{E(x) = E_k(x) \Rightarrow \exists \varepsilon > 0\ \exists \{x_n\}_{n \in \mathbb{N}} \nearrow +\infty \text{ s.t. } \forall n\ |E(x_n)| > \varepsilon.}$$

Let $g_k = x^{q_k} \cdot (\log x)^{n_k}$. If $f \sim \sum E_k \cdot g_k$ and $f \sim \sum \tilde{E}_k \cdot g_k$, then

$$|E_0(x) - \tilde{E}_0(x)| = \frac{1}{g_0}\left(|E_0 \cdot g_0 - f| + |f - \tilde{E}_0 \cdot g_0|\right) = \frac{1}{g_0} \cdot o(g_0) \xrightarrow[x \to +\infty]{} 0, \text{ but } E := E_0 - \tilde{E}_0 \not\to 0 \text{ !!}$$

So $E_0 = \tilde{E}_0$ and inductively $E_k = \hat{E}_k$ $\boxtimes$

PROOF OF THM 2 $(\hat{\mathcal{F}} = \sum \mathcal{D} \cdot \Gamma)$: Sophisticated version of these same tools.

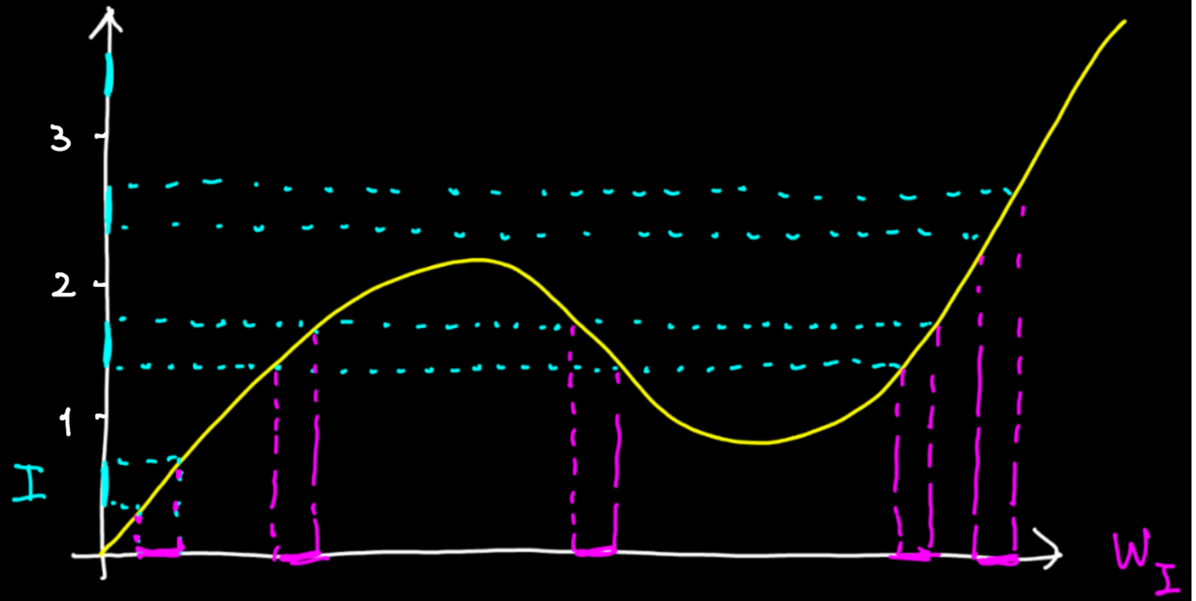
CONTINUOUSLY UNIFORMLY DISTRIBUTED MODULO 1 MAPS (HOW?)

DEF. A map $p=(p_1,\dots,p_m):[0,+\infty)\to\mathbb{R}^m$ is C.U.D. mod 1 if

$\forall$ box $I\subseteq[0,1)^m$, if $W_I=\{x\in\mathbb{R}:\{p(x)\}\in I\}$, then

fractional part

$$\lim_{T\to+\infty}\frac{\mathrm{vol}_1(W_I\cap[0,T])}{T}=\mathrm{vol}_m(I)$$



Remark. p C.U.D. mod 1 $\Rightarrow\{\{p(x)\}:x\in\mathbb{R}\}$ is dense in $[0,1)^m$, but $\not\Leftarrow$

Every $I\subset[0,1)^m$ is filled with the same density

FACT (Weyl) If p is a polynomial map with $\mathbb{Q}$-linearly independent components and no constant terms, then p is C.U.D. mod 1.

