SOME REMARKS ON THE DIFFERENTIAL PRIMITIVE ELEMENT THEOREM AND HILBERT BASIS THEOREM

KATHLEEN LAMONT

Student ID: 10723712

Supervisor: Dr. Omar León Sánchez

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Department of Mathematics, The University of Manchester

Abstract

Pogudin's *A primitive element theorem for fields with commuting derivations and automorphisms* [Pog19] improves the differential primitive element theorem put forth in Kolchin's *Differential Algebra and Algebraic Groups* [Kol73] for fields of characteristic 0, showing that only the field extension need be non-degenerate as opposed to the base field. One of the goals of this dissertation is to give a detailed presentation of the proofs of both Kolchin and Pogudin. Another goal is to show that Pogudin's results cannot be expanded to positive characteristic in the same way.

Additionally, a differential analog of Hilbert's basis theorem is proven, which states that for a differentially perfect field, every differential radical ideal of the field's ring of differential polynomials is finitely generated. Kolchin proves this differential analog of Hilbert's basis theorem utilizing multiple commuting derivations. This dissertation provides an account of a differential analog of Hilbert's basis theorem in the case of a single derivation, which simplifies much of the machinery needed.

Contents

Introduction

Kolchin's differential primitive element theorem states that a finitely differentially generated differentially separable field extension has a primitive element if the base field is non-degenerate. However, this condition was relaxed for fields with characteristic zero in Pogudin's *A primitive element theorem for fields with commuting derivations and automorphisms* [Pog19] stated below:

Theorem 0.1. *Suppose that $(E, \delta)/(F, \delta)$ is finitely differentially generated with $\mathrm{char}F = 0$ and that E is differentially algebraic over F. Then if (E, δ) is non-degenerate, there exists $\alpha \in E$ such that $E = F{<}\alpha{>}$.*

In this dissertation, Pogudin's proof is walked through and we attempt to extend to the case of positive characteristic; first we show that in positive characteristic, E being non-degenerate is equivalent to F being non-degenerate; in which case the proof follows by Kolchin's account. We thus look into, instead, if Poguin's primitive element theorem can be extended to positive characteristic if rather than considering non-degenerate field extensions, we consider field extensions with non-constant elements. As being non-degenerate is equivalent to having non-constant elements in zero characteristic, this distinction is only relevant in positive characteristic. However, we show that this condition does not guarantee a primitive element.

Additionally, this dissertation surveys a differential analog of the algebraic version of the Hilbert Basis Theorem stated below:

Theorem 0.2. *Let F be a differentially perfect field. Then every radical differential ideal in $F\{X\}$ is finitely generated.*

The literature surrounding these results, predominately Kolchin's *Differential Algebra and Algebraic Groups* [Kol73], focuses on fields having multiple commuting derivations, which can often be harder to understand, or on fields of characteristic zero only such as Marker's *Model Theory of Differential Fields* [Mar96]. In Marker's account, he relies on the fact that in characteristic zero, differential polynomials are guaranteed to a non-zero separant; as this is not true for all polynomials in positive characteristic, we show first that when (F, δ) is differentially perfect, a minimal rank differential polynomial of an ideal does have this property and thus extend Marker's prove to arbitrary characteristic whilst also providing a clear presentation of the results in Kolchin's book in the single-derivation case, which often simplifies much of the machinery needed.

The first section refreshes the reader on the algebraic versions of the primitive theorem as well as the Hilbert basis theorem. The second focuses on giving the reader an understanding of the basics of differential algebra by defining key terms as well as showing key results. Section 3 introduces preliminaries on differential fields, including notions of differentially algebraic and differentially separable. The following section continues to build on key definitions needed to state the differential primitive theorem, namely that of non-degenerate derivations on a field. In section 5, the proof of Pogudin's version of the primitive element is presented and the case is made as to why it cannot be extended to positive characteristic. In final section, the machinery needed to prove a differential version of Hilbert's basis theorem is given before proving the theorem.

