Primitive Element Theorem for fields with commuting derivations and automorphisms

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$$\sqrt{2} = \frac{\alpha^3 - 9\alpha}{2} \quad \text{and} \quad \sqrt{3} = \frac{11\alpha - \alpha^3}{2}, \text{ where } \alpha := \sqrt{2} + \sqrt{3}.$$

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Thus, $E = F(\sqrt{2} + \sqrt{3})$.

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Moral: allow derivatives $\implies$ get a primitive element

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$$\alpha := \Gamma(x) + e^x \implies \sigma(\alpha) - e\alpha = \Gamma(x)(x - e)$$
$$\implies \frac{\sigma^2(\alpha) - e\sigma(\alpha)}{\sigma(\alpha) - e\alpha} = \frac{(x+1-e)x}{x-e}.$$

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Moral: allow shifts and derivatives $\implies$ get a primitive element

What do we want?

Prove PETs for field extensions that

- might involve transcendental functions
- but have extra structure
  (derivatives and/or shifts).

Formal setup: one derivation

Definitions

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$$(ab)' = a'b + ab' \text{ for every } a, b \in F.$$
- Let $F \subset E$ be an extension of differential fields. Then $a \in E$ is differentially algebraic over F if it satisfies a polynomial differential equation (DE) over F.
For example, $\sqrt{x}$, e^x, and $\sin x$ are differentially algebraic over $\mathbb{C}$.

State of the art: one derivation

Theorem (Kolchin, 1942)

Let $F \subset E$ be an extension of differential fields such that

- E is generated over F by finitely many elements and their derivatives;
- every $a \in E$ is differentially algebraic over F;
- there is $b \in F$ such that $b' \neq 0$.

Then there exists $\alpha \in E$ such that E is generated over F by α and its derivatives.

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Theorem (P., 2015)
In Kolchin's theorem, $F \to E$.

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- Let $F \subset E$ be an extension of partial differential fields. Then $a \in E$ is partial differentially algebraic over F if it satisfies a polynomial PDE over F.

Theorem (Kolchin, 1942)

Let $F \subset E$ be an extension of partial differential fields such that

- E is finitely generated over F using the derivatives;
- every $a \in E$ is partial differentially algebraic over F;
- the restrictions of the derivations on F are F-linearly independent

Then there exists $\alpha \in E$ such that E is over F by α using the derivatives.

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- Let $F \subset E$ be an extension of difference fields. Then $a \in E$ is difference algebraic over F if the elements of its orbit satisfy an algebraic equation over F
  For example, if σ is a shift $\sigma(f(x)) = f(x+1)$, then $\sqrt{x}$ and $\Gamma(x)$ are difference algebraic over $\mathbb{C}$.

State of the art: one shift

Theorem (Cohn, 1965)

Let $F \subset E$ be an extension of difference fields such that

- E is generated over F by finitely many elements and their orbits;
- every $a \in E$ is difference algebraic over F;
- the automorphism has infinite order on F.

Then there exists $\alpha \in E$ such that $E = F(\sigma^i(\alpha) \mid i \in \mathbb{Z})$.

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Applications of prior results

- Fundamental theoretical tools in differential/difference algebra (e.g., Galois theory of differential and difference equations, model theory of differential/difference fields);
- Representation of solution sets of differential-algebraic equations (Cluzeau-Hubert);
- Effective bounds in for differential/difference equations;
- Used to construct normal forms of systems in control theory (Fliess).

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$\mathbb{C} \subset \mathbb{C}(x, e^x, \Gamma(x), \Gamma'(x), \Gamma''(x) \ldots)$	not covered at all

What exactly do we want

- Remove conditions on F.
  Why?
  - Applicable to $\mathbb{C} \subset \mathbb{C}$(some functions).
  - Applicable to extensions coming from affine varieties equipped with a vector field or an automorphism.
- Allow several automorphisms and derivations.
  Why? Partial difference and difference-differential equations.

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    - they all commute.
- Let $F \subset E$ be an extension of $\Delta\Sigma$-fields. Then $a \in E$ is $\Delta\Sigma$-algebraic if there is an algebraic relation between its images under $\delta_1, \ldots, \delta_s, \sigma_1, \ldots, \sigma_t$.

Main result: statement

Theorem (P., 2019)
Let $F \subset E$ be an extension of $\Delta\Sigma$-fields such that

- E is generated over F by finitely many elements using Δ and Σ;
- every $a \in E$ is $\Delta\Sigma$-algebraic over F;
- $\delta_1, \ldots, \delta_s$ are E-linearly independent on E;
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Remarks

- Generalizes theorems of Kolchin and Cohn.
- All the examples are covered.

What was an issue?

Strategy (Artin, Kolchin, Cohn)

- take a pair of generators a and b of E;
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But $x, \ln x$, and $\ln(1-x)$ are algebraically independent, so $\operatorname{trdeg}_F E = 3$.

What if not a linear combination?

Strategy (P., 2015)

- take a pair of generators a and b of E;
- replace with $a + \sum_{i=1}^{N} \lambda_i b^i$, where N is large enough λ_i's are generic enough from F.

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- take a pair of generators a and b of E;
- replace with $a + f(b)$, where f is a function.

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Special case of the proof

Setup

- just one derivation;
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   where $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_4\}$ are new transcendental constants.

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2. Goal: $b \in L := F(\Lambda, c, c', \ldots)$.
3. c and c' are alg. dependent over $F(\Lambda)$ $\implies$ there are nonzero $v_1, v_2 \in L$ such that

$$\begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} 1 & b & b^2/2 & b^3/3! & b^4/4! \\ 0 & b' & (b^2)'/2 & (b^3)'/3! & (b^4)'/4! \end{pmatrix} \in L^5$$

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- We remove restrictions from the base field. This allows, for example, to consider solutions of autonomous equations.

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