INTRODUCTION TO ω-STABLE THEORIES
TOPICS COURSE (CODE MATH81021)
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What is model theory? It is a branch of math logic that studies mathematical structures by looking at their first-order properties. A principal theme of the subject is to

look at theories with interesting properties and prove
structural theorems about their models.

This can be illustrated by the *existence and uniqueness* of prime models in ω-stable theories (in analogy to algebraic closure in the context of fields). We will view ω-stable theories as an abstraction of classical algebraic geometry where the objects are Zariski closed set (i.e., solutions to systems of polynomials eq's) and more generally Zariski constructible (Boolean combination of Zariski closed).

The main point is: As in algebraic geometry, in any ω-stable theory we can associate to every *definable set* a well behaved notion of "dimension".

Acknowledgements. Most of the material presented here is based on the very nice introduction to model theory given in [1]. For this I thank Dave.

1. Lecture 1 (Oct 12th)

First-order languages. To study groups we consider sets equipped with a distinguished element and binary function

$$(G, e, \cdot)$$

Similarly, to study rings we consider sets with distinguished elements and functions

$$(R, 0, 1, +, -, \cdot)$$

and to study ordered rings we add the relation symbol $<$. The point is that in each of these categories we are fixing a *language*. More generally,

Definition 1.1. A language $\mathcal{L}$ is given by specifying:

(1) A set of constant symbols $\mathcal{C}$
(2) A set of function symbols with a given arity $\mathcal{F} = \{(f, n_f) \in \mathcal{F}\}$
(3) A set of relation symbols with a given arity $\mathcal{R} = \{(R, n_R) \in \mathcal{R}\}$

Example 1.2.

- The language of groups is $\mathcal{L}_{groups} = \{e, \cdot\}$ with $e \in \mathcal{C}$ and $\cdot \in \mathcal{F}$ of arity 2.
- The language of rings is $\mathcal{L}_{rings} = \{0, 1, +, -, \cdot\}$ with $0, 1 \in \mathcal{C}$, $+, -, \cdot \in \mathcal{F}$ all of arity 2.
- The language of ordered rings $\mathcal{L}_{ord} = \mathcal{L}_{rings} \cup \{<\}$ with $<\in \mathcal{R}$ of arity 2.
- The language of differential rings $\mathcal{L}_{\delta} = \mathcal{L}_{rings} \cup \{\delta\}$ with $\delta \in \mathcal{F}$ or arity 1.

Just as groups are given by specifying a set and how e and $\cdot$ are interpreted in that set, we have:

Definition 1.3. Let $\mathcal{L}$ be a first-order language. An $\mathcal{L}$-structure $\mathcal{M}$ is given by a nonempty set M and an *interpretation* for the symbols in $\mathcal{L}$:

(1) An element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$
(2) A function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $(f, n_f) \in \mathcal{F}$
(3) A relation $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $(R, n_R) \in \mathcal{R}$

Example 1.4. In the language of groups, $\mathcal{L}_{groups}$-structures are of the form $\mathcal{G} = (G, e^{\mathcal{G}}, \cdot^{\mathcal{G}})$

(1) $(\mathbb{C}, 0, +)$, here the interpretations are $e^{\mathbb{C}} = 0$ and $\cdot^{\mathbb{C}} = +$ (usual complex number addition).
(2) $(\mathbb{C}, 1, *)$, here the interpretations are $e^{\mathbb{C}} = 1$ and $\cdot^{\mathbb{C}} = \cdot$ (usual complex number multiplication).

Remark 1.5. The first example is an actual group while the second is **not**. In general, $\mathcal{L}_{groups}$-structures will not be groups unless we impose some (first-order) "conditions" (for instance, $\forall x \exists y \; xy = yx = e$).

Just as we study group homomorphisms (or ring homomorphisms), we have

Definition 1.6. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. An $\mathcal{L}$-embedding $\varphi : \mathcal{M} \to \mathcal{N}$ is an injective map $\varphi : M \to N$ that preserves the $\mathcal{L}$-structure; that is

(1) $\varphi(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for all $c \in \mathcal{C}$
(2) $\varphi(f^{\mathcal{M}}(a_1, \ldots, a_{nf})) = f^{\mathcal{N}}(\varphi(a_1), \ldots, \varphi(a_{n_f}))$ for all $f \in \mathcal{F}$ and a_i's in M
(3) $(a_1, \ldots, a_{n_R}) \in R^{\mathcal{M}}$ iff $(\varphi(a_1), \ldots, \varphi(a_{n_R})) \in R^{\mathcal{N}}$ for all $R \in \mathcal{R}$ and a_i's in M

Moreover, if φ is bijective we call it an $\mathcal{L}$-isomorphism (**Note** that in this case $\varphi^{-1} : \mathcal{N} \to \mathcal{M}$ will be an $\mathcal{L}$-embedding).

If $M \subseteq N$ and the inclusion map from M to N is an $\mathcal{L}$-embedding we say that $\mathcal{M}$ is a substructure of $\mathcal{N}$, written $\mathcal{M} \subseteq \mathcal{N}$. Note that in this case we have $c^{\mathcal{M}} = c^{\mathcal{N}}$ for all $c \in \mathcal{C}$, $f^{\mathcal{M}}(a_1, \dots, a_{n_f}) = f^{\mathcal{M}}(a_1, \dots, a_{n_f})$ for all $f \in \mathcal{F}$ and a_i's in M, and $R^{\mathcal{M}} = R^{\mathcal{N}} \cap M^{n_R}$ for all $R \in \mathcal{R}$.

Remark 1.7. In the language of groups, an $\mathcal{L}_{groups}$-substructure of an actual group is generally just a submonoid (since we are not including the inverse function in the language).

First-order formulas. We use the language $\mathcal{L}$ to build formulas that will describe properties of $\mathcal{L}$-structures. For the $\mathcal{L}_{groups}$-structure $(\mathbb{C}^*, 1, *)$ we have that the property

$$\forall x \forall y \; x \cdot y = y \cdot x.$$

In general, $\mathcal{L}$-formulas are string of symbols built from $\mathcal{L}$ and

(1) $=$ (equality)
(2) $x_1, x_2, \dots$ (a list of variables)
(3) $\wedge, \vee, \neg$ (logical connectives)
(4) $\exists, \forall$ (quantifiers)
(5) $(,)$ (parenthesis)

Formally,

Definition 1.8. The set of $\mathcal{L}$-terms $\mathfrak{T}$ is defined inductively as follows

- $c \in \mathfrak{T}$ for all $c \in \mathcal{C}$
- $x_i \in \mathfrak{T}$ for $i = 1, 2, \dots$
- for each $(f, n_f) \in \mathcal{F}$, if $t_1, \dots, t_{n_f} \in \mathfrak{T}$ then $f(t_1, \dots, t_{n_f}) \in \mathfrak{T}$

Example 1.9. In $\mathcal{L}_{rings}$, $\cdot(x_1, -(x_2, 1))$ is an term and we usually write as $x_1 \cdot (x_2 - 1)$. Also, $+(1, 1)$ is a term usually written $1 + 1$.

We can now define

Definition 1.10. An atomic $\mathcal{L}$-formula is a string of symbols of the form

(1) $t_1 = t_2$ where t_i's are terms, or
(2) $R(t_1, \dots, t_{n_R})$ where the t_i's are terms and $R \in mathcalR$

The set of $\mathcal{L}$-formulas is inductively defined as: It contains the atomic formulas and, if ϕ, ψ are $\mathcal{L}$-formulas, then

$$\neg\phi, \quad \phi \wedge \psi, \quad \phi \vee \psi, \quad \exists x_i \phi, \quad \forall x_i \phi$$

are also $\mathcal{L}$-formulas.

For example, in the language of rings

- $(x_1 = 0) \vee (x_1 = 1)$
- $\forall x_1 \, (x_1 = 0) \vee (\exists x_2 \; x1 \cdot x_2 = 1)$

Note that in the first formula the variable x_1 is free (i.e., not bounded by a quantifier), while in the second formula x_1, x_2 are bounded. An $\mathcal{L}$-formula with no free variables is called an $\mathcal{L}$-*sentence*.

We now define what does it mean for an $\mathcal{L}$-formula to hold in an $\mathcal{L}$-structure $\mathcal{M}$. First, for $t(x_1, \dots, x_n)$ an $\mathcal{L}$-term we interpret it in $\mathcal{M}$ as the function $t^{\mathcal{M}} : M^n \to M$ defined as follows:

- if $t = c$, with $c \in \mathcal{C}$, then $t^{\mathcal{M}}$ is the constant function on M^n equal to $c^{\mathcal{M}}$
- if $t = x_i$, then $t^{\mathcal{M}}$ is the i-th coordinate function on M^n
- if $t = f(t_1, \dots, t_{n_f})$ where $(f, n_f) \in \mathcal{F}$ and we have already defined the interpretation for the t_i's, then
$$t^{\mathcal{M}}(a_1, \dots, a_n) = f^{\mathcal{M}}(t_1(a_1, \dots, a_n), \dots, t_{n_f}(a_1, \dots, a_n))$$
for all $(a_1, \dots, a_n) \in M^n$

Example 1.11. In $\mathcal{L}_{rings}$, suppose t is $x_1 \cdot (x_2 - 1)$. In the structure $(\mathbb{C}, +, -, *)$ this term defines the polynomial function $t^{\mathbb{C}} : \mathbb{C}^2 \to \mathbb{C}$ given by

$$(z_1, z_2) \mapsto z_1(z_2 - 1)$$

In fact, in the language of rings, all terms can be interpreted as polynomial functions over the integers.

Definition 1.12. Fix a language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathcal{M}$. Let $\phi(x)$ with $x = (x_1, \dots, x_n)$ be an $\mathcal{L}$-formula and $a \in M^n$. We inductively define $\mathcal{M} \models \phi(a)$, which reads as "$\phi(a)$ holds in $\mathcal{M}$" or "$\mathcal{M}$ satisfies $\phi(a)$",

1. if ϕ is of the form $t_1 = t_2$, then $\mathcal{M} \models \phi(a)$ if $t_1^{\mathcal{M}}(a) = t_2^{\mathcal{M}}(a)$
2. if ϕ is of the form $R(t_1, \dots, t_{n_R})$, then $\mathcal{M} \models \phi(a)$ if
$$(t_1^{\mathcal{M}}(a), \dots, t_{n_R}^{\mathcal{M}}(a)) \in R^{\mathcal{M}}$$
3. if ϕ is of the form $\neg\psi$, and we have defined $\mathcal{M} \models$ for ψ, then $\mathcal{M} \models \phi(a)$ if $\mathcal{M} \not\models \psi(a)$
4. if ϕ is of the form $\psi \wedge \theta$, and we have defined $\mathcal{M} \models$ for ψ and θ, then $\mathcal{M} \models \phi(a)$ if $\mathcal{M} \models \psi(a)$ and $\mathcal{M} \models \theta(a)$. Similarly, for ϕ of the form $\psi \vee \theta$.
5. if ϕ is of the form $\exists y\, \psi(x, y)$ and we have defined $\mathcal{M} \models$ for $\psi(x, y)$, then $\mathcal{M} \models \phi(a)$ if there is $b \in M$ such that $\mathcal{M} \models \psi(a, b)$. Similarly for ϕ of the form $\forall y\, \psi(x, y)$.

Example 1.13. In $\mathcal{L}_{rings}$, consider the formula $\phi(x)$ given by $\exists y\, x = y^2$. Then, in the structure $(\mathbb{R}, 0, 1, +, -, *)$, we have that $\mathbb{R} \models \phi(a)$ iff $a \geq 0$.