1. Basic Theorems in Algebra

In this section, we recall the algebraic primitive element theorem as well as Hilbert's basis theorem.

Theorem 1.1. The Primitive Element Theorem *For any finite separable extension E of F, there exists $\alpha \in E$ such that $E = F(\alpha)$.*

Proof. If F is a finite field, then as the extension is finite, E is also a finite field and hence the multiplicative group of E is cyclic. Thus for some $\alpha \in E$ and $E = F(\alpha)$.

Let F be infinite. Suppose $E = F(\alpha, \beta)$ for some $\alpha, \beta \in E$. We introduce a new variable denoted by t and consider the element $\gamma_t = \alpha + t\beta \in E[t]$. Since α, β are separable over F, they are separable over $F(t)$. Obviously, t is separable over $F(t)$. Thus γ_t is separable over $F(t)$. We thus have that $F(t, \gamma_t)/F(t)$ is a finite-separable extension.

Let $f(x) \in F(t)[x]$ be the minimal polynomial of γ_t over $F(t)$. We can thus write $f(x) = a_o(t) + a_1(t)x + a_2(t)x^2 + \ldots + a_n(t)x^n$ and by definition, $f(\gamma_t) = a_o(t) + a_1(t)\gamma_t + a_2(t)(\gamma_t)^2 + \ldots + a_n(t)(\gamma_t)^n = 0$. Moreover,

$$\begin{aligned} 0 &= \frac{df(x)}{dt} \\ &= \sum_{i=0}^{n} \frac{da_i(t)}{dt}(\gamma_t)^i + \sum_{i=0}^{n} \beta a_i(t) i (\gamma_t)^{i-1} \\ &= \sum_{i=0}^{n} \frac{da_i(t)}{dt}(\gamma_t)^i + \left.\frac{df(x)}{dx}\right|_{x=\gamma_t} \end{aligned}$$

Since $F(t, \gamma_t)/F(t)$ is separable, $\left.\frac{df(x)}{dx}\right|_{x=\gamma_t} \neq 0$. Thus

$$\beta = \left(-\sum_{i=0}^{n} \frac{da_i(t)}{dt}(\gamma_t)^i\right)\left(\left.\frac{df(x)}{dx}\right|_{x=\gamma_t}\right)^{-1} = \frac{p(\gamma_t)}{q(\gamma_t)}$$

for some $p, q \in F[t]$. From Lemma 4.1, since E/F is a field extension, there exists some $c \in F$ such that $q(\alpha + c\beta) \neq 0$. Let $\gamma_c = \alpha + c\beta$. We thus have that $\beta = \frac{p(\gamma_c)}{q(\gamma_c)} \in F(\alpha + c\beta)$. It follows that $\alpha + c\beta - c\beta = \alpha \in F(\alpha + c\beta)$. Thus $E = F(\alpha + c\beta)$.

Now suppose the statement holds for some $n \geq 1$ and consider

$$\begin{aligned} E &= F(\alpha_1, \ldots, \alpha_{n+1}) \\ &= F(\alpha_1, \ldots, \alpha_n)(\alpha_{n+1}) \\ &= F(\alpha', \alpha_{n+1}) \end{aligned}$$

by the inductive hypothesis. Proceeding as in the base case, it thus follows that $E = F(\xi)$ where $\xi = \alpha_1 + c_2\alpha_2 + \ldots + c_{n+1}\alpha_{n+1}$ for some $c_i \in F$ □

Theorem 1.2. Hilbert Basis Theorem *Let R be a ring such that every ideal is finitely generated. Then every ideal of $R[x]$ is finitely generated. In particular, if F is a field, then every ideal of $F[x]$ is finitely generated.*

Proof. Let I be an ideal of $R[x]$. Let $f_0 \in I$ be a polynomial of minimal degree. Then for $k \geq 1$, we choose f_k to be of minimal degree in $I \backslash (f_1, \ldots, f_{k-1})$. Note that by construction, $\deg(f_k) \geq \deg(f_{k-1})$ for all $k \geq 1$. Denote the leading coefficient of f_k by a_k for $k \geq 0$. Let $I_{\text{coeff}} \coloneqq \{a_0, a_1, \ldots\}$. It's easy to show that I_{coeff} is an ideal of R and is thus finitely generated. Thus $I_{\text{coeff}} = (a_0, \ldots, a_n)$ for some $n \in \mathbb{N}$. We show $I = (f_0, \ldots, f_n)$.

