Transserial solutions of Abel's equation

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Informal definition.

A **transseries** is a formal series whose monomials involve *exponentials* and *logarithms*.

M. Aschenbrenner, L. van den Dries, J. van der Hoeven (2017):*Asymptotic Differential Algebra and Model theory of transseries,* Ann. Math. Studies (ADH).

Example of transseries

$$\begin{aligned}\varphi(x) = & -3e^{e^x} + e^{\frac{e^x}{\log x} + \frac{e^x}{\log^2 x} + \frac{e^x}{\log^3 x} + \cdots} - x^{11} + 7 \\ & + \frac{\pi}{x} + \frac{1}{x\log x} + \frac{1}{x\log^2 x} + \frac{1}{x\log^3 x} + \cdots \\ & + \frac{2}{x^2} + \frac{6}{x^3} + \frac{24}{x^4} + \frac{120}{x^5} + \frac{720}{x^6} + \cdots \\ & + e^{-x} + 2e^{-x^2} + 3e^{-x^3} + e^{-x^4} + \cdots,\end{aligned}$$

where $x > \mathbb{R}$.

Various constructions of fields of transseries

Construction 1 (S. Kuhlmann, 2000). exponential-logarithmic (EL) series : $\mathbb{S}$
Construction 2 (Ecalle, 1992; van den Dries, Macintyre, Marker, 2001). logarithmic-exponential (LE) series : $\mathbb{T}$
Kuhlmann, Tressl 2012. $\mathbb{T}$ embeds in $\mathbb{S}$, and $\mathbb{S}$ doesn't embed in $\mathbb{T}$.
Hence the two constructions produce non isomorphic models of $\mathrm{Th}\left(\mathbb{R}_{\mathrm{an,exp}}\right)$.
Roughly : with $\ell_n = \log_n x := (\log \circ \cdots \circ \log)(x)$,

$$\sum_{n \geq 0} \ell_n$$

belongs to $\mathbb{S}$ but not to $\mathbb{T}$.
Remark.

1. ADH considers $\mathbb{T}$, as an ordered differential valued field.
2. **Question**. Does $\boldsymbol{\omega} = \frac{1}{\ell_0^2} + \frac{1}{\ell_0^2 \ell_1^2} + \frac{1}{\ell_0^2 \ell_1^2 \ell_2^2} + \cdots$ belong to $\mathbb{S}$?

Example 1, oscillatory integrals.

$$\mathbb{R} \ni x \longmapsto \mathcal{I}(x) = \int_{y\in\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\varphi(y)} f(y)\,\mathrm{d}y,$$

1. *Amplitude* f: $\mathcal{C}^\infty$ with compact support K
2. *Phase* φ: analytic, $\varphi(0) = 0$, 0 unique singularity of φ in $\mathring{K}$.

Asymptotics of $\mathcal{I}(x)$ when $x \to +\infty$, via *resolution of singularities* of φ:

$$\exists r \in \mathbb{N}^*, \qquad \mathcal{I}(x) \underset{x\to+\infty}{\sim} \sum_{p\in\mathbb{N}^*} x^{-p/r} \sum_{k=0}^{n-1} a_{p,k} \log^k x,$$

$a_{p,k} \in \mathbb{R}$.

Transseries in "real life"

Example 2, Computing observables in QFT, via perturbation theory. The observable F, $z \gg 1$, leads to a the "transserial expansion with 1-parameter σ" :

$$\mathcal{F}(z,\sigma) = \sum_{n=0}^{\infty} \frac{F_n}{z^{n+1}} + \sum_{\ell=0}^{\infty} \sigma^{\ell} \mathrm{e}^{-\ell A z} z^{\beta_\ell} \sum_{k=0}^{+\infty} \frac{F_k^{(\ell)}}{z^k} \in \mathbb{C}\left[\left[z^{-1}, \sigma \mathrm{e}^{-Az}\right]\right].$$

Chapter 12 of ADH: Triangular automorphisms: a piece of "iteration theory"