Proposition 1.14. *Suppoer $\mathcal{M} \subseteq \mathcal{N}$, $a \in M^n$ and $\phi(x)$ is a quantifier free formula. Then,*

$$\mathcal{M} \models \phi(a) \iff \mathcal{N} \models \phi(a)$$

Idea of the proof. First, one shows (inductively on the complexity on terms) that for each $\mathcal{L}$-term t we have $t^{\mathcal{M}}(a) = t^{\mathcal{N}}(a)$. Then one proceeds by induction on the complexity of the quantifier free formula ϕ (i.e., first consider the case when it is an atomic formula, and then when it has logical connectives). □

Remark 1.15. The above proposition is not generally true if ϕ is not q.f.. For instance, consider the $\mathcal{L}_{groups}$-sentence σ given by $\forall x \forall y\, x \cdot y = y \cdot x$. Now take your favourite nonabelian group G, then

$$Z(G) \models \sigma \quad \text{but} \quad G \not\models \sigma$$

where $Z(G)$ denotes the centre of the group. For another example, consider the $\mathcal{L}_{rings}$-formula $\phi(x)$ given by $\exists y\, x = y^2$ then with their standard ring structures we have

$$\mathbb{R} \not\models \phi(-1) \quad \text{while} \quad \mathbb{C} \models \phi(-1)$$

2. Lecture 2 (Oct 19th)

Definition 2.1. Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are said to be elementarily equivalent, $\mathcal{M} \equiv \mathcal{N}$, if for all $\mathcal{L}$-sentences σ we have

$$\mathcal{M} \models \sigma \iff \mathcal{N} \models \sigma$$

It is easy to check that if $\mathcal{M}$ and $\mathcal{N}$ are isomorphic, $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$. But note that the converse is not generally true (for instance, in $\mathcal{L}_{rings}$, $\mathbb{Q}^{alg} \equiv \mathbb{C}$ but they are of course not isomorphic). In case $\mathcal{M} \cong \mathcal{N}$ via $\psi : M \to N$ we actually have that for all formulas $\phi(x)$, with $x = (x_1, \dots, x_n)$, and $a \in M^n$ we have

$$(*) \mathcal{M} \models \phi(a) \iff \mathcal{N} \models \phi(\psi(a))$$

NOTE: If $\psi : \mathcal{M} \to \mathcal{N}$ is just an $\mathcal{L}$-embedding then $(*)$ does not generally hold. An extension of Proposition 1.14 above shows that it does hold whenever ϕ is quantifier free.

It is of course important to understand the cases when $(*)$ holds. We even give it a name:

Definition 2.2. An $\mathcal{L}$-embedding $\psi : \mathcal{M} \to \mathcal{N}$ is called elementary if $(*)$ holds for all formulas $\phi(x)$ and $a \in M^n$. A substructure $\mathcal{M} \subseteq \mathcal{N}$ is called elementary if the inclusion map from M to N is elementary, that is, if

$$\mathcal{M} \models \phi(a) \iff \mathcal{N} \models \phi(a)$$

for all ϕ and $a \in M^n$. In this case we write $\mathcal{M} \preceq \mathcal{N}$

Clearly, if $\mathcal{M} \preceq \mathcal{N}$ then $\mathcal{M} \equiv \mathcal{N}$. We have the following test for elementary substructures

Theorem 2.3 (Tarski-Vaught test)**.** *Suppose $\mathcal{M} \subseteq \mathcal{N}$. Then, $\mathcal{M} \preceq \mathcal{N}$ iff for every formula $\phi(x, \bar{y})$ and $\bar{a} \in M^n$ we have*

$$\mathcal{N} \models \exists x \, \phi(x, \bar{a}) \iff \mathcal{M} \models \exists \, \phi(x, \bar{a})$$

Sketch of the proof. We check right-to-left. To show $\mathcal{M} \preceq \mathcal{N}$, let $\psi(\bar{y})$ be an arbitrary formula and $\bar{a} \in M^n$ we must show that $\mathcal{M} \models \psi(\bar{a})$ iff $\mathcal{N} \models \psi(\bar{a})$. By Proposition 1.14 we know this holds for q.f. formulas. Thus, it suffices to prove the case when ψ is of the form $\exists x \, \phi(x, \bar{y})$ where the claim holds for $\phi(x, \bar{y})$. We have

$$\mathcal{M} \models \psi(\bar{a}) \iff \mathcal{M} \models \exists x \phi(x, \bar{a}) \iff \mathcal{N} \models \exists x \phi(x, \bar{a}) \iff \mathcal{N} \models \psi(\bar{a})$$

where the second $\iff$ uses our assumption (for the right-to-left direction). $\square$

Theories. Fix a language $\mathcal{L}$. An $\mathcal{L}$-theory T is simply a set of $\mathcal{L}$-sentences. We say that an $\mathcal{L}$-structure $\mathcal{M}$ is a model of T, $\mathcal{M} \models T$ if $\mathcal{M} \models \sigma$ for all $\sigma \in T$.

Definition 2.4. An $\mathcal{L}$-theory T is said to be satisfiable if has model. A class $\mathcal{K}$ of $\mathcal{L}$-structures is elementary (or axiomatizable) if there is an $\mathcal{L}$-theory T such that $\mathcal{K} = \{\mathcal{M} : \mathcal{M} \models T\}$.

Example 2.5.

(1) In $\mathcal{L}_{groups}$, the class of all groups, $GROUPS$, can be axiomatize with the sentence

$$\forall x \forall y \forall z \, (x \cdot (y \cdot z) = (x \cdot y) \cdot z) \wedge (e \cdot x = x \cdot e = x) \wedge (\exists w \, x \cdot w = w \cdot x = e)$$

(2) The class of abelian groups, $AbGROUPS$, can be axiomatized with the axioms of groups together with

$$\forall x \forall y \, x \cdot y = y \cdot x$$

(3) The class of rings and the class of fields can be axiomatized in a natural fashion in the language $\mathcal{L}_{rings}$. Moreover, the class of fields of characteristic zero can be axiomatized with the axioms of fields together the infinite scheme of axioms

$$\{\exists x \, px \neq 0 : p \text{ is prime}\}$$

here px stands for the addition of x p-times. Also, the class of algebraically closed fields can be axiomatized with the scheme

$$\{\forall a_0 \dots \forall a_{n-1} \exists x \, x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0 : n = 1, 2, \dots\}$$

This theory is denoted by ACF and if we specify the characteristic by ACF_p (for $p = 0$ or prime).

(4) In the language of differential rings $\mathcal{L}_\delta = \mathcal{L}_{ringa} \cup \{\delta\}$, the theory DF_0 given by the theory of fields of characteristic zero together with

$$\forall x \forall y (\delta(x + y) = \delta(x) + \delta(y)) \wedge (\delta(x \cdot y) = \delta(x) \cdot y + x \cdot \delta(y))$$

axiomatizes the class of differential fields of characteristic zero.

The compactness theorem. We say that a theory T is *finitely satisfiable* if for every finite $\Sigma \subseteq T$ the theory Σ is satisfiable.

Theorem 2.6 (Compactness Theorem)**.** *Let T be an $\mathcal{L}$-theory. T is satisfiable iff T is finitely satisfiable.*

Proof. We use ultraproducts. We prove right-to-left. We may assume T is infinite. Let $I = \{\Sigma \subset T : \Sigma \text{ is finite}\}$. For each $\sigma \in T$, let $X_\sigma = \{\Sigma \in I : \sigma \in \Sigma\}$. Also, let $D = \{Y \subseteq I : X_\sigma \subseteq Y \text{ for some } \sigma \in T\}$. It is easy to check that D is a filter. Let $\mathcal{U}$ be an ultrafilter extending D. For each $\Sigma \in I$, let $\mathcal{M}_\Sigma$ be a model of Σ (its existence is guaranteed by our assumption).

We finish the proof by showing that the ultraproduct

$$\mathcal{M} := \prod_{\Sigma \in I} M_\Sigma / \mathcal{U}$$

is a model of T. To see this, let $\sigma \in T$. We must show that $\mathcal{M} \models \sigma$. By Łoś's theorem it suffices to check that $\{\Sigma \in I : \mathcal{M}_\Sigma \models \sigma\}$ is in $\mathcal{U}$. But this is clear since

$$X_\sigma \subseteq \{\Sigma \in I : \mathcal{M}_\Sigma \models \sigma\}$$

and so the latter set is in D, which is a subset of $\mathcal{U}$. □

Using Henkin constructions one can prove the following strengthening of the compactness theorem:

Proposition 2.7. *If T is a finitely satisfiable $\mathcal{L}$-theory, then there is $\mathcal{M} \models T$ with $|M| \leq |\mathcal{L}| + \aleph_0$.*

Corollary 2.8. *Let T be a theory with an infinite model and κ a cardinal $\geq |\mathcal{L}| + \aleph_0$. If T is finitely satisfiable, then there is a model of T is size exactly κ.*

Proof. Let $\mathcal{L}^* = \mathcal{L} \cup \{c_\alpha : \alpha < \kappa\}$ with each c_α a constant symbol. Let

$$T^* := T \cup \{c_\alpha \neq c_\beta : \alpha < \beta < \kappa\}$$

We claim that T^* is finitely satisfiable. Let Σ be a finite subset of T^*. Then $\Sigma \subseteq \Sigma_0 \cup \{c_\alpha \neq c_\beta : \alpha < \beta < n\}$ for some finite $\Sigma_0 \subseteq T$ and $n < \omega$. Let $\mathcal{N}$ be an infinite model of T (its existence is part of our assumptions). We can interpret c_α in N for all $\alpha < n$ such that for $\alpha < \beta < n$ the interpretations of c_α and c_β are distinct. Call this $\mathcal{L}^*$-structure $\mathcal{N}^*$. Then, $\mathcal{N}^* \models \Sigma$. Thus, T^* is indeed finitely satisfiable. By the strong form of compactness (Proposition 2.7), there is a model $\mathcal{M}^*$ of T^* of size at most $|\mathcal{L}^*| + \aleph_0 = \kappa$. On the other hand, since $\mathcal{M}^*$ has distinct interpretations for each of the c_α's, it must have size equal to κ. Letting $\mathcal{M}$ be de reduct of $\mathcal{M}^*$ to the language $\mathcal{L}$ we get the desired model of T. □

We now aim towards showing that ACF_0 is a "complete" theory.

Definition 2.9. Let T be an $\mathcal{L}$-theory. An $\mathcal{L}$-sentence σ is a consequence of T if $\mathcal{M} \models \sigma$ for all $\mathcal{M} \models T$. We say that T is a complete theory if for any sentence σ we have that either $T \models \sigma$ or $T \models \neg\sigma$.

Lemma 2.10. *If T is a complete theory and $\mathcal{M}, \mathcal{N}$ are models of T, then $\mathcal{M} \equiv \mathcal{N}$.*

Proof. Easy exercise left to the reader. □

Example 2.11. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. The theory of $\mathcal{M}$ is defined as

$$Th(\mathcal{M}) = \{\sigma : \mathcal{M} \models \sigma\}$$

This is always a complete theory.

Definition 2.12. Let κ be an infinite cardinal. We say that a theory T is κ-categorical if any two models of T of the same size are isomorphic.

Theorem 2.13 (Vaught's test)**.** *Let T be a theory with no finite models that is κ-categorical for some infinite $\kappa \geq |\mathcal{L}|$. Then T is a complete theory.*

Proof. Towards a contradiction assume T is not complete. Then there is σ such that $T \not\models \sigma$ and $T \not\models \neg\sigma$. It follows that the theories $T_1 = T \cup \{\neg\sigma\}$ and $T_2 = T \cup \{\sigma\}$ are both satisfiable. Also, since T has no finite models, T_i has no finite models for $i = 1, 2$. By Corollary 2.8, there are models $\mathcal{M}_i \models T_i$, $i = 1, 2$, such that $|\mathcal{M}_1| = \kappa = |\mathcal{M}_2|$. In particular, $\mathcal{M}_1$ and $\mathcal{M}_2$ are models of T. Now, by κ-categoricity, $\mathcal{M}_1 \cong \mathcal{M}_2$. In particular, $\mathcal{M}_1 \equiv \mathcal{M}_2$, but this is impossible as $M_1 \models \neg\sigma$ while $\mathcal{M}_2 \models \sigma$. □

Corollary 2.14. *The theory* ACF_0 *is complete.*

Proof. By Vaught's test, it suffices to show that ACF_0 is κ-categorical for some infinite κ. Indeed this is true for any uncountable κ since any two algebraically closed fields of characteristic zero are isomorphic if and only if they have the same transcendence degree over $\mathbb{Q}$ (also recall that a field of cardinality κ, for uncountable κ, has transcendence degree κ). □

3. Lecture 3 (Oct 26th)

Definable sets. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. A subset $X \subseteq M^n$ is said to be definable if there is an $\mathcal{L}$-formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and $\bar{b} \in M^m$ such that

$$X = \phi^{\mathcal{M}}(\bar{x}, \bar{b}) := \{\bar{\in} M^n : \mathcal{M} \models \phi(\bar{a}, \bar{b})\}$$

We say that $\phi(\bar{x}, \bar{b})$ defines X and that X is $\bar{b}$-definable. If X does not the need parameters (i.e., $X = \phi^{\mathcal{M}}(\bar{x})$ for an $\mathcal{L}$-formula $\phi(\bar{x})$) we say that X is 0-definable.

Example 3.1.

(1) In $\mathcal{L}_{rings}$, suppose $\mathcal{M}$ is an actual ring, say $(R, 0, 1, +, -, \cdot)$. In this case, $\mathcal{L}$-terms are equivalent to polynomials (over $\mathbb{Z}$). Moreover, it can be checked that subsets of R defined by atomic formulas with parameters are solution sets of polynomial equations over R.