Assume $I \not\subset (f_0, \ldots, f_n)$, then by construction, $f_{n+1} \notin (f_0, \ldots, f_n)$. However, $a_{n+1} \in (a_0, \ldots, a_n)$. Thus there exists some $c_i \in R$ such that $a_{n+1} = \sum_{i=0}^{n} c_i a_i$. Consider $g(x) \coloneqq \sum_{i=0}^{n} c_i a_i f_i(x) x^{\deg(f_{n+1}) - \deg(f_i)}$. Then $\deg(g) = \deg(f_{n+1})$ and g has the same leading coefficient as f_{n+1}. Thus $f_{n+1} - g$ has strictly lesser degree than f_{n+1} and thus $f_{n+1} - g \in (f_0, \ldots, f_n)$. As $g \in (f_0, \ldots, f_n)$, $f_{n+1} \in (f_0, \ldots, f_n)$, which is a contradiction. Thus $I \subset (f_0, \ldots, f_n)$ and it follows that $I = (f_0, \ldots, f_n)$. □

2. Basic Differential Algebra

We present the basic notions of differential algebra, defining what a derivation and a differential ring is before introducing differential polynomials and field extensions. We assume that a ring R is commutative and has unity.

A **differential ring** (R, δ) is a ring R equipped with a **derivation**:

$$\delta : R \to R$$

namely, a map satisfying

$$\begin{aligned} \delta(a + b) &= \delta(a) + \delta(b) \\ \delta(ab) &= a\delta(b) + \delta(a)b \end{aligned}$$

for all $a, b \in R$

Example 2.1.

(1) Every ring R can be equipped with the trivial derivation, where for all $r \in R, \delta(r) = 0$
(2) For any polynomial ring $R[x]$, $\delta(\sum_{i=0}^{n} a_i x^i) = \sum_{i=1}^{n} i a_i x^{i-1}$ defines a derivation.

If (R, δ) is an integral domain or a field, we call R a differential integral domain or differential field respectively. If R' is a subring of R such that $\delta R' \subset R'$, then we say that R' is a differential subring of R. If R, R' are both differential fields, we say that R' is a differential subfield of R. A *differential ideal*, I, is an ideal of R such that $\delta(I) \subset I$.

For differential fields $(F,\delta),(E,\delta)$ such that F is a differential subfield of E, we say that E is a differential field extension of F, denoted by E/F or $(E,\delta)/(F,\delta)$. For some set Σ, $F<\Sigma>$ is the smallest differential subfield of E containing F and Σ. Thus, $F<\Sigma> = F((\delta^j(\Sigma))_{j\in\mathbb{N}})$ as any differential field containing Σ must also contain $\delta^j(\Sigma)$ for $j \in N$. We say that a E is *finitely generated* over F as a field if there exists some finite set of generators such that $E = F(\alpha_1,\dots,\alpha_m)$ for some $\alpha_j \in E$. For $\alpha \in E$, the differential field $F<\alpha>$ being finitely generated over F is equivalent to $F<\alpha> = F(\alpha,\delta(\alpha),\dots,\delta^n(\alpha))$ for some $n \in \mathbb{N}$. This follows from the fact that since $F<\alpha> = F(\alpha_i,\dots,\alpha_m)$ for some $\alpha_i \in F<\alpha>$, then for each i, $\alpha_i = a_{i_0}\alpha + \dots a_{i_n}\delta^{i_n}\alpha$ for some $a_{i_j} \in F<\alpha>$. Letting $n = \max\limits_{1\le i\le m}(i_n)$, the claim follows. We say a differential field extension E is *finitely differentially generated* over F, if there exists some $\alpha_i \in E$ such that $E = F<\alpha_1,\dots,\alpha_m>$.