Consider $f(x) = x + a_{m+1}x^{m+1} + \cdots \in x + x^{m+1}\mathbb{R}[[x]]$, $a_{m+1} \neq 0$.
Theorem (Jabotinski, 1947 ; Baker, 1960). For any $t \in \mathbb{R}$, there exists a *unique*:

$$f^{[t]}(x) = x + ta_{m+1}x^{m+1} + \sum_{n=m+2}^{\infty} b_n(t)\, x^n,$$

$b_n(t)$ polynomials of degree at most $n-m$, and $f \circ f^{[t]} = f^{[t]} \circ f$.
Moreover:

1. if $t \in \mathbb{N}$, then $f^{[t]} = f \circ \cdots \circ f$, t times ; in particular, $f^{[1]} = f$ and $f^{[0]} = \mathrm{id}$;
2. $f^{[t]} \circ f^{[s]} = f^{[s]} \circ f^{[t]} = f^{[t+s]}$: f *embeds* in the *flow* $\left\{f^{[t]}, t \in \mathbb{R}\right\}$, f is the *time* 1 of the *flow* $f^{[t]}$.

BONUS. $\left\{f^{[t]}\right\}$ is the flow of a *vector field* $X = h(x)\dfrac{\mathrm{d}}{\mathrm{d}x}$, $h(x) \in x^2\mathbb{R}[[x]]$:

$$\text{for } g \in \mathbb{R}[[x]], \quad X(g) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\left(g \circ f^{[t]}\right).$$

f determines h, and h detemines f.

Chapter 12 of ADH: Triangular automorphisms: a piece of “iteration theory”

$$f \in x + x^2\mathbb{R}[[x]] \longrightarrow \text{Flow } \left\{f^{[t]}\right\}, t \in \mathbb{R} \longrightarrow \text{Vector field } X = h\frac{\mathrm{d}}{\mathrm{d}x}.$$

Notation. $h = \operatorname{itlog}(f)$ (following Ecalle).
Proposition. for $f, g \in x + x\mathbb{R}[[x]]$,

$$\operatorname{itlog}(f \circ g) = \operatorname{itlog}(f) + \operatorname{itlog}(g), \quad \text{hence } \operatorname{itlog}\left(f^{[n]}\right) = n \cdot \operatorname{itlog}(f), \ n \in \mathbb{N}.$$

Exercise. $f(x) = x + x^2$. Compute $f^{[t]}$ and $\operatorname{itlog}(f)$, find $f^{[1/2]}$ such that $f^{[1/2]} \circ f^{[1/2]} = f$.

Exercise. $f(x) = \exp(x) - 1 = x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} \cdots$ or

$f^{-1}(x) = \log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots$. See Chapter 12 in ADH.

Problems of convergence

Question. If the series f has a nonzero radius of convergence, what about $f^{[t]}$ and $h = \text{itlog}(f)$?
Example. $f_a(x) = \dfrac{x}{1+ax}$, $a \in \mathbb{R}$. $f_a^{[t]}(x) = \dfrac{x}{1+tax}$,
$\text{itlog}(f_a)(x) = -ax^2$.
In general (Baker, 1962) : Given a series $f \in x + x^2\mathbb{R}[[x]]$, *the set* $\mathcal{T}$ *of* t *for which* $f^{[t]}$ *has a nonzero radius of convergence* is $\{0\}$, or a lattice generated by one or two generators, or $\mathbb{R}$.
Example. $f(x) = e^x - 1$. $1 \in \mathcal{T}$. $f^{[t]}$ is convergent$\Longleftrightarrow t \in \mathbb{Z}$. No examples with two generators are known.
Theorem (Aschenbrenner, 2015). The power series

$\text{itlog}(e^z - 1) = \dfrac{z^2}{2} - \dfrac{z^3}{12} + \dfrac{z^4}{48} - \dfrac{z^5}{180} + \cdots$ is differentially transcendental over the ring $\mathbb{C}\{z\}$ of convergent series at the origin.
Theorem (Ecalle, 1976). $f \in \mathbb{C}\{z\} \Rightarrow \text{itlog}(f)$ is *Borel-summable.*

A generalization of analytic germs: "Dulac germs"

Now, $x \approx 0$ and $x > 0$.