(2) Again in the language of rings, consider the real field $\mathbb{R}$ and the formula $\phi(x)$ given by

$$\exists y\, x = y^2$$

Then $X = \phi^{\mathbb{R}}(x)$ is equal to the nonnegative real numbers. This is an infinite and coinfinite set. Can we find a infinite and coinfinite definable set inside of $\mathbb{C}$? Note that $\phi^{\mathbb{C}}(x) = \mathbb{C}$, and so in this case ϕ is equivalent to the quantifier formula $x = x$. We will see that all definable sets of $\mathbb{C}$ are indeed quantifier free definable (and thus $\mathbb{C}$ can not have a infinite and coinfinite definable subset).

Quantifier elimination. The idea is that once we get rid of quantifiers, definable sets are somewhat easier to understand.

Definition 3.2. A theory T is said to have q.e. (quantifier elimination) if for every $\mathcal{L}$-formula $\phi(\bar{x})$ there is a q.f. formula $\psi(\bar{x})$ such that

$$T \models \forall \bar{x}\, \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$$

We give a few consequences of quantifier elimination.

Definition 3.3. A theory T is said to be model-complete if for every two models $\mathcal{M}, \mathcal{N}$ we have that

$$\mathcal{M} \subseteq \mathcal{N} \implies \mathcal{M} \preceq \mathcal{N}$$

Lemma 3.4. *If T has q.e., then T is model-complete.*

Proof. Let $\mathcal{M}, \mathcal{N}$ be models of T with $\mathcal{M} \subseteq \mathcal{N}$. Let $\phi(\bar{x})$ be a formula and $\bar{a}$ from M, we must show that $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{N} \models \phi(\bar{a})$. Since T has q.e., there is q.f. $\psi(\bar{x})$ such that $T \models \forall \bar{x}\, \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$. This yields

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{M} \models \psi(\bar{a}) \iff \mathcal{N} \models \psi(\bar{a}) \iff \mathcal{N} \models \phi(\bar{a})$$

where the second equivalence uses Proposition 1.14. □

Lemma 3.5. *Let T be a theory with q.e. Suppose there is an $\mathcal{L}$-structure $\mathcal{M}_0$ that embeds into every model of T. Then T is a complete theory.*

Proof. We will use the fact that a theory is complete iff any two models are elementarily equivalent. Now, let $\mathcal{M}, \mathcal{N}$ be models of T and σ an $\mathcal{L}$-sentence. By q.e., there is a q.f. sentence ψ such that $T \models \sigma \leftrightarrow \psi$. We then have

$$\mathcal{M} \models \sigma \iff \mathcal{M} \models \psi \iff \mathcal{M}_0 \models \psi \iff \mathcal{N} \models \psi \iff \mathcal{N} \models \sigma$$

where in the second and third equivalence we used Proposition 1.14 (or its extension rather to structures embeddable in other structures). ☐

We now give a useful test for q.e. The proof can be found in §3.1 of [1].

Theorem 3.6 (Test for q.e.)**.** *Let T be an $\mathcal{L}$-theory. Suppose that for every q.f. $\mathcal{L}$-formula $\phi(\bar{x}, y)$ the following property holds:*

(*) *For any $\mathcal{M}, \mathcal{N}$ models of T with a common substructure A, if there are $\bar{a}$ from A and $b \in M$ such that $\mathcal{M} \models \phi(\bar{a}, b)$, then there is $c \in \mathcal{N}$ such that $\mathcal{N} \models \phi(\bar{a}, c)$.*

Then T has q.e.

We now focus in the theory ACF.

Theorem 3.7. *ACF has q.e.*

Proof. We use the test for q.e. Let $\phi(\bar{x}, y)$ be a q.f. formula and $K, L \models ACF$ with R a common subring. Suppose there is a tuple $\bar{a}$ form R and $b \in K$ such that $K \models \phi(\bar{a}, b)$. We must find $c \in L$ such that $L \models \phi(\bar{a}, c)$. Let $F = Frac(R)$, then F is a common subfield of K and L, and the tuple $\bar{a}$ is from F.

Now, we may assume that ϕ is of the form

$$\vee_i \left(\wedge_j (p_{i,j}(\bar{x}, y) = 0) \wedge (q_i(\bar{x}, y) \neq 0)\right)$$

Moreover, we may assume that there are no disjunctions, and so we have that in K

$$\wedge_j (p_j(\bar{a}, b) = 0) \wedge (q(\bar{a}, b) \neq 0)$$

It suffices to show that there $c \in L$ such that $p_i(\bar{a}, c) = 0$ for all i and $q(\bar{a}, c) \neq 0$. There are two cases to consider

(1) If one of the $p_i(\bar{a}, y)$'s is not the zero polynomial. Then b is algebraic over F. Let h be the minimal polynomial of h over F. Since the solutions of h are generic specializations of b over F, if we let $c \in L$ be any solution of h, then c will have the desired properties (note that such c exists since $L \models ACF$).
(2) If all the $p_i(\bar{a}, x)$'s are zero. Then, we simply need to find $c \in L$ such that $q(\bar{a}, c) \neq 0$. But $q(\bar{a}, y)$ is not the zero polynomial (since $q(\bar{a}, b) \neq 0$), so the equation $q(\bar{a}, y) = 0$ has finitely many solutions, as L is infinite, we can find $c \in L$ that does not solve this equation.

☐

A theory is said to be *strongly minimal* if for every $\mathcal{M} \models T$, every definable $X \subseteq \mathcal{M}$ is either finite or cofinite. We have the following consequences of q.e.

Corollary 3.8.
(1) *ACF is model-complete and strongly minimal.*
(2) *ACF_p is a complete theory for $p = 0$ or prime. (We saw this one before using Vaught's test)*

Proof. (1) is immediate from Lemma 3.4 and q.e. (2) follows from Lemma 3.5 noting that in characteristic p the prime field ($\mathbb{Q}$ or $\mathbb{F}_p$, respectively) embeds into every field. ☐

<u>**Basic notions from classical algebraic geometry.**</u> Fix $\mathcal{U} \models ACF_0$ and $K \leq \mathcal{U}$ (a subfield). For $S \subseteq K[\bar{x}]$, with $\bar{x} = (x_1, \ldots, x_n)$, we let

$$V(S) := \{a \in \mathcal{U}^n : p(a) = 0 \text{ for all } p \in S\}$$

A subset $X \subseteq \mathcal{U}^n$ is said to be *Zariski-closed* (over K) if $X = V(S)$ for some $S \subseteq K[\bar{x}]$. The collection of Zariski-closed sets of $\mathcal{U}^n$ forms the closed sets of a noetherian topology (i.e., every descending chain of closed sets is finite). This fact follows from Hilbert's basis theorem (every ideal of $K[\bar{x}]$ is finitely generated). A (finite) Boolean combination of Zariski-closed sets of $\mathcal{U}^n$ is called a *Zariski-constructible* set. We have the following immediate consequence of q.e. for ACF.

Corollary 3.9. *A set $X \subseteq \mathcal{U}^n$ is definable iff it Zariski-constructible (over $\mathcal{U}$).*

A theory is said to be *strongly minimal* if for every $\mathcal{M} \models T$, every definable $X \subseteq \mathcal{M}$ is either finite or cofinite. An immediate consequence of the above corollary is that ACF_0 is strongly minimal (as polynomials in one variable have finitely solutions).

In the proof of the following proposition we will make use of the following general fact (the proof is an easy exercise left to the reader).

Fact 3.10. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. If $X \subseteq M^n$ is A-definable, then every $\mathcal{L}$-automorphism of $\mathcal{M}$ that fixes A pointwise must fix X setwise.*

Proposition 3.11. *Let $\mathcal{U} \models ACF_0$. If $X \subseteq \mathcal{U}^n$ is a definable set and $f : X \to \mathcal{U}$ is a definable function, then there are X_i's constructible sets and rational functions $f_i : X_i \to \mathcal{U}$ such that $X = \cup_i X_i$ and $f|_{X_i} = f_i$.*

Proof. Let $\mathcal{L}^* = \mathcal{L}_{\mathcal{U}} \cup \{c_1, \ldots, c_n\}$ where the c_i's are new constant symbols. We will write $\bar{c}$ instead of $(c_1, \ldots, c_n)$. Consider the $\mathcal{L}^*$-theory

$$T^* = \{f(\bar{c}) \neq \rho(\bar{c}) : \rho \text{ is a rational function over } \mathcal{U}\} \cup \{\bar{c} \in X\} \cup ACF \cup Diag(\mathcal{U})$$

We claim that T is **not** satisfiable. Towards a contradiction, suppose it is, say witnessed by a model L. Then $L \models ACF \cup Diag(\mathcal{U})$, and so, by model completeness, $\mathcal{U} \preceq L$. This implies that the interpretation of f in L, written f^L, is a definable function from X^L to L. Let $\bar{b}$ be $\bar{c}^L$. Then we have that $f^L(\bar{b}) \neq \rho(\bar{b})$ for any rational function ρ over $\mathcal{U}$. This yields $f^L(\bar{b}) \notin \mathcal{U}(\bar{b})$. Since we are in characteristic zero, there is a field automorphism α of L fixing $\mathcal{U}(\bar{b})$ pointwise such that $\alpha(f^L()\bar{b}) \neq f^L(\bar{b})$. However, the graph of f^L is a $\mathcal{U}$-definable set, thus, by Fact 3.10, α must fix this graph pointwise, and since it also fixes $\bar{b}$ (pointwise) it must fix $f^L(\bar{b})$. We have reached the desired contradiction, and hence T^* is not satisfiable.

By compactness, there is a finite $\Sigma \subseteq T^*$ which is not satisfiable. We have that there are rational functions $\rho_1 \ldots, \rho_m$ over $\mathcal{U}$ such that

$$\Sigma \subseteq \{f(\bar{c}) \neq \rho_1(\bar{c}), \ldots, f(\bar{c}) \neq \rho_m(\bar{c})\} \cup \{\bar{c} \in X\} \cup ACF \cup Diag(\mathcal{U})$$

Now let $\bar{a} \in \mathcal{U}^n$ be an arbitrary element of X. Make $\mathcal{U}$ into an $\mathcal{L}^*$-structure by interpreting $\bar{c}$ as $\bar{a}$. Then $\mathcal{U} \models \{\bar{c} \in X\} \cup ACF \cup Diag(\mathcal{U})$. Since Σ is unsatisfiable, we must have that $\mathcal{U} \models (f(\bar{c}) = \rho_1(\bar{c})) \vee \cdots \vee (f(\bar{c}) = \rho_m(\bar{c}))$. In other words, $f(\bar{a}) = \rho_i(\bar{a})$ for some i. Since $\bar{a}$ was chosen arbitrarily, we have shown that

$$\mathcal{U} \models \forall \bar{x} \in X \; (f(\bar{x}) = \rho_1(\bar{x})) \vee \cdots \vee (f(\bar{x}) = \rho_m(\bar{x}))$$

Letting now $X_i = \{\bar{x} \in X : f(\bar{x}) = \rho_i(\bar{x})\}$ we get the desired definable sets (the desired rational functions are of course $\rho_i, \ldots, \rho_m$). □

Theorem 3.12 (Hilbert's Nullstellensatz)**.** *Let $\mathcal{U} \models ACF_0$ and $K \leq \mathcal{U}$. Suppose $f_1(\bar{x}) = 0, \dots, f_s(\bar{x}) = 0$ is a system of polynomial equations over K such that the ideal $(f_1, \dots, f_s)$ is properly contained in $K[\bar{x}]$. Then the system has a solution in $\mathcal{U}$.*

Proof. Let P be a maximal ideal containing $(f_1, \dots, f_s])$. Let $F = (K[\bar{x}]/P)^{alg}$, and $a_i := x_i \, modP$. Then, in F, we have $f(\bar{a}) = 0$ for all $f \in P$. In particular,

$$F \models \exists \bar{y} \, (f_1(\bar{y}) = 0) \wedge \cdots \wedge (f_s(\bar{y}) = 0)$$

Using q.e. for ACF_0 and the usual transfer trick (up and down from subtructures to extensions) we get that

$$\mathcal{U} \models \exists \bar{y} \, (f_1(\bar{y}) = 0) \wedge \cdots \wedge (f_s(\bar{y}) = 0)$$

as desired. □

We now define the theory DCF_0 (differentially closed fields of characteristic zero). We work in the language of differential rings $\mathcal{L}_\delta = \mathcal{L}_{rings} \cup \{\delta\}$. Recall that DF_0 is the theory of fields of characteristic zero together with

$$\forall x, y \, (\delta(x+y) = \delta(x) + \delta(y)) \wedge (\delta(xy) = \delta(x)y + x\delta(y))$$

So, the models of DF_0 is the class of differential fields of characteristic zero. Let $(K, \delta) \models DF_0$. The ring of differential polynomials in $\bar{x} = (x_1, \dots, x_n)$ is

$$K\{\bar{x}\} := K[x_1, \dots, x_n, \delta x_1, \dots, \delta x_n, \delta^2 x_1, \dots, \delta^2 x_n, \dots]$$

For example, an expression of the form $\delta(x_1) + x_1\delta^3(x_2)$ is a differential polynomial (or δ-polynomial). In this example the order of the δ-polynomial is 3. In general, the order of a δ-polynomial is the largest m such that $\delta^m x_i$ appears (for some i). We have that $(K, \delta) \models DCF_0$ if and only if

for every pair f, g of δ-polynomials over K in one variable with $ord(f) > ord(g)$ and $g \neq 0$, there exists $a \in K$ such that $f(a) = 0$ and $g(a) \neq 0$

Theorem 3.13. *DCF_0 has q.e.*

Proof. The proof is similar to the one for ACF (using the q.e. test). However, there is a subtlety when we talk of the "minimal" δ-polynomial of an δ-algebraic element. In general, differential ideals of $K\{x\}$ are not finitely generated (though radical differential ideals are, by the Ritt-Raudenbush basis theorem). Nonetheless, there is a suitable elimination theory for differential polynomials, the issue is that in general one has to saturate by a "separant" to get "generic" solutions. The details of these ideas are carefully presented in [2, Chap.2, §1 and §2], and we encourage the interested reader to check this reference out. □

It follows from Lemmas 3.4 and 3.5, that DCF_0 is a complete theory (as $(\mathbb{Q}, \delta = 0)$ embeds into every model) that is also model-complete.