Proposition 2.2. *For a differential ring* (R,δ)*:*

(a) $\delta(r^n) = nr^{n-1}\delta(r)$ *for all* $r \in R$ *and for all* $n \in \mathbb{N}$
(b) $\delta(0) = 0$
(c) $\delta(1) = 0$
(d) *For a unit* $b \in R$, $\delta\left(\frac{a}{b}\right) = \frac{b\delta(a)-a\delta(b)}{b^2}$
(e) *If* R *is a domain,* δ *extends uniquely to* Frac(R)

Proof.

(a) We proceed via induction n. If $n = 1$, the statement is trivially true. Assuming the statement holds for some $n \in \mathbb{N}$, consider

$$\begin{aligned}\delta(r^{n+1}) &= \delta(r\cdot r^n)\\ &= r\delta(r^n) + \delta(r)\cdot r^n\\ &= r\cdot(nr^{n-1}\delta(r)) + \delta(r)\cdot rn\\ &= (n+1)r^n\delta(r)\end{aligned}$$

(b)

$$\begin{aligned}\delta(0) &= \delta(0+0)\\ &= \delta(0)+\delta(0)\\ &= 2\delta(0)\end{aligned}$$

Thus $\delta(0) = 0$

(c)

$$\begin{aligned}\delta(1) &= \delta(1\cdot 1)\\ &= 1\cdot\delta(1) + \delta(1)\cdot 1\\ &= 2\delta(1)\end{aligned}$$

Thus $\delta(1) = 0$

(d)

$$\begin{aligned}\delta(a) &= \delta\left(\frac{a}{b}\cdot b\right)\\ &= \frac{a}{b}\delta(b) + \delta\left(\frac{a}{b}\right)b\end{aligned}$$

Thus $\delta\left(\frac{a}{b}\right) = \frac{1}{b}\delta(a) + \frac{a}{b^2}\delta(b) = \frac{b\delta(a)-a\delta(b)}{b^2}$

(e) follows from (d)

□

The **field of constants** of a differential ring is precisely the kernel of δ. The constants of a differential ring (R, δ) are denoted by R_0 and for a differential field (F, δ), we denote the constants by F_0.

Proposition 2.3.

(1) For a finite field F the field of constants F_0 is the entire field.

(2) For a differential field F, such that char$F = p \neq 0$, F^p is a subfield of the field of constants

Proof.

(1) Since F is finite, char $F = p$ for some prime p and $|F| = p^n$ for some $n \in \mathbb{N}$. Moreover, $|F^*| = p^n - 1$ and is cyclic. Let α be a generator for F^x. Then $\alpha^{p^n-1} = 1$. Thus $\delta(\alpha^{(p^n-1)}) = \delta(1) = 0$ and $(p^n - 1)(\alpha^{(p^n-2)})\delta(\alpha) = 0$. Since $p \nmid (p^n - 1)$, $\delta(\alpha) = 0$. As for all $r \neq 0 \in F$, $r = \alpha^m$ for some $m \in N$, by Proposition 2.2, it follows that $\delta(r) = 0$.

(2) Let $\alpha \in F^p$. Then $\alpha = \beta^p$ for some $\beta \in F$. Thus $\delta(\alpha)) = p\beta^{p-1}\delta(\beta) = 0$ since char $= p$.