Definition

1. A ***Dulac series*** is $\hat{f}(x) = \sum_{i=1}^{\infty} x^{\alpha_i} P_i(\log x)$, where P_i are polynomials, and the numbers $\alpha_i > 0$ are either finitely many, or belong to a finitely generated sub-semigroup of $\mathbb{R}_+$ and $\alpha_i \nearrow +\infty$.
2. A ***Dulac germ*** is the germ of an analytic map on an open interval $(0, d)$, extended continuously by $f(0) = 0$, which admits a ***Dulac series asymptotic expansion*** $\hat{f}(x)$ at 0:

$$\forall n \ : \ f(x) - \sum_{i=1}^{n} P_i(\log x)\, x^{\alpha_i} = o\left(x^{\alpha_n}\right).$$

3. f admits an extension on a "big" complex domain, which guarantees ***quasianalyticity*** : if $f \neq x$ then the series $\widehat{f} \neq x$.
4. The germ f is called ***parabolic*** if f is *tangent to identity* : $\hat{f} = x + \cdots$ The parabolic Dulac germs form a group for $\circ$.

The main result : embedding in a flow for Dulac germs

Joint work with P. Mardesic, M. Resman and V. Zupanovic, 2018

Notation: $\boldsymbol{\ell} := -\dfrac{1}{\log x}$, $\boldsymbol{\ell}_2 = \boldsymbol{\ell} \circ \boldsymbol{\ell}, \ldots$

Theorem. Let $f(x) = x - ax^{\alpha}\ell^{k} + o\left(x^{\alpha}\ell^{k}\right)$, $\alpha > 1$, be a parabolic Dulac germ, with $f(0) = 0$. Then f embeds in a *flow* $\left\{f^{[t]}\right\}$ of analytic germs on $(0, d)$, which admit at the origin an asymptotic logarithmic expansion $\widehat{f}^{[t]}(x) = \sum_{\alpha} x^{\alpha} \sum_{k} a_{\alpha,k}(t)\,\boldsymbol{\ell}^{k}$, and $t \mapsto a_{\alpha,k}(t) \in \mathbb{R}$ is $\mathcal{C}^{1}$. Moreover, the supports of the transseries $\widehat{f}^{[t]}$ are contained in a common well-ordered subset of $\mathbb{R} \times \mathbb{Z}$.

$$f^{[1]} = f, \quad f^{[t]} \circ f^{[s]} = f^{[t+s]}, \quad f^{[0]} = \mathrm{id}.$$

Important remark. In each "block" $x^{\alpha}P_{\alpha}(\log x)$ of *Dulac series*, the monomial x^{α} is multiplied by a polynomial $P_{\alpha}(\log x)$. Now, in each "block" of the transseries $\hat{f}^{[t]}(x)$, the monomial x^{α} is multiplied by a (possibly divergent) series $\sum_{k} a_{\alpha,k}(t)\,\boldsymbol{\ell}^{k}$.

Question. What does it mean for a transseries to be the (trans) asymptotic expansion of a germ of function?

Embedding in a flow and Abel's equation

$f : (0, d) \to (0, d)$ embeds in $\left\{ f^{[t]} \right\}$, $f(0) = 0$, unique fixed point in $[0, d)$.
Let $x_0 \in (0, d)$. Two natural systems of coordinates on $(0, d)$:

1. The x-coordinate.
2. The time t s.t. $f^{[t]}(x_0) = x$. We put $\Psi(x) = t$: *Fatou coordinate of* x.

Example. $f(x) = \dfrac{x}{1-x}$, $f^{[t]}(x) = \dfrac{x}{1-tx}$, $\Psi(x) = \dfrac{x - x_0}{x \cdot x_0}$.