Notions from differential algebraic geometry. Suppose $(\mathcal{U}, \delta) \models DCF_0$ and $(K, \delta) \leq (\mathcal{U}, \delta)$ (a differential subfield). For $S \subseteq K\{\bar{x}\}$, we let $V(S) = \{a \in \mathcal{U}^n : p(a) = 0 \text{ for all } p \in S\}$.

Definition 3.14. A set $X \subseteq \mathcal{U}^n$ is said to be Kolchin-closed if $X = V(S)$ for some $S \subseteq K\{\bar{x}\}$.

The collection of Kolchin-closed sets of $\mathcal{U}^n$ forms the closed sets of a noetherian topology (i.e., every descending chain of closed sets is finite). This fact follows from the Ritt-Raudenbush basis theorem (every increasing chain of radical differential ideals of $K\{\bar{x}\}$ is finite). A (finite) Boolean combination of Kolchin-closed sets of $\mathcal{U}^n$ is called a *Kolchin-constructible* set. We have the following immediate consequence of q.e. for DCF_0.

Corollary 3.15. *A set $X \subseteq \mathcal{U}^n$ is definable iff it Kolchin-constructible (over $\mathcal{U}$).*

We also have a differential version of the Nullstellensatz (the proof is similar to the algebraic case):

Theorem 3.16 (Differential Nullstellensatz)**.** *Let $(\mathcal{U}, \delta) \models DCF_0$ and $(K, \delta) \leq (\mathcal{U}, \delta)$. Suppose $f_1(\bar{x}) = 0, \ldots, f_s(\bar{x}) = 0$ is a system of δ-polynomial equations over K such that the differential ideal generated $f_1, \ldots, f_s$ is properly contained in $K\{\bar{x}\}$. Then the system has a solution in $\mathcal{U}$.*

Remark 3.17. DCF_0 is **not** strongly minimal. Indeed, the constants of every model is an infinite and coinfinite set. More precisely, if $(\mathcal{U}, \delta) \models DCF_0$ and $U^\delta = ker(\delta)$ we claim that $\mathcal{U}^\delta$ is infinite and coinfinite. Consider the δ-polynomial $f(x) = \delta x$. For any finite set of points $a_1, \ldots, a_m$ of $\mathcal{U}^\delta$ let $g(x)$ be the polynomial $(x - a_1) \cdots (x - a_m)$. Then by the axioms of DCF_0, $\mathcal{U}$ has an element a such that $f(a) = 0$ and $g(a) \neq 0$. This says that $a \in U^\delta$ but a is not equal to any of the a_i's. Thus, $\mathcal{U}^\delta$ is infinite. On the other hand, for any finite set of points $a_1, \ldots, a_m$ of $\mathcal{U} \setminus \mathcal{U}^\delta$ if we let $p(x) = \delta^2 x$ and $q(x) = \delta x(x - a_1) \cdots (x - a_m)$, then, again by the axioms, $\mathcal{U}$ has an element a such that $p(a) = 0$ and $q(a) \neq 0$. The inequation says that $a \notin \mathcal{U}^\delta$ and is not equal to any of the $a_i's$. Thus, $\mathcal{U}^\delta$ is coinfinite.

4. Lecture 4 (Nov 9th)

Types. Given a collection $\Theta(\bar{x})$ of $\mathcal{L}$-formulas, in variables $\bar{x} = (x_1, \dots, x_n)$, we say that Θ is satisfiable in an $\mathcal{L}$-structure $\mathcal{M}$ if there is $\bar{b} \in M^n$ such that $\mathcal{M} \models \theta(\bar{b})$ for all $\theta \in \Theta$. In this case we say that $\bar{b}$ realizes Θ in $\mathcal{M}$.

The compactness theorem yields the following:

Fact 4.1. *Given an $\mathcal{L}$-theory T, if $\Theta(\bar{x})$ is finitely satisfiable in a model of T, then Θ is satisfiable in a model of T.*

Let A be a (parameter) set of new constants. Let $\mathcal{L}_A = \mathcal{L} \cup A$. Fix a complete satisfiable $\mathcal{L}_A$-theory T.

Definition 4.2.

(1) Let $p(\bar{x})$ be a collection of $\mathcal{L}_A$-formulas in variable $\bar{x} = (x_1, \dots, x_n)$, we say that p is an n-type if p is satisfiable in a model of T (equivalently, by Fact 4.1, finitely satisfiable in a model of T).
(2) We call p a complete type if $\phi \in p$ or $\neg\phi \in p$ for all $\mathcal{L}_A$-formula $\phi(\bar{x})$.
(3) The Stone space $S_n(A)$ (or $S_n^T(A)$) is the set of complete n-types equipped with the topology generated by the basic open sets
$$[\phi] := \{p \in S_n(A) : \phi \in p\}$$
where $\phi(\bar{x})$ is an $\mathcal{L}_A$-formula.

Remark 4.3.

(1) Note that a type p is finitely satisfiable in any model of T. Indeed, let $\mathcal{M} \models T$ and λ a finite subset of p. By taking conjunctions we may assume that λ is a single $\mathcal{L}_A$-formula. By definition, there is $\mathcal{N} \models T$ that satisfies p; in particular, $\mathcal{N} \models \exists\bar{x}\lambda(\bar{x})$. Since T is complete, $\mathcal{M} \equiv \mathcal{N}$, and so $\mathcal{M} \models \exists\bar{x}\lambda(\bar{x})$. Note that is can happen that is not satisfiable in $\mathcal{M}$. In this case we say that $\mathcal{M}$ *omits* p.
(2) Any complete type p is of the form
$$p = tp^{\mathcal{N}}(\bar{b}/A) := \{\phi : \mathcal{N} \models \phi(\bar{b}), \phi \text{ is an } \mathcal{L}_A\text{-formula}\}$$
for some $\mathcal{N} \models T$. Indeed simply take $\mathcal{N}$ any model satisfying the type p. In this case we say that $\bar{b}$ is a realization of p (in $\mathcal{N}$).
(3) Basic open sets $[\phi]$ of $S_n(A)$ are also closed (and so clopen). To see this simply note that $[\phi] = S_n(A) \setminus [\neg\phi]$. This shows that the topology on the Stone space is totally disconnected (i.e., given $p \neq q$ there exists a clopen containing p but not q).
(4) The compactness theorem implies that $S_n(A)$ is a **compact** topological space.

We recall that in a topological space X a point $p \in X$ is isolated if $\{p\}$ is open. It is easy to check that for $p \in S_n(A)$, we have that p is isolated iff $\{p\} = [\phi]$ for some $\mathcal{L}_A$-formula (we say that ϕ isolates p).

Remark 4.4. If $p \in S_n(A)$ is isolated by ϕ, then ϕ 'determines' p in at least two senses:

(i) If $q \in S_n(A)$ and $\phi \in q$, then $q = p$.
(ii) For any $\mathcal{L}_A$-formula ϕ we have that
$$\psi \in p \iff T \models \forall\bar{x}(\phi(\bar{x}) \implies \psi(\bar{x}))$$

Let $p \in S_n(A)$ with $p = tp^{\mathcal{N}}(\bar{b}/A)$ where $\mathcal{N} \models T$. Define $p^{\mathcal{N}}$ to be the set of realizations of p in $\mathcal{N}$. We then have

$$p^{\mathcal{N}} = \{\bar{b} \in \mathcal{N} : \mathcal{N} \models \theta(\bar{b}) \text{ for all } \theta \in p\} = \cap_{\theta \in p} \theta^{\mathcal{N}}$$

where recall that $\theta^{\mathcal{N}}$ is the A-definable set of tuples satisfying θ in $\mathcal{N}$. We see that generally $p^{\mathcal{N}}$ is an intersection of A-definable sets. It is not hard to see that $p^{\mathcal{N}}$ is a definable set iff p is isolated. In this case $p^{\mathcal{N}} = \phi^{\mathcal{N}}$ where ϕ is an $\mathcal{L}_A$-formula isolating p.

Lemma 4.5. *Suppose $p \in S_n(A)$ is isolated. Then p is realized in all models of T.*

Proof. Let ϕ isolate p, and let $\mathcal{M} \models T$. By Remark 4.3(1), $\mathcal{M} \models \phi(\bar{b})$ for some $\bar{b} \in M^n$. By Remark 4.4(ii), $\mathcal{M} \models \psi(\bar{b})$ for all $\psi \in p$. Hence, $\bar{b}$ realizes p. □

The above lemma shows that an isolated type cannot be omitted in a model of T. In a countable language, the converse is true.

Theorem 4.6 (Omitting type theorem)**.** *Suppose $\mathcal{L}_A$ is countable and T is a complete satisfiable $\mathcal{L}_A$-theory. If $p \in S_n(A)$ is nonisolated, then p is omitted in some model of T.*

We will use a Henkin type construction. For this we need some preparation.

Definition 4.7. Let $\mathcal{C}$ be a set of new constant symbols and $\mathcal{L}^* = \mathcal{L} \cup \mathcal{C}$. An $\mathcal{L}^*$-theory T^* has the witness property (w.r.t. $\mathcal{C}$) if for every $\mathcal{L}^*$-formula $\phi(x)$ (in one variable) there is a constant $c \in \mathcal{C}$ such that

$$T^* \models (\exists x \phi(x)) \to \phi(c)$$

Let T^* be a complete satisfiable $\mathcal{L}^*$-theory with the witness property. We now aim to build a model $\mathcal{M}^*$ using the new constant set $\mathcal{C}$. For $c, d \in \mathcal{C}$, set $c \sim d$ iff $T \models c = d$. Then $\sim$ is an equivalence relation. We let the underlying set of our structure be $M^* = \mathcal{C}/\sim$. We now make M^* into an $\mathcal{L}^*$-structure:

(i) **Constants** For $c \in \mathcal{C}$, set $c^{\mathcal{M}} = c/\sim$. For d a constant symbol from the language $\mathcal{L}$, note that (by the witness property) there is $d' \in C$ such that $T^* \models d = d'$, then set $d^{\mathcal{M}} = d'/\sim$.

(ii) **Functions** For f a function symbol and $\bar{b} = (b_1, \ldots, b_n) \in M^*$ with $b_i = c_i/\sim$, we need to specify the value of $f^{\mathcal{M}}(\bar{b})$. By the witness property, there is c such that $T^* \models f(c_1, \ldots, c_n) = c$. Set $f(\bar{b}) = c/\sim$.

(iii) **Relations** For R a relation symbol and $\bar{b} = (b_1, \ldots, b_n) \in M^*$ with $b_i = c_i/\sim$, we set

$$\bar{b} \in R^{\mathcal{M}} \iff T^* \models R(c_1, \ldots, c_n)$$

One can check that all of the above are well defined (independent of representatives). The importance of this construction is given by the following fact (see [1, Chapter 2]).

Fact 4.8. *If $\mathcal{M}^*$ is the $\mathcal{L}^*$-structure built above, then $\mathcal{M}^* \models T^*$. We call $\mathcal{M}^*$ the Henkin model of T^*.*

Proof of the Omitting type theorem. Let $\mathcal{C}$ be an infinite countable set of new constants and set $\mathcal{L}^* = \mathcal{L}_A \cup \mathcal{C}$. We will construct a sequence of $\mathcal{L}^*$-sentences $\theta_0, \theta_1, \ldots$ such that $\models \theta_{s+1} \to \theta_s$ and if we set $T^* = T \cup \{\theta_i : i = 0, 1, \ldots\}$ then T^* will be a complete satisfiable $\mathcal{L}^*$-theory with the witness property. Moreover, the Henkin model of T^* will omit the type p.