□

For a given differential ring (R, δ), we define the **ring of differential polynomials**, denoted by $R\{X\}$, to be $R[x_0, x_1, \ldots]$ and extend the derivation δ by $\delta(x_i) = x_{i+1}$. For any $f \in R\{X\}$, the **order** of f is defined to be the largest n such that x_n appears in f. This f can then be written in the form:

$$f(X) = \sum_{i=0}^{m} g_i(x_1, \ldots, x_{n-1})(x_n)^i$$

with $g_i \in R\{X\}$ and $g_m \neq 0$. m is then called the **degree** of f. The rank of f is (n, m) where rank is ordered first by the polynomial's order and then by its degree. The separant of f, denoted by s_f, is $\frac{df}{dx_n}$. For instance, the differential polynomial $f(X) = x_1 + x_1x_2^2 + x_2x_3^3 \ldots + x_{n-1}x_n^n$ has separant $s_f = nx_{n-1}x_n^{n-1}$.

We can extend this definition of a ring of differential polynomials to multiple differential variables by defining $R\{X_1, \ldots, X_n\} = R\{X_1\}\{X_2\} \ldots \{X_n\}$

For an element α in a differential field extension $(E, \delta)/(F, \delta)$, we define the **defining differential ideal of** α over F to be $I_\alpha \coloneqq \{f \in F\{X\} \mid f(\alpha) = 0\}$. It's clear that this is an ideal and it is differential as $\delta(f(a)) = (\delta f)(a)$ for all $a \in E$. Thus if $f(\alpha) = 0$, it follows that $(\delta f)(\alpha) = 0$. Note that while we mostly denote a differential polynomial by $f(X)$, we sometimes write it as $f(x_0, \ldots, x_n)$ to denote that f has order at most n.

Lemma 2.4. *Let E/F be a differential field extension and let $a \in E$. Let $I \coloneqq \{g \in F[x] \mid g(a) = 0\}$. If there exists $b \in F(a)$ such that*

$$g'(a)b + g^\delta(a) = 0$$

for all $g \in I$ where g^δ is obtained from applying δ to the coefficients of g. Then there is a derivation $\delta : F(a) \mapsto E$ such that $\delta(a) = b$.

Proof. Let $a \in E$ and suppose such a b exists. Consider $h \in F[a]$. Then $h = q(a)$ for some $q \in F[t]$. Consider $\delta : F[a] \to F(a)$. Let $\delta(h) = q'(a)b + q^\delta(a)$. We first check that this definition is well-defined. Suppose $h = p(a)$ for some $p \in F[t]$. Consider:

$$[q'(a)b + q^\delta(a)] - [p'(a)b + p^\delta(a)] = (q-p)'(a)b + (p+q)^\delta(a)$$

□

Note that $(p-q)(a) = 0$, thus $p - q \in I$. By supposition,

$$(q-p)'(a)b + (p+q)^\delta(a) = 0$$

and thus δ is well-defined. We now show this δ is additive and satisfies Leibniz rule. Let $h_1, h_2 \in F(a)$. Then $h_1 = q_1(a), h_2 = q_2(a)$ for some $q_1, q_2 \in F[t]$.

$$\begin{aligned} \delta(h_1 + h_2) &= \delta((q_1+q_2)(a)) \\ &= (q_1+q_2)'(a)b + (q_1+q_2)^\delta(a) \\ &= q_1'(a)b + q_1^\delta(a) + q_2'(a)b + q_2^\delta(a) \\ &= \delta(h_1) + \delta(h_2) \end{aligned}$$

$$\begin{aligned} \delta(h_1h_2) &= \delta(q_1q_2(a)) \\ &= (q_1q_2)'(a)b + (q_1q_2)^\delta(a) \\ &= q_1q_2'(a)b + q_1'q_2(a)b + q_1q_2^\delta(a) + q_1^\delta q_2(a) \\ &= q_1(a)[q_2'(a)b + q_2^\delta(a)] + q_2(a)[q_1'(a)b + q_1^\delta(a)] \\ &= h_2\delta(h_1) + h_1\delta(h_2) \end{aligned}$$

Note that this can be extended to a derivation $\delta : F(a) \to F(a)$ by Proposition 2.2 part (e). Thus δ is a derivation on $F(a)$ and $\delta(a) = b$.