Important property. As $f^{[1]} = f$:

$$\text{Abel's equation}: \ \Psi(f(x)) = \Psi(x) + 1, \quad \begin{array}{ccc} x & \xrightarrow{f} & f(x) \\ \downarrow \Psi & & \Psi \downarrow \\ y & \xrightarrow{T_1} & y+1 \end{array}$$

Conversely. From Ψ, $\begin{array}{ccc} x & \xrightarrow{f^{[t]}} & f(x) \\ \downarrow \Psi & & \Psi \downarrow \\ y & \xrightarrow{T_t} & y+t \end{array}$ produces a flow $\left\{ f^{[t]} \right\}$ in which f embeds:

$$f^{[t]}(x) = \Psi^{-1}(\Psi(x) + t).$$

Solving Abel's equation for a Dulac germ

We start from $f(x) = x + x^{\alpha_1} P_1\left(\ell^{-1}\right) + x^{\alpha_2} P_2\left(\ell^{-1}\right) + o\left(x^{\alpha_2}\right)$. We look for Ψ s.t. $\Psi(f(x)) = \Psi(x) + 1$, and, *at the same time*, for $\widehat{\Psi}$ s.t. $\widehat{\Psi}\left(\widehat{f}(x)\right) = \widehat{\Psi}(x) + 1$ (we use Q.A. !).

Search $\widehat{\Psi}(x) = \widehat{\Psi}_1(x) + \widehat{R}_1(x)$, $\widehat{\Psi}_1(x) = x^{\beta_1} \sum_k b_{\beta_1,k} \boldsymbol{\ell}^k$: first block.

$$\widehat{\Psi}\left(\widehat{f}(x)\right) = \widehat{\Psi}(x + \widehat{g}(x)) = \widehat{\Psi}(x) + \widehat{\Psi}'(x)\,\widehat{g}(x) + \cdots = \widehat{\Psi}(x) + 1$$

$$\widehat{g}(x) = x^{\alpha_1} P_1\left(\ell^{-1}\right) + \cdots \Longrightarrow \widehat{\Psi}_1' \cdot x^{\alpha_1} P_1\left(\ell^{-1}\right) = 1 \Longrightarrow \widehat{\Psi}_1'(x) = \frac{1}{x^{\alpha_1} P_1\left(\ell^{-1}\right)}$$

$\widehat{\Psi}_1(x) = \int^x t^{-\alpha_1} \frac{1}{P_1\left(\ell^{-1}\right)} \mathrm{d}t$, *usual* asym. exp. of $\Psi_1(x) = \int_d^x t^{-\alpha_1} \frac{1}{P_1\left(\ell^{-1}\right)} \mathrm{d}t$.

Then we continue with $\widehat{R}_1(x)$, which is solution of $\widehat{R}_1\left(\widehat{f}(x)\right) - \widehat{R}_1(x) = \widehat{\delta}_1(x)$, etc...

Solving Abel's equation for a Dulac germ

We put:

$$\Psi(x) = \Psi_1(x) + \Psi_2(x) + \cdots,$$

solution of Abel's equation $\Psi(f(x)) = \Psi(x) + 1$, with the "block by block transasymptotic expansion" $\widehat{\Psi}(x) = \widehat{\Psi}_1(x) + \widehat{\Psi}_2(x) + \cdots$.

Example. $f(x) = x + x^2 \log x = x - x^2 \boldsymbol{\ell}^{-1}$.

$$\widehat{\Psi}_1(x) = \int \frac{\mathrm{d}x}{x^2 \log x} = x^{-1} \sum_{n=1}^{\infty} n! \boldsymbol{\ell}^n, \text{ and } \Psi_1(x) = \int_d^x \frac{\mathrm{d}t}{t^2 \log t}.$$

$\widehat{R}_1 = \widehat{\Psi}_2 + \widehat{\Psi}_3 + \cdots$ satisfies
$\widehat{R}_1\left(\widehat{f}(x)\right) - \widehat{R}_1(x) = 1 - \left(\widehat{\Psi}_1\left(\widehat{f}(x)\right) - \widehat{\Psi}_1(x)\right) = \widehat{\delta}_1(x)$, while
$\Psi(x) = \Psi_1(x) + R_1(x)$, we have :

$$R_1(f(x)) - R_1(x) = 1 - \int_x^{f(x)} \frac{\mathrm{d}t}{x^2 \boldsymbol{\ell}^{-1}}.$$