Enumerate the $\mathcal{L}^*$-sentences as $\phi_0, \phi_1, \dots$, and the n-tuples from $\mathcal{C}$ as $\bar{b}_1, \bar{b}_2, \dots$. Start by setting θ_0 to be a tautology (e.g. $\forall x \; x = x$). Now assume we have $\theta_0, \theta_1, \dots, \theta_{3i}$. We have three stages to consider

$\underline{3i+1}$ (at this stage we ensure completeness) If $T \cup \{\theta_{3i}, \phi_i\}$ is satisfiable set $\theta_{3i+1} = \theta_{3i} \wedge \phi_i$; otherwise, set $\theta_{3i+1} = \theta_{3i} \wedge \neg\phi$.

3i+2 (at this stage we ensure the witness property) If ϕ_i is of the form $\exists x \psi(x)$ for some $\mathcal{L}^*$-formula and $\theta_{3i+1} \models \phi_i$, then set $\theta_{3i+2} = \theta_{3i+1} \wedge \psi(c)$ where c is any element of $\mathcal{C}$ that does not occur in any of the $\theta_0, \dots, \theta_{3i+1}$. Otherwise (i.e., ϕ_i is not of the form $\exists x \psi(x)$ or $\theta_{3i+1} \not\models \phi_i$), set $\theta_{3i+2} = \theta_{3i+1}$.

3i+1 (at this stage we ensure that we omit p) Let $\bar{b}_i = (e_1, \dots, e_n)$, and let $\psi(\bar{x})$ be the $\mathcal{L}_A$-formula obtained from θ_{3i+2} by replacing e_i with x_i, and replace each $c \in \mathcal{C} \setminus \{e_1, \dots, e_n\}$ with a new variable x_c and add $\exists x_c$ at the front. Since p is nonisolated, there is an $\mathcal{L}_A$-formula $\lambda(\bar{x}) \in p$ such that

$$T \not\models \forall \bar{x} \; (\psi(\bar{x}) \rightarrow \lambda(\bar{x}))$$

Otherwise ψ would isolate p. Set $\theta_{3i+3} = \theta_{3i+2} \wedge \neg\lambda(\bar{b}_i)$.

Now, if se set $T^* := T \cup \{\theta_i : i = 0, 1, \dots\}$, then T^* is a complete satisfiable $\mathcal{L}^*$-theory with the witness property. Let $\mathcal{M}^*$ be the Henkin model of T^*. We finish by arguing that the reduct $\mathcal{M}$ of $\mathcal{M}^*$ to $\mathcal{L}_A$ is model of T omitting p. $\mathcal{M}$ is clearly a model of T. Let $\bar{a}$ be an arbitrary n-tuple from M. Then $\bar{a} = \bar{b}_i^{\mathcal{M}^*}$ for some i. In stage $3i+3$ we guaranteed that $T^* \models \neg\lambda(\bar{b}_i)$ for some $\lambda \in p$. So $\mathcal{M} \models \neg\lambda(\bar{a})$. As $\bar{a}$ was an arbitrary tuple from $\mathcal{M}$, we see that $\mathcal{M}$ cannot realize p. $\square$

Remark 4.9. The OTT (Omitting type theorem) does not generally hold when the language is uncountable. We leave it as an exercise (to the interested reader) to build such a (counter)example.

5. Lecture 5 (Nov 16th)

Prime models. Fix a set of parameters A and let T be a complete satisfiable $\mathcal{L}_A$-theory. A model $\mathcal{M} \models T$ is said to be *prime* if for every $\mathcal{N} \models T$ there is an elementary embedding from $\mathcal{M}$ into $\mathcal{N}$. The model $\mathcal{M}$ is said to be *atomic* if $tp^{\mathcal{M}}(\bar{b}/A)$ is isolated for all $\bar{a}$ from M.

Example 5.1. If $T = ACF_0$, then $\mathbb{Q}^{alg}$ is a prime model. Moreover, it is also atomic. Indeed if for $\bar{a}$, a tuple from $\mathbb{Q}^{alg}$, the type of $\bar{a}$ is isolated by $h_1(x) = 0 \wedge \cdots \wedge h_n(x) = 0$ where h_i is the minimal polynomial of a_i over $\mathbb{Q}$.

Definition 5.2. Given two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$, and $B \subseteq M$. A map $f : B \to N$ is call a pem (partial elementary map) if for all $\mathcal{L}$-formulas $\phi(\bar{x})$ and $\bar{b}$ from B we have that

$$\mathcal{M} \models \phi(\bar{b}) \iff \mathcal{N} \models \phi(f(\bar{b}))$$

Remark 5.3. Suppose T has quantifier elimination, $\mathcal{M},\mathcal{N} \models T$, and B is an $\mathcal{L}$-substructure of $\mathcal{M}$. Then, any $\mathcal{L}$-embedding $f : B \to \mathcal{N}$ is a pem (but not necessarily an elementary map).

Theorem 5.4. *Let $\mathcal{L}_A$ be countable and T a complete $\mathcal{L}_A$-theory with infinite models. Then a model of T is prime iff it is countable and atomic.*

Proof. ($\Rightarrow$) Let $\mathcal{M}$ be a prime model. By the strong form of the compactness theorem (Corollary 2.8), T has a countable model, and so, as $\mathcal{M}$ embeds in that model, $\mathcal{M}$ must be countable. Let $\bar{a}$ a tuple from M and set $p = tp^{\mathcal{M}}(a/A)$. Let $\mathcal{N} \models T$. Since $\mathcal{M} \preceq \mathcal{N}$, we have that $p = tp^{\mathcal{N}}(\bar{a}/A)$, and so p is realized in $\mathcal{N}$. This shows that p cannot be omitted. Hence, by the OTT, it must isolated.

($\Leftarrow$) Let $\mathcal{M}$ be a countable atomic model, and $\mathcal{N} \models T$. We will be an elementary embedding from $\mathcal{M}$ into $\mathcal{N}$. Let $M = \{m_0, m_1, \dots\}$. For each i, let ϕ_i isolate $tp^{\mathcal{M}}(m_0, \dots, m_i/A)$. We now build a sequence of pems $(f_i)_{i<\omega}$ with $Dom(f_i) = \{m_0, \dots, m_{i-1}\}$. Set $f_0 = \emptyset$, this is a pem since $\mathcal{M} \equiv \mathcal{N}$ (recall that T is complete). Suppose we have build $f_0, \dots, f_s$. We have $\mathcal{M} \models \phi_s(m_0, \dots, m_s)$ and so, since f is a pem, $\mathcal{N} \models \phi_s(f_s(m_0), \dots, f_s(m_{s-1}), e)$ for some $e \in N$. Set $f_{s+1} = f_s \cup (m_s \mapsto e)$. Since ϕ_s isolates $tp^{\mathcal{M}}(m_0, \dots, m_s/A)$ we have that

$$tp^{\mathcal{M}}(m_0, \dots, m_s/A) = tp^{\mathcal{N}}(f_{s+1}(m_0), \dots, f_{s+1}(m_s))$$

so f_{s+1} is indeed a pem.

To finish, simply set $f = \cup_{i<\omega} f_i$. Then f is the desired elementary map. □

Remark 5.5. Let $\mathcal{M} \models T$ and $\bar{a}$ from $\mathcal{M}$. If $tp(\bar{a}/A)$ is isolated then the type of any subtuple of $\bar{a}$ is also isolated. For instance, say $tp(a, b/A)$ is isolated by $\phi(x, y)$, then $tp(a/A)$ is isolated by $\exists y \phi(x, y)$. On the other hand, the converse is not generally true; that is, one can find isolated types $tp^{\mathcal{M}}(a/A)$ and $tp^{M}(b/A)$ such that $tp^{M}(a, b/A)$ is not isolated. We leave it as an exercise to build such an example.

Theorem 5.6. *Suppose $\mathcal{L}_A$ is a countable language and T is a complete $\mathcal{L}_A$-theory with infinite models. TFAE*

1. *T has a prime model*
2. *T has an atomic model*
3. *For all n, the isolated types in $S_n(A)$ are dense.*

Proof. As a prime model embeds in any atomic model, we get that the existence of an atomic model implies the existence of a countable one. Thus, by Theorem 5.4, (1) $\Leftrightarrow$ (2).

(2) $\Rightarrow$ (3) Let $\mathcal{M}$ be an atomic model. Let $[\phi]$ be a nonempty basic open set of $S_n(A)$. Then $T \models \exists \bar{x}\phi(\bar{x})$, and so there is $\bar{b}$ in M such that $\mathcal{M} \models \phi(\bar{b})$. This implies that the isolated type $tp^{\mathcal{M}}(\bar{b}/A)$ is in $[\phi]$.

(3) $\Rightarrow$ (2) We build an atomic model using a Henkin construction (as in the OTT). Let $\mathcal{C} = \{c_0, c_1, \dots\}$ be an infinite countable set of new constants and $\mathcal{L}^* = \mathcal{L}_A \cup \mathcal{C}$. We will build $\mathcal{L}^*$-sentences $\theta_0, \theta_1, \dots$ in such a way that the $\mathcal{L}^*$-theory $T^* := T \cup \{\theta_i : i = 0, 1, \dots\}$ is complete satisfiable with the witness property (w.r.t $\mathcal{C}$). Along the way we will ensure that the Henkin model of T^* yields an atomic model of T. As in the proof of the OTT, suppose we have $\theta_0, \dots, \theta_{3i}$, and that at stage $3i+1$ we ensured completeness and at stage $3i+2$ we ensure the witness property. For the next stage we have

<u>**3i+3**</u> (here we ensure atomicity) Let $n \geq i$ such that the constants from $\mathcal{C}$ appearing in θ_{3i+2} come from $\{c_0, \dots, c_n\}$. Let $\psi(\bar{x})$ an $\mathcal{L}$-formula such that $\theta_{3i+2} = \psi(\bar{c})$ where $\bar{c} = (c_1, \dots, c_n)$. From our assumption, namely that isolated types are dense, there is an isolated $p \in [\psi]$. Let λ isolate p, and set $\theta_{3i+3} = \lambda(\bar{c})$

Now let $\mathcal{M}$ be the reduct to $\mathcal{L}$ of the Henkin model of T^*. We claim that $\mathcal{M}$ is atomic. Let $\bar{b}$ be from M. Let n such that $\bar{b} \subseteq \{c_0, \dots, c_n\}$ and this n appears in stage $3i+3$. By Remark 5.5, it suffices to check that $tp^{\mathcal{M}}(\bar{c}/A)$ is isolated. Since $T \models \theta_{3i+3}$, by the shape of θ_{3i+3}, we have that $\mathcal{M} \models \lambda(\bar{c})$ where this λ is the formula isolating p from stage $3i+3$. But then $tp^{\mathcal{M}}(\bar{c}/A) = p$ and the later is isolated. □

We now prove that in a countable language if the number of type is not maximal ($< 2^{\aleph_0}$) then we have prime models.

Theorem 5.7. *Suppose T is a complete satisfiable theory in a countable language $\mathcal{L}_A$. If $S_n(A) < 2^{\aleph_0}$, then the isolated types in $S_n(A)$ are dense.*

Proof. Suppose not. Then there is an $\mathcal{L}_A$-formula ϕ such that $[\phi]$ is nonempty and contains no isolated types. Now, note that there exists an $\mathcal{L}_A$-formula ψ such that $[\phi \wedge \psi]$ and $[\phi \wedge \neg\psi]$ are nonempty and contain no isolated types. Indeed, since $[\phi]$ contains at least two types, say p and q, we can find $\psi \in p \setminus q$. This ψ does the job.

We can continue the above process and build a binary tree (with infinite branches) of $\mathcal{L}_A$-formulas $(\phi_\sigma : \sigma \in \cup_{n<\omega} 2^n)$. Let me just say that to build this tree you define inductively $\phi_{\sigma,1} = \phi_\sigma \wedge \psi$ and $\phi_{\sigma,0} = \phi_\sigma \wedge \neg\psi$ where ψ is an $\mathcal{L}_A$-formula such that $[\phi_\sigma \wedge \psi]$ and $[\phi_\sigma \wedge \neg\psi]$ are nonempty and contain no isolated types. Let $f \in 2^{\mathbb{N}}$ be a branch of the tree (we view this as a map $f : \mathbb{N} \to 2$). We denote by $f|_n$ the truncation of f at n. We have a descending chain

$$[\phi_{f|_0}] \supseteq [\phi_{f|_1}] \supseteq \cdots$$

Since $S_n(A)$ is compact, there is $p_f \in \cap_{n=0}^{\infty} [\phi_{f|_n}]$. It is easy to check that if $f \neq q$ then $p_f \neq p_g$. Thus $f \mapsto p_f$ yields an injective map from $2^{\mathbb{N}} \to S_n(A)$. This contradicts the original assumption. □

Definition 5.8. An $\mathcal{L}$-structure $\mathcal{M}$ is said to be homogeneous if whenever $A \subset M$, with $|A| < |M|$, $f : A \to M$ is a pem and $a \in M$, there is pem $f : A \cup \{a\} \to M$ extending f.