We derive several useful consequences of this lemma.

Corollary 2.5. *Let (F, δ) be a differential field and E/F a separably algebraic extension. Then δ extends uniquely to E.*

Proof. Let $a \in E$. It suffices to show that δ extends to $F(a)$. Note that $F(a)/F$ is separable and algebraic as E/F is. Let $I := \{f \in F[x] \mid f(a) = 0\}$ and let f be the minimal polynomial of a over F. We show that $b = -\frac{f^\delta(a)}{f'(a)}$ satisfies the desired property in Lemma 2.4. Since $g \in I$, then $f|g$ and $g = pf$ for some $p \in F[t]$. Then

$$\begin{aligned} g'(a)b + g^\delta(a) &= -pf'(a)(\frac{f^\delta(a)}{f'(a)}) - p'f(a)(\frac{f^\delta(a)}{f'(a)}) + pf^\delta(a) + p^\delta f(a) \\ &= [p'(a)(-\frac{f^\delta(a)}{f'(a)}) + p^\delta(a)]f(a) \end{aligned}$$

As $f(a) = 0$, $g'(a)b + g^\delta(a) = 0$ as desired. Thus, by Lemma 2.4, a derivation $\widetilde{\delta}$ exists on $F(a)$. Moreover, for all $a \in F, \widetilde{\delta}(a) = \delta(a)$ and thus $\widetilde{\delta}$ extends δ.

We now show this extension is unique. Let $a \in E$. Since E/F is separable and algebraic, $F(\alpha)/F$ is as well. Let $f(x) = \sum_{i=0}^n c_i x^i$ be the minimal polynomial of a over F. Let $\widetilde{\delta}$ be an extension of δ. Then

$$\begin{aligned} 0 &= \widetilde{\delta}(f(a)) \\ &= \widetilde{\delta}(\sum_{i=0}^n c_i a^i) \\ &= \sum_{i=0}^n \delta(c_i)a^i + \sum_{i=1}^n c_i\widetilde{\delta}(a)a^{i-1} \end{aligned}$$

As f is separable, $\widetilde{\delta}(a) = -\frac{f^\delta(a)}{f'(a)}$. □

Corollary 2.6. *Let (F, δ) be a differential field and $(E, \delta)/(F, \delta)$ a differential field extension. Assume E/F is separably algebraic extension. Then if δ acts trivially on F, it also acts trivially on E.*

Proof. Let $a \in E$. As E/F is separably algebraic, $\delta(a) = -\frac{f^\delta(a)}{f'(a)}$ where f is the minimal polynomial of a over f. As δ is trivial on f, $f^\delta \equiv 0$ and thus $\delta(a) = 0$. □

Corollary 2.7. *Let (F, δ) be a differential field such that $\delta(F) \equiv 0$ and E/F a field extension. Suppose $a \in E$ is non-separable over F. Then for any $b \in E$, there exists a derivation $\delta : F(a) \mapsto E$ extending $\delta \equiv 0$ on F such that $\delta(a) = b$.*

Proof. Let $a, b \in E$ with a non-separable over F and let $I := \{f \in F[x] \mid f(a) = 0\}$. Let $g \in I$. Since a is non-separable over F, $g'(a) = 0$. Moreover, as $\delta \equiv 0$ on F, $g^\delta \equiv 0$. Thus $g'(a)b + g^\delta(a) = 0$ and the claim follows from Lemma 2.4. □

Corollary 2.8. *Let (F, δ) be a differential field. Then for any $b \in F(t)$, δ can be extended to a derivation on $F(t)$ such that $\delta(t) = b$.*