(HOW?)

PROPOSITION.

$$E(x) = \sum_{j=1}^{N} c_j e^{i p_j (x^{1/d})} \Rightarrow \exists \varepsilon > 0, \exists \{x_n\} \nearrow +\infty \text{ s.t. } \forall n \ |E(x_n)| > \varepsilon$$

proof.

$E(x) \not\equiv 0 \Rightarrow \exists x_0$ s.t. $|E(x_0)| \geq \varepsilon$. Since $e^{2\pi i \cdot b} = e^{2\pi i \cdot \{b\}}$, we are done, modulo:

Claim. $\exists \{x_n\} \nearrow +\infty$ s.t. $\{p(x_n)\}$ is $\frac{\varepsilon}{2}$-close to $a := \{p(x_0)\}$.

proof. If not, then $\{x : \{p(x)\}$ is $\frac{\varepsilon}{2}$-close to $a\}$ is bounded from above.

Let I be a box around a of radius $\frac{\varepsilon}{2}$ and $W_I = \{x \in \mathbb{R} : \{p(x)\} \in I\}$.

Then $\lim_{T \to +\infty} \frac{\mathrm{vol}_1(W_I \cap [0,T])}{T} = 0$, because W_I is bounded.

But p C.U.D. $\Rightarrow \lim_{T \to +\infty} \frac{\mathrm{vol}_1(W_I \cap [0,T])}{T} = |I| = \varepsilon > 0$, absurd $\boxtimes$

COROLLARY. $e^{-|x|} = \mathrm{Fourier}\left(\frac{1}{1+4\pi^2 x^2}\right) \in \hat{\mathcal{J}} \setminus \mathcal{D}$.

otherwise, by THM 1, $e^{-|x|} = \sum E_k \cdot x^{q_k} \cdot (\log x)^{m_k}$ and, since $e^{-|x|} \not\equiv 0$,

$\exists k_0 \in \mathbb{N}$ s.t. $E_{k_0} \not\equiv 0$ and $e^{-|x|} \underset{+\infty}{\sim} E_{k_0}(x) \cdot x^{q_{k_0}} \cdot (\log x)^{m_{k_0}}$.

But then $E_{k_0}(x) \sim e^{-|x|} \cdot x^{-q_{k_0}} \cdot (\log x)^{-m_{k_0}} \longrightarrow 0$, which contradicts the prop. $\boxtimes$

SUMMING UP

$\mathcal{S} = \mathcal{D}ef\,(\mathbb{R}an)$, $\mathcal{C} = \sum \mathcal{S}\prod \log \mathcal{S}$, $\mathcal{F} = \mathcal{S} \cup e^{i\mathcal{S}}$, $\mathcal{D} = \sum \mathcal{C}\cdot e^{i\mathcal{S}}$

$\hat{\mathcal{F}} :=$ closure of $\mathcal{F}$ under parametric integration $= \sum \mathcal{D}\cdot\Gamma$, where

$$\Gamma := \left\{ \gamma_{h,s}(\bar{x}) = \int_{\mathbb{R}} h(\bar{x},t)\cdot(\log|t|)^s\cdot e^{it}\,dt \;:\; h \in \mathcal{S},\ h(\bar{x},\cdot)\in L^1(\mathbb{R}),\ s\in\mathbb{N} \right\}.$$

Asymptotic expansions

- $\mathcal{D} \ni g(x) = \sum E_k(x)\cdot x^{q_k}\cdot(\log x)^{m_k}$, $E_k(x) = \sum c_j e^{ip_j(x^{1/d})} \not\to 0$
- $\Gamma \ni \gamma$ $\nearrow$ $\gamma \sim_{+\infty} \sum E_k\cdot x^{q_k}(\log x)^{m_k}$ $\uparrow +\infty$
  $\searrow$ γ "flatter" than $x^q\cdot(\log x)^m$ (exponentially? doubly exp?)
- $e^{-|x|} \in \hat{\mathcal{F}}$ but $e^{|x|} \notin \hat{\mathcal{F}}$.

$$\boxed{\text{Let } \mathcal{G} = \mathcal{S} \cup e^{i\mathcal{S}} \cup e^{\mathcal{S}}.\ \text{Who is } \hat{\mathcal{G}}\ ??}$$