Lemma 5.9. *Let T be a complete satisfiable $\mathcal{L}_A$-theory. If $\mathcal{M}$ is countable atomic model, then $\mathcal{M}$ is homogeneous.*

Proof. Let $\bar{a} \in M$ and $f : \bar{a} \to M$ a pem. Let $c \in M$ and $\phi(\bar{x}, y)$ isolate $tp^{\mathcal{M}}(\bar{a}, c/A)$. Since f is a pem, there is $d \in M$ such that $\mathcal{M} \models \phi(f(\bar{a}), d)$. By choice of ϕ, we get $tp^{\mathcal{M}}(\bar{a}, c/A) = tp^{\mathcal{M}}(f(\bar{a}), d/A)$ and so the map $f^* = f \cup (c \mapsto d)$ is the desired pem. □

Theorem 5.10. *Let T be a complete satisfiable theory in a countable language $\mathcal{L}_A$. If $\mathcal{M}$ and $\mathcal{N}$ are prime models of T, then they are isomorphic.*

Proof. We have seen that $\mathcal{M}$ and $\mathcal{N}$ are atomic. We build an isomorphism $f : \mathcal{M} \to \mathcal{N}$ by a back-and-forth argument. Let $M = \{a_0, a_1, \dots\}$ and $N = \{b_0, b_1, \dots\}$. We build a sequence of pems as follows: Let $f_0 = \emptyset$, then f_0 is a pem as T is complete. Suppose we have built $f_0, f_1, \dots, f_{2i}$ with $Dom(f_2 i) = \{a_0, \dots, a_{i-1}\}$ and $Im(f_{2i}) = \{b_0, \dots, b_{i-1}\}$. In the stage $2i + 1$ we ensure that $a_i \in Dom(f_{2i+1})$. Indeed, let $p = tp^{\mathcal{M}}(\bar{a}, a_i/A)$ where $\bar{a} = (a_0, \dots, a_{i-1})$, as in the proof of Lemma 5.9 we can find $e \in N$ such that $tp^{\mathcal{M}}(\bar{a}, a_i/A) = tp^{\mathcal{N}}(f_{2i}(\bar{a}), e/A)$. Hence we set $f_{2i+1} = f_i \cup (a_i \mapsto e)$, this is a pem. At stage $2i + 2$ we do a similar argument to ensure that $b_i \in Im(f_{2i+2})$. Finally, $f = \cup_{i<\omega} f_i$ is the desired isomorphism. □

6. Lecture 6 (Nov 23rd)

Prime model extensions. We now consider the case when the parameter set A is arbitrary (not necessarily countable). We fix a complete theory T in a countable language $\mathcal{L}$. If $\mathcal{M} \models T$ and $A \subseteq M$, we write $S_n^{\mathcal{M}}(A)$ for the space of complete n-types over A with respect to the theory $Th_A(\mathcal{M})$. When the model $\mathcal{M}$ is understood we omit it from the superscript.

Definition 6.1. Let $\mathcal{M} \models T$ and $A \subseteq M$.

(1) We say that $\mathcal{M}$ is a prime model over A if whenever there is $\mathcal{N} \models T$ and a pem $f : A \to \mathcal{N}$ we can extend f to an elementary embedding from $\mathcal{M}$ into $\mathcal{N}$.
(2) We say that $\mathcal{M}$ is atomic over A is $tp(\bar{b}/A)$ is isolated for every tuple $\bar{b}$ from M.

Example 6.2. Recall that if T has q.e. and $A \subseteq M$ is a substructure, then every $\mathcal{L}$-embedding $f : A \to \mathcal{N}$ is a pem (but not necessarily an elementary map).

(1) Let $T = ACF_0$. For any field F of characteristic zero, the algebraic closure F^{alg} is a prime and atomic model over F.
(2) Let $T = DCF_0$. Let (F, δ) be a differential subfield of $(K, \delta) \models DCF_0$. Is there a differential closure of F? That is, is there $F < F^{diff} \models DCF_0$ such that for every $F < L \models DCF_0$ there is a differential embedding from F^{diff} into L fixing F? This is equivalent to: Is there a prime model over F? If so, is it atomic over F? We will see that the answer is YES to both questions.

Theorem 6.3. *Suppose $\mathcal{M} \models T$ has the property that for every $B \subseteq M$ the isolated types in $S_n^{\mathcal{M}}(B)$ are dense (for all n). Then, for every $A \subseteq M$, there is $\mathcal{M}_0 \preceq \mathcal{M}$ which is a prime and atomic model over A.*

Proof. Consider the following contruction

(1) $A_0 = A$
(2) If no element of $M \setminus A_\alpha$ realizes an isolated type over A_α we stop and set $\delta = \alpha$. Otherwise, let $a_\alpha \in M \setminus A_\alpha$ such that it realizes an isolated type over A_α and set $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$.
(3) If α is a limit ordinal, set $A_\alpha = \cup_{\beta<\alpha} A_\beta$.

One can check that A_δ is a substructure of $\mathcal{M}$. We claim that $\mathcal{M}_0 := A_\delta$ is the desired model. We first show that $\mathcal{M}_0 \preceq \mathcal{M}$. We use Tarski-Vaught test. Suppose $\mathcal{M} \models \exists x\ \phi(x, a)$ with a from $\mathcal{M}_0$. By our assumption, there is an isolated type in $[\exists x\ \phi(x, a)] \subseteq S_n^{\mathcal{M}}(A_\delta)$; that is, there is $b \in M$ such that $\mathcal{M} \models \phi(b, a)$ and $tp^{\mathcal{M}}(b/A_\delta)$ is isolated. By construction of A_δ, $b \in A_\delta = \mathcal{M}_0$, and so $\mathcal{M}_0 \preceq \mathcal{M}$.

We now prove that $\mathcal{M}_0$ is a prime model over A. Let $\mathcal{N} \models T$ and $f : A \to \mathcal{N}$ a pem. We will show that there is a sequence $f = f_0 \subseteq f_1 \subseteq \dot{\subseteq} f_\alpha \subseteq \cdots \subseteq f_\delta$ with each $f_\alpha : A_\alpha \to \mathcal{N}$ a pem. We can then set $f = \cup_{\alpha \leq \delta} f_\alpha$ to get the desire elementary map from $\mathcal{M}_0$ into $\mathcal{N}$. Suppose we have built $f_\alpha : A_\alpha \to \mathcal{N}$. Let $\phi(x, a)$ isolate $tp(a_\alpha/A_\alpha)$. Since f_α is a pem, we can find $b \in N$ such that $\mathcal{N} \models \phi(b, f_\alpha(a))$. If we set $f_{\alpha+1} = f_\alpha \cup (a_\alpha \mapsto b)$, then it is not hard to check that this is s pem. For limit ordinal α, set $f_\alpha = \cup_{\beta<\alpha} f_\beta$.

We leave as an exercise to check that $\mathcal{M}_0$ is atomic over A. Let me just mention that to do this one can use the fact that if $tp(a/B)$ is isolated by $\phi(x, b)$ and $tp(b/C)$ is isolated by $\psi(y, c)$, then $tp(a/C)$ is isolated by $\exists y\ \phi(x, y) \wedge \psi(y, c)$. ☐

Now, how can we guarantee that for every $B \subseteq M$ the isolated types in $S_n(B)$ are dense? That is, under what conditions on T can we guarantee that the assumption of Theorem 6.3 are satisfied?

Definition 6.4. A complete theory T in a countable language is said to be ω-stable if for every $\mathcal{M} \models T$ and countable $A \subseteq M$ we have that $|S_n^{\mathcal{M}}(A)| \leq \aleph_0$ for all n.

Proposition 6.5. *Suppose T is ω-stable. Then, for every $\mathcal{M} \models T$ and $A \subseteq M$, the isolated types in $S_n^{\mathcal{M}}(B)$ are dense.*

Proof. If the isolated types are not dense, then, as in the proof of Theorem 5.7, we get a a binary tree of $\mathcal{L}_A$-formulas $(\phi_\sigma : \sigma \in \cup_{n=0}^{\infty} 2^n)$. Let A_0 be the smallest set containing all the parameters of the formulas in the tree. Then A_0 is countable. Then, again as in the proof of Theorem 5.7, we can build an injective function from $2^{\mathbb{N}}$ into $S_n(A_0)$, implying that $|S_n(A_0)| = 2^{\mathbb{N}}$. But this contradicts ω-stability. □

Putting together the above proposition and theorem we get

Corollary 6.6. *Suppose T is ω-stable. If $\mathcal{M} \models T$ and $A \subseteq M$, then there is a prime and atomic model over A.*

Later on we will discuss about the *uniqueness* of prime models over A.

Saturated models. Let κ be an infinite cardinal. An $\mathcal{L}$-structure $\mathcal{M}$ is κ-saturated if for any $A \subseteq M$ with $|A| < \kappa$ all types over A (w.r.t. $Th_A(\mathcal{M})$) are realized in $\mathcal{M}$. We say that $\mathcal{M}$ is saturated if it is $|M|$-saturated.

Fact 6.7. *Suppose $\mathcal{M}$ is saturated and $A \subset M$ with $|A| < |M|$.*

(1) *If $f : A \to \mathcal{M}$ is a pem, then there is an extension of f to an automorphism of $\mathcal{M}$.*
(2) *$\bar{a}, \bar{b} \in M$ have the same type over A iff there is $\sigma \in Aut(\mathcal{M}/A)$ such that $\sigma(\bar{a}) = \bar{b}$.*
(3) *A definable set X is A-definable iff for every $\sigma \in Aut(\mathcal{M}/A)$ we have $\sigma(X) = X$ setwise.*
(4) *For any $\mathcal{N} \equiv \mathcal{M}$ with $|N| \leq |M|$, there is an elementary embedding from $\mathcal{N}$ into $\mathcal{M}$.*

For a proof of the following see [1, Chap. 6.5].

Theorem 6.8. *Let T be an ω-stable theory. Then, for any infinite cardinal κ, there is a saturated model of T of size κ.*

Notation and assumptions. From now on we work with a ω-stable theory T. We fix a *large* saturated $\mathcal{U} \models T$ (we will sometimes refer to $\mathcal{U}$ as a monster model). We make the following assumptions:

- All models we consider are elementary substructures of $\mathcal{U}$.
- All parameter sets A come from $\mathcal{U}$.
- All types over A are with respect to the theory $Th_A(\mathcal{U})$
- Definable means definable with parameters (from $\mathcal{U}$) and formula mean $\mathcal{L}_{\mathcal{U}}$-formula.

Morley rank. Let X be a definable set we define $RM(X)$ inductively on ordinals as follows:

(1) $RM(X) \geq 0$ iff X is nonempty.

(2) $RM(X) \geq \alpha + 1$ iff there pairwise disjoint definable sets $X_1, X_2, \dots \subset X$ such that $RM(X_i) \geq \alpha$.
(3) For limit ordinal α, $RM(X) \geq \alpha$ iff $RM(X) \geq \beta$ for all $\beta < \alpha$.

We say $RM(X) = \alpha$ if $RM(X) \geq \alpha$ but $RM(X) \not\geq \alpha + 1$. We set $RM(X) = -1$ when X is empty. If $RM(X) \geq \alpha$ for all ordinal α we write $RM(X) = \infty$ and say that X has unbounded Morley rank. If $RM(X) = \alpha$, the *Morley degree* of X, $\deg(X)$, is the largest natural number d such that X contains d-many pairwise disjoint definable sets each of rank α.