Proof. As t is a transcendental, $I := \{f \in F[x] \mid f(a) = 0\} = \emptyset$. Thus for any $b \in F(t)$,

$$g'(a)b + g^\delta(a) = 0$$

for all $g \in I$. By Lemma 2.4, the desired derivation exists. □

3. On Differential Field Extensions

Let E/F be a field extension. A set of elements $(\alpha)_{i\in I} \subset E$ is **algebraically dependent** over F if there exists some non-zero $f \in F[(X_i)_{i\in I}]$ such that f vanishes at $(\alpha)_{i\in I} \subset E$. If the set of elements $(\alpha)_{i\in I} \subset E$ is a singleton, denoted by α, we say that α is **algebraic** over F. A set of elements $(\alpha)_{i\in I} \subset E$ is **separably dependent** over F if there exists some $f \in F[(X_i)]_{i\in I}$ such that f vanishes at $(\alpha)_{i\in I} \subset E$ and for some $i \in I$, $\frac{\delta f}{\delta X_i}$ does not vanish there. If the set of elements $(\alpha)_{i\in I} \subset E$ is a singleton, denoted by α, we say that α is **separable** over F. The extension E/F is **separable** if for every $(\alpha)_{i\in I} \subset E$, $(\alpha)_{i\in I}$ being algebraically dependent implies that the set is separably dependent.

Lemma 3.1. *Let $u_1, \dots, u_r, v_1, \dots, v_s$ be elements of a field extension E/F and suppose that $r < s$ and each v_i is separably dependent over $F(u_1, \dots, u_r)$. Then $(v_1, \dots, v_s)$ is separably dependent over F*

Proof. When char $F = 0$, as algebraic implies separable, it suffices to show that $(v_1, \dots, v_s)$ are algebraically dependent over F. We can assume that $\{u_1, \dots, u_r\}$ is an algebraically independent set. Suppose, toward contradiction that $(v_1, \dots, v_s)$ were not algebraically dependent over F. By assumption, as v_1 is algebraic over $F(u_1, \dots, u_r)$, there exists some polynomial f_1 such that $f_1(u_1, \dots, u_s, v_1) = 0$. By assumption, there exists some i such that a term containing u_i appears. Without loss of generality, assume $i = 1$. Then u_1 is algebraic over $F(v_1, u_2, \dots, u_s)$. Now suppose that for some $1 < k < s$, we have found $v_1, \dots v_k$ such that $F(u_1, \dots, u_k)$ is algebraic over $F(v_1, \dots, v_k, u_{k+1}, \dots, u_s)$. Then as v_{k+1} is algebraic over $F(u_1, \dots, u_k)$, there exists some $f \in F[X_1, \dots, x_{s+1}]$ such $f(v_{k+1}, v_1, \dots, v_k, u_{k+1}, \dots, u_s) = 0$. As the v_is are algebraically independent, it follows that a term containing u_i must appear for some i and without loss of generality, we can assume that $i = k+1$. It thus follows by induction that $F(u_1, \dots, u_s)$ is algebraic over $F(v_1, \dots, v_s)$. Then as v_{s+1}

is algebraic over $F(u_1, \dots, u_s)$, it follows that v_{s+1} is algebraic over $F(v_1, \dots, v_s)$ which is a contradiction. Thus $(v_1, \dots, v_s)$ are algebraically dependent as desired.