The following are left as exercises:

Lemma 6.9. *Let X and Y be definable sets*

(1) $RM(X) = 0$ *iff X is finite*
(2) *If $X \subseteq Y$, then $RM(X) \leq RM(Y)$*
(3) $RM(X \cup Y) = \max\{RM(X), RM(Y)\}$
(4) *If $f : X \to Y$ is a definable bijection, then $RM(X) = RM(Y)$*

Theorem 6.10. *If T is ω-stable, then $RM(X)$ is bounded for all definable X.*

Proof. Suppose not. Then there is X with $RM(X) = \infty$. Let

$$\beta = \sup\{RM(Z) : Z \text{ is definable with } RM(Z) < \infty\}$$

Since $RM(X) \geq \beta + 2$, we can find a definable subset Y of X with $RM(Y) \geq \beta + 1$ and $RM(X \setminus Y) \geq \beta + 1$. Hence, $RM(Y) = RM(X \setminus Y) = \infty$. As we have done a couple of times now, we can iterate this process to build a binary tree of $\mathcal{L}_{\mathcal{U}}$-formulas $(\phi_\sigma : \sigma \in \cup_{n=0}^{\infty} 2^n)$. Letting A be a countable set containing all the parameters of the formulas in the tree, we get an injective function from $2^{\mathbb{N}}$ into $S_n(A)$. This contradicts ω-stability. □

7. Lecture 7 (Nov 30th)

Rank of types. For $p \in S_n(A)$ we define

$$RM(p) = \inf\{RM(\phi) : \phi \in p\}.$$

and

$$\deg(p) = \inf\{\deg(\phi) : \phi \in p, RM(\phi) = RM(p)\}$$

For a tuple $\bar{a}$, we write $RM(\bar{a}/A)$ to mean $RM(tp(\bar{a}/A))$. Note that since any set of ordinals is well ordered, for any $p \in S_n(A)$ there is $\phi_p \in p$ such that $RM(p) = RM(\phi_p)$ and $\deg(p) = \deg(\phi)$. We call such ϕ_p a minimal formula for p.

Lemma 7.1.

(1) *If X is A-definable, then $RM(X) = \sup\{RM(a/A) : a \in X\}$.*

(2) *Let $p \in S_n(A)$. For any $\mathcal{L}_A$-formula ψ we have that*

$$\psi \in p \iff RM(\psi \wedge \phi_p) = RM(p)$$

Consequently, if $p \neq q \in S_n(A)$, then $RM(\phi_p \wedge \phi_q) < \max\{RM(p), RM(q)\}$.

(3) *For any $\mathcal{L}_A$-formula ϕ we have*

$$\deg(\phi) \geq |\{p \in S_n(A) : \phi \in p \text{ and } RM(p) = RM(\phi)\}|$$

Proof. (1) For any $a \in X$ we have

$$RM(a/A) = \inf\{RM(Y) : a \in Y \text{ for } A\text{-definable } Y\} \leq RM(X)$$

On the other hand, consider the collection of $\mathcal{L}_A$-formulas $\Theta(x)$ given by

$$(x \in X) \wedge (x \notin Y : Y \text{ is } A\text{-definable with } RM(Y) < \alpha)$$

We claim this is satisfiable. Take a finite subset $(x \in X) \wedge (x \notin Y_1) \wedge \cdots \wedge (x \notin Y_s)$ Since $RM(Y_i) < \alpha$, we get $RM(Y_1 \cup \cdots \cup Y_s) < \alpha)$. So $\cup_{i=1}^{s} Y_i \cap X$ is properly contained in X, and so we can find a realization. Hence, there is a realizing $\Theta(x)$. By choice of a, we have $a \in X$ and $RM(a/A) = RM(X)$. This proves the desired equality.

(2) First, suppose $\psi \in p$. Then $\psi \wedge \phi_p \in p$ and so by minimality of ϕ_p we get

$$RM(\psi \wedge \phi_p) = RM(\phi_p) = RM(p).$$

On the other hand, suppose $RM(\psi \wedge \phi_p) = RM(\phi_p)$ but $\psi \notin p$. Then $\neg\psi \in p$, by the above we get that $RM(\neg\psi \wedge \phi_p) = RM(\phi_p)$. But this implies that $\deg(\neq \psi \wedge \phi_p) < \deg(\phi_p)$ contradicting minimality of ϕ_p.

For the 'consequently' clause, let $\psi \in q \setminus p$, then

$$RM(\phi_p \wedge \phi_q) = RM((\phi_p \wedge \phi_q \wedge \psi) \vee (\phi_p \wedge \phi_q \wedge \neg\psi)) = \max\{RM(\phi_p \wedge \phi_q \wedge \psi), RM(\phi_p \wedge \phi_q \wedge \neg\psi)\}$$

But $\phi_q \wedge \psi \notin p$ and so, by the above, $RM(\phi_p \wedge \phi_q \wedge \psi) < RM(p)$. Similarly, $\phi_p \wedge \neg\psi \notin q$ and so $RM(\phi_p \wedge \phi_q \wedge \neg\psi) < RM(q)$. The desired inequality follows.

(3) Let $d = \deg(\phi)$. Suppose we have distinct $p_1, \ldots, p_{d+1} \in S_n(A)$ such that $\phi \in p_i$ and $RM(p_i) = RM(\phi)$. Let ϕ_i be a minimal formula for p_i. From the consequently in (2), we see that $RM(\phi_i \wedge \phi_j) < RM(\phi)$. From these, we can build $(d+1)$-many pairwise disjoint definable subsets of $\phi^{\mathcal{U}}$ all of Morley rank $RM(\phi)$. This yields $\deg(\phi) > d$ which is of course a contradiction. □

In the context of ACF_0 Morley rank is a well known rank. Suppose $K \models ACF_0$ and $V \subseteq K^n$ is a Zariski closed set. Then $RM(V) = \dim V$ and the Morley degree of V equals the number of (absolutely) irreducible components of V of top dimension. Also, for a subfield k we have $RM(a/k) = trdeg_k k(a)$.

Now, recall that in (classical) algebraic geometry one has a well behaved notion of independence: a is independent from b over k if $trdeg_k k(a) = trdeg_{k(b)} k(b)(a)$ (i.e., the tuples a and b and algebraically disjoint over k). We now aim to define a notion of independence in an arbitrary ω-stable theory. We will mimic the notion from algebraic geometry using RM instead of $trdeg$.

Forking and Independence. We continue to work with an ω-stable theory T and $\mathcal{U}$ our monster model.

Definition 7.2. Let $A \subset B$, $p \in S_n(A)$ and $q \in S_n(B)$ with $p \subseteq q$. We say that q is a nonforking extension (nfe) of p if RM(q)=RM(p); otherwise, we say that q is a forking extension or that q forks over A.

The idea of nonforking is that B does not add much more information than what we had from A, or that q is as free as an extension of p can be.

We now show that nonforking extensions always exist.

Proposition 7.3. *(Existence of nonforking extensions) Suppose $p \in S_n(A)$ and $A \subseteq B$.*

(1) *There is a nonforking extension $q \in S_n(B)$ of p*
(2) *There at most* $\deg p$ *nfe's of p to B*

Proof. (1) Let $\alpha := RM(p)$. Let ψ be an $\mathcal{L}_U$-formula such that $\psi^{\mathcal{U}} \subset \phi_p^{\mathcal{U}}$, $RM(\psi) = \alpha$ and $\deg \psi = 1$. We claim that

$$q := \{\theta(x) : \theta \text{ is an } \mathcal{L}_B\text{-formula and } RM(\theta \wedge \psi) = \alpha\} \tag{7.1}$$

does the job. First, q is a type: take a finite subset $\theta_1, \dots, \theta_s$, by induction and the fact that $\deg \psi = 1$, one can check that $RM(\theta_1 \wedge \cdots \wedge \theta_s \wedge \psi) = \alpha \geq 0$, thus we can realize $\theta_1 \wedge \cdots \wedge \theta_s$. Also, q is a complete type: for any $\mathcal{L}_B$-formula λ either $RM(\psi \wedge \lambda) = \alpha$ or $RM(\psi \wedge \neg\lambda) = \alpha$. It is an extension of p: let $\lambda \in p$, if $\lambda \notin q$ then $\neg\lambda \in q$ and so $\alpha = RM(\psi \wedge \neg\lambda) \leq RM(\phi_p \wedge \neg\lambda)$ which implies that $\neg\lambda \in p$ (by Lemma 7.1(2)) which is impossible. Finally, for any $\theta \in q$, we have $RM(\theta \wedge \psi) = \alpha$ and so $RM(\theta) = \alpha$. This shows that $RM(q) = \alpha$ and so q is a nfe of p.

(2) Since ϕ_p is an $\mathcal{L}_B$-formula we have by Lemma 7.1(3)

$$\deg(\phi_p) \geq |\{q \in S_n(B) : \phi \in q \text{ and } RM(q) = RM(\phi_p)\}|$$

All nfe's of p to B are accounted for in the latter set. So there are at most $\deg p$ of them. □

Remark 7.4.

(1) Note that if $\deg p = 1$ then p has a unique forking extension q to any B and $\phi_q = \phi_p$.
(2) We leave as an exercise to check that any forking extension of p has the form (7.1) for some $\mathcal{L}_{\mathcal{U}}$-formula ψ with $\psi^{\mathcal{U}} \subseteq \phi^{\mathcal{U}}$, $RM(\psi) = RM(\phi_p)$ and $\deg \psi = 1$. In particular, if ψ is an $\mathcal{L}_B$-formula and q is the associated (by (7.1)) nfe of p to B then $\phi_q = \psi$. It follows that if B contains the parameters of $\deg p$ many definable disjoint subsets of $\phi_p^{\mathcal{U}}$ of rank $RM(p)$ then p has exactly $\deg p$ nfe's to B.

Definition 7.5 (Definability of types)**.** We say that $p \in S_n(A)$ is definable over B if for each $\mathcal{L}$-formula $\psi(x,y)$ there is an $\mathcal{L}_B$-formula $d_p\psi(y)$ such that for all $a \in A$

$$\psi(x,a) \in p \iff \mathcal{U} \models d_p\psi(a)$$

The following is the key to prove definability of types in ω-stable theories. The proof can be found in [1, Chapter 6.3].

Theorem 7.6. *Let $\phi(x)$ be an $\mathcal{L}_A$-formula. Then, for every $\mathcal{L}$-formula ψ, we have that the set*

$$\{b \in \mathcal{U} : RM(\phi(x) \wedge \psi(x,b)) = RM(\phi(x))\}$$

is A-definable. Call the $\mathcal{L}_A$-formula $d_\phi\psi$.

Corollary 7.7. *If $p \in S_n(A)$ then there is a finite $A_0 \subseteq A$ such that p is definable over A_0.*

Proof. Let A_0 be the parameters of ϕ_p. Given any $\mathcal{L}$-formula ψ we have by Lemma 7.1(2) that for any a from A

$$\psi(x,a) \in p \iff RM(\phi_p(x) \wedge \psi(x,a)) = RM(\phi_p)$$

The latter is equivalent to $\mathcal{U} \models d_{\phi_p}\psi(a)$ by Theorem 7.6. □

Definition 7.8. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. An A-definable set is said to be stably embedded if every definable set $Y \subseteq X^n$ is $(A \cup X)$-definable.

Corollary 7.9. *If $X \subseteq \mathcal{U}^m$ is A-definable, then it is stably embedded.*

Proof. Let $Y \subseteq X^n$ be definable by $\psi(x,b)$ where ψ is an $\mathcal{L}$-formula. Then $c \in Y$ iff $\mathcal{U} \models \psi(c,b)$ iff $c \in X^n$ and $\psi(c,y) \in tp(b/X)$ iff $c \in X^n$ and $\mathcal{U} \models d_p\psi(c)$ where $p = tp(b/X)$. The condition $c \in X^n$ is A-definable while the $d_p\psi$ is X-definable. □

Note that if $\deg p = 1$ then, by the proof of Proposition 7.3(1), the unique nonforking extension of p to B is given by

$$q = \{\psi(x,b) : b \in B,\ RM(\phi_p(x) \wedge \psi(x,b)) = RM(\phi_p)\}$$

But this set is equal to

$$\{\psi(x,b) : b \in B,\ \mathcal{U} \models d_p\psi(b)\}$$

We thus have

Lemma 7.10. *Suppose $\deg p = 1$, then the unique nfe of p to B is given by*

$$q = \{\psi(x,b) : b \in B,\ \mathcal{U} \models d_p\psi(b)\}$$

In particular, for any $\psi(x,y)$, $d_p\psi = d_q\psi$ and so q is definable over A.

Independence. Let $A \subseteq B$. We say that a is independent from B over A if $RM(a/A) = RM(a/B)$. We write this as $a \mathop{\smile\!\!\!\!\!\!|}_A B$.

Lemma 7.11. *Independence has the following properties:*

1. *(Automorphism invariant) For all $\sigma \in Aut(\mathcal{U})$, if $a \mathop{\smile\!\!\!\!\!\!|}_A B$ then $\sigma(a) \mathop{\smile\!\!\!\!\!\!|}_{\sigma(A)} \sigma(B)$.*
2. *(Transitivity) Suppose $B \subseteq C$. Then $a \mathop{\smile\!\!\!\!\!\!|}_A C$ iff $a \mathop{\smile\!\!\!\!\!\!|}_A B$ and $a \mathop{\smile\!\!\!\!\!\!|}_B C$*
3. *(finite character) $a \mathop{\smile\!\!\!\!\!\!|}_A B$ iff $a \mathop{\smile\!\!\!\!\!\!|}_A B_0 \cup A$ for all finite subsets B_0 of B*
4. *(finite basis) for any a there is a finite $A_0 \subseteq A$ such that $a \mathop{\smile\!\!\!\!\!\!|}_{A_0} A$*
5. *(existence of nfe's) for any a there is b with $tp(a/A) = tp(b/a)$ and such that $b \mathop{\smile\!\!\!\!\!\!|}_A B$*

(6) *(symmetry)* $a \mathop{\smile\!\!\!\!\!\!\;|}_A b$ *iff* $b \mathop{\smile\!\!\!\!\!\!\;|}_A a$

We leave the proofs of (1) to (5) as exercises. The proof of (6) requires more work, we refer the reader to [1, Chapter 4.3]. Let us mention that if a theory has a notion of independence satisfying the above six properties and such that types have finitely many nonforking extensions, then the theory is ω-stable.

8. Lecture 8 (Dec 7th)

In this lecture we will show that in a ω.stable theory any two prime models over A are are isomorphic over A. We fix a complete satisfiable theory T in a countable language.

Constructible models. Let $\mathcal{M} \models T$ and $A \subset M$. Let $(a_\alpha : \alpha < \delta)$ be a sequence of elements of M where δ is an ordinal. Set $A_\alpha = (a_\beta : \beta < \alpha)$. We say that $(a_\alpha : \alpha < \delta)$ is a *construction over* A if $a_\alpha \notin A \cup A_\alpha$ and $tp(a_\alpha/A\alpha)$ is isolated for all $\alpha < \delta$. We say that $B \subseteq M$ is *constructible over* A is there is a construction $(a_\alpha : \alpha < \delta)$ over A such that $B = A \cup (a_\alpha : \alpha < \delta)$.

Recall that to prove the existence of prime model extensions in ω-stable theories we did the following:

1. Given $\mathcal{M} \models T$ and $A \subset M$, we built a constructible over A elementary substructure $\mathcal{M}_0$ of $\mathcal{M}$.
2. We saw that any model which is constructible over A is prime and atomic over A.

We will prove that any two constructible over A models of T are isomorphic over A. Then, we will see that in a ω-stable theory prime models are always constructible. First we need the notion of *sufficient*.

Definition 8.1. Suppose $(a_\alpha : \alpha < \delta) \subseteq M$ is a construction over A and let $\theta_\alpha(x)$ be an $\mathcal{L}_{A_\alpha}$ that isolates $tp(a_\alpha/A_\alpha)$. A subset C of M is called *sufficient* if whenever $a_\alpha \in C$ then all the parameters of $\theta_\alpha \in C$.

Lemma 8.2. *Suppose $M \models T$ is constructible over $A \subset M$.*

1. *If $\bar{b}$ is a finite subset from M, then there is finite sufficient subset $\bar{c}$ containing $\bar{b}$.*
2. *If C is sufficient, then M is constructible over $A \cup C$.*

Proof. We only check (1). Let $(a_\alpha : \alpha < \delta)$ be construction of M over A. We prove, by induction on α, that if $\bar{b}$ is in A_α then there is a finite sufficient $\bar{c}$ in A_α containing $\bar{b}$.

If $\alpha = 0$, then $\bar{b}$ is in A and so $\bar{b}$ is of course sufficient. If α is a limit ordinal, then $\bar{b}$ is in A_β for some $\beta < \alpha$ and so, by induction, there is finite sufficient $\bar{c}$ in $A_\beta \subset A_\alpha$ containing $\bar{b}$. So now assume $\alpha = \beta + 1$. If $\bar{b}$ is in A_β we are done by induction, so we may asumme that $\bar{b}$ is of the form $\bar{b}_0 \cup \{a_\beta\}$. Let $\bar{c}'$ be the parameters of θ_β. Then $\bar{b}_0 \cup \bar{c}'$ is in A_β, and so, by induction, there is a finite sufficient $\bar{c}$ in A_β containing $\bar{b}_0 \cup \bar{c}'$. But then $\bar{c} \cup \{a_\beta\}$ is a finite sufficient in A_α containing $\bar{b}$. □

We will make use of the following fact whose proof we leave as an exercise.

Fact 8.3. *For tuples a and b, the type $tp(a, b/A)$ is isolated iff $tp(aA)$ and $tp(b/A, b)$ are both isolated.*

Theorem 8.4 (Ressayre's theorem)**.** *Let $\mathcal{M}, \mathcal{N}$ be elementary substructures of a monster model $\mathcal{U}$ of T. Let A be a common subset of M and N. If $\mathcal{M}$ and $\mathcal{N}$ are constructible over A, then they are isomorphic over A.*

Proof. Consider the set

$$\mathcal{I} = \{f : X \to N : A \subseteq X \subseteq M, f \text{ is a pem over } A, X \text{ is sufficient in } \mathcal{M} \text{ and } f(X) \text{ is sufficient in } \mathcal{N}\}$$

Note that Id_A is in $\mathcal{I}$. Moreover, given increasing chain of elements of $\mathcal{I}$, its union is again in $\mathcal{I}$. Thus, by Zorn's lemma, $\mathcal{I}$ has a maximal element $f : X \to N$ (with respect to inclusion). We claim that f is the desired isomorphism. So we must prove that $X = M$ and $f(X) = N$. Towards a contradiction assume there is $a \in M \setminus X$ (we only consider this case since the case $N \setminus f(X) \neq \emptyset$ can be treated with a similar argument as what follows). Let $\bar{c}_0$ be finite sufficient of M containing a. By Lemma 8.2(2), M is constructible over X and so $tp(\bar{c}_0/X)$ is isolated. Thus, we can extend f to a pem $f_0 : X \cup \bar{c} \to N$. Note that, while $X \cup \bar{c}_0$ is sufficient, $Im(f_0) = f_(X) \cup f_0(\bar{c}_0)$ might not be sufficient. So now let δ_0 be finite sufficient of N containing $f_0(\bar{c}_0)$. Because N is constructible over $f(X)$, the type $tp(\delta_0/X)$ is isolated, and so, by Fact 8.3, $tp(\delta/f(X) \cup f_0(\bar{c}_0))$ is also isolated. We can extend f_0 to a pem $f'_0 : X \cup \bar{c}'_0 \to N$ for some finite $\bar{c}'_0$ containing $\bar{c}_0$ and such that $Im(f'_0) = f(X) \cup \delta_0$. Note that now $Im(f'_0)$ is sufficient but $Dom(f'_0) = X \cup \bar{c}'_0$ might not be. However, we can continue this process and construct finite sufficient sets $\bar{c}_0 \subset \bar{c}_1 \subset \cdots$ in M and a sequence $f_0 \subset f_1 \subset \cdots$ where each $f_i : X \cup \bar{c}_i \to N$ is a pem with $f_i(\bar{c}_i)$ contained in a sufficient subset of $f_{i+1}(\bar{c}_{i+1})$. Letting $g = \cup_i f_i$, we get that $Dom(g)$ and $Im(g)$ are sufficient in M and N, respectively. So that $g \in \mathcal{I}$, this contradicts maximality of f. □

The proof of the following is in [1, Chapter 6.4].

Lemma 8.5. *Suppose T is ω-stable. If $p \in S_n(A)$ has a isolated nonforking extension, then p is itself isolated.*

We will need the following fact which follows easily using finite character, symmetry, and transitivity of independence.

Fact 8.6. *$a, b \mathop{\smile\!\!\!\!\!\!|}_A C$ iff $a \mathop{\smile\!\!\!\!\!\!|}_A C$ and $b \mathop{\smile\!\!\!\!\!\!|}_{A,a} C$*

The following proposition shows that in ω-stable theories prime models are constructible.

Proposition 8.7. *Suppose T is ω-stable. If $M \models T$ is constructible over $A \subset M$ and $A \subset B \subset M$, then B is constructible over A.*

Proof. Let $(a_\alpha : \alpha < \delta)$ be a construction of M over A. The idea of the proof is to build a sequence $(B_\alpha : \alpha < \delta)$ such that $B_0 = A$, $\cup_{\alpha<\delta} B_\alpha = B$, and with the property that $B_{\alpha+1}$ is constructible over B_α. By iterating these constructions we get a construction of B over A.

We first build an auxiliary sequence $(X_\alpha : \alpha < \delta)$ of sufficient subsets of M with the following properties

(1) $X_0 = A$, $X_\alpha \subset X_\beta$ for $\alpha < \beta$
(2) $a_\alpha \in X_\alpha$ (this yields $\cup_\alpha X_\alpha = M$)
(3) $|X_{\alpha+1} \setminus X_\alpha| \leq \aleph_0$
(4) if $\bar{d}$ is from X_α then $\bar{d} \mathop{\smile\!\!\!\!\!\!|}_{X_\alpha \cap B} B$

When α is a limit ordinal we set $X_\alpha = \cup_{\beta<\alpha} X_\beta$. So assume we have built X_α, we now build $X_{\alpha+1}$. Let $\bar{c}_0$ be a finite sufficient containing a_α. By finite character of independence, there is a finite $\bar{b}$ in B such that $\bar{c}_0 \mathop{\smile\!\!\!\!\!\!|}_{X_\alpha, \bar{b}} B$. Now let $\bar{c}_1$ be a finite sufficient containing $\bar{c}_0 \cup \bar{b}$. Note that now $\bar{c}_0 \mathop{\smile\!\!\!\!\!\!|}_{X_\alpha, B \cap \bar{c}_1} B$. We continue this process and build a sequence $\bar{c}_0 \subset \bar{c}_1 \subset \cdots$ of finite sufficient such that

$$\bar{c}_n \mathop{\smile\!\!\!\!\!\!|}_{X_\alpha, B \cap \bar{c}_{n+1}} B$$

Set $X_{\alpha+1} = X_\alpha \cup (\cup_{n=0}^{\infty} \bar{c}_n)$. Clearly this $X_{\alpha+1}$ satisfies properties (1)-(3). Property (4) requires a bit more work, we leave as an exercise hinting to use Fact 8.6.

To finish set $B_\alpha = X_\alpha \cap B$. Then $B_\alpha \subset B_{\alpha+1}$ and $\cup_\alpha B_\alpha = B$ (by properties (1) and (2) above). The result follows once we show that $B_{\alpha+1}$ is constructible over B_α. Let $b_0, b_1, \dots$ be an enumeration of $B_{\alpha+1} \setminus B_\alpha$ (here we use property (3) above). We must show that $tp(b_n/B_\alpha, b_0, \dots, b_{n-1})$ is isolated for every n. By Fact 8.3, it suffices to show that $tp(b_0, \dots, b_n/B_\alpha)$ is isolated. Let $\bar{b} = (b_0, \dots, b_n)$. Because X_α is sufficient, M is constructible over X_α (see Lemma 8.2(2)), and so $tp(\bar{b}/X_\alpha)$ is isolated, say by $\phi(x, d)$. Then ϕ also isolates $tp(\bar{b}/B_\alpha, d)$. By property (4) above, $d \mathop{\smile\!\!\!\!\!\!\;|}_{B_\alpha} B$, then by symmetry and transitivity we get $\bar{b} \mathop{\smile\!\!\!\!\!\!\;|}_{B_\alpha} d$. Thus, the type $tp(\bar{b}/B_\alpha, d)$ is a nfe of $tp(\bar{b}/B_\alpha)$. By Lemma 8.5, the latter type must be isolated as desired. And we are done! $\square$

The fact that prime model extensions in ω-stable theories are all isomorphic now follows from the above results:

Corollary 8.8 (Uniqueness of prime model extensions)**.** *Suppose T is ω-stable, $\mathcal{U}$ is our monster model, and A is a common subset of $\mathcal{M}, \mathcal{N} \preceq \mathcal{U}$. If $\mathcal{M}$ and $\mathcal{N}$ are prime models over A, then they are isomorphic over A. Consequently, $\mathcal{M}$ is atomic over A.*

Proof. We have seen that there is a constructible prime and atomic model $\mathcal{M}_0$ over A. Since $\mathcal{M}, \mathcal{N}$ are prime models over A, we can embed them into $\mathcal{M}_0$ fixing A. By Proposition 8.7, $\mathcal{M}$ and $\mathcal{N}$ are both constructible over A. By Ressayre's theorem (Theorem 8.4 above), $\mathcal{M}$ and $\mathcal{N}$ are isomorphic over A, as desired. For the 'consequently' clause, note that Ressayre's theorem also yields $\mathcal{M} \cong_A \mathcal{M}_0$. Since the latter is atomic over A the former is as well. $\square$

References

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