Suppose char $F = p > 0$. We proceed by induction on r. Suppose $r = 1$. For all $i \leq s$, since v_i is separably algebraic over $F(u_1)$, there exists some $f_i(X_1, Y_i) \in F[X_1, Y_i]$ such that $f_i(u_1, v_i) = 0$ and $\frac{df}{dy_i}(u_1, v_i) \neq 0$. Let $n_i = deg_{Y_i}(f_i)$ over $F(u_1)$. If v_i is algebraic over F then we can find a polynomial of some degree m such that $deg_{X_i}(f_i) < m$ and the coefficient of Y^{n_i} is one. Thus v_i is separably algebraic over F and the claim follows. We may thus assume that v_i is transcendental over F. Let ν denote the largest natural number such that $f_i \in F[X_1^{p^\nu}, Y_i]$. Define the function $\varphi_i(X_1^{p^\nu}, Y_i) := f_i(X_1, Y_i)$. As ν was taken to be maximal, there exists some i such that $\varphi_i \notin F[X_1^p, Y_i]$. Without loss of generality, we can assume this happens for $i = 1$. Thus $\frac{d\varphi_1}{dX_1} \neq 0$. Let $t = u_1^{p^\nu}$. If t is algebraic over F, say of degree m, then as $\deg_{X_1} \frac{d\varphi_1}{dX_1} < m$, $\frac{d\varphi_1}{dX_1}(t, Y_1) \neq 0$. As the coefficient in φ_1 of Y^{n_1} is 1, $\deg_{X_1} \frac{d\varphi_1}{dX_1} < n_1$. Thus $\frac{d\varphi_1}{dX_1}(t, v_1) \neq 0$. Thus t is separably algebraic over $F(v_1)$. If t is transcendental, then $\varphi(t, Y_1)$ does not divide $\frac{d\varphi_1}{dX_1}(t, Y_1)$ in $F[t, Y_1]$ and thus in $F(t)[Y_1]$. Thus t is separably algebraic over $F(v_1)$. As v_2 is separably algebraic over $F(t)$ and t is separable algebraic over $F(v_1)$, it follows that v_2 is separably algebraic over $F(v_1)$ and the desired claim thus follows.

Let $r > 1$ and suppose the claim holds for all natural numbers less than r. Then $(v_1, \dots, v_s)$ is separably dependent over $F(u_1)$. Thus v_s is separably algebraic over $F(u_1, v_1, \dots, v_{s-1})$. Similarly, $(v_1, \dots, v_{s-1})$ is separably dependent over $F(u_1)$ and thus v_{s-1} is separably algebraic over $F(u_1, v_1, \dots, v_{s-2})$. Thus both v_s, v_{s-1} are separably algebraic over $F(v_1, \dots, v_{s-2})(u_1)$. By the inductive hypothesis, (v_s, v_{s-1}) is separably dependent over $F(v_1, \dots, v_{s-2})$. Thus, $(v_1, \dots, v_s)$ is separably dependent over F. ☐

For a differential field extension E/F with derivation operator δ, we say $(\alpha)_{i \in I} \subset E$ is **differentially algebraic** over F if $(\delta^j(\alpha_i))_{i \in I, j \in \mathbb{Z}^{\geq 0}}$ is algebraically dependent over F. If the set of elements $(\alpha_i)_{i \in I} \subset E$ is a singleton, denoted by α, we say that α is **differentially algebraic** over F. We say $(\alpha_i)_{i \in I} \subset E$ is **differentially separable** over F if $(\delta^j(\alpha_i))_{i \in I, j \in \mathbb{Z}^{\geq 0}}$ is separably dependent over F. If the set of elements $(\alpha_i)_{i \in I} \subset E$ is a singleton, denoted by α, we say that α is **differentially separable** over F. The differential field E is differentially separable over F if for all $\alpha \in E$, α is differentially separable over F.

Lemma 3.2. *Let α, β be elements of a differential extension of F. If β is differentially separable over $F<\alpha>$ and α is differentially separable over F then β is differentially separable over F.*

Proof. Since α is differentially separable over F, by definition there exists some $j \in \mathbb{N}$ such that $\alpha, \dots, \delta^{j-1}(\alpha)$ is separably dependent over F. Similarly, there exists $i \in \mathbb{N}$ such that $\beta, \dots, \delta^{i-1}(\beta)$ is separably dependent over $F(\alpha, \dots, \delta^{j-1}(\alpha))$. Without loss of generality, we can take $i > j$. By Lemma 3.1, the set $\{\beta, \dots, \delta^{i-1}(\beta)\}$ is separably dependent over F and thus β is differentially separable over F. ☐

Lemma 3.3. *Let E/F be a differential field extension. Let E_{diff} be the set of elements that are differentially separable over F. Then E_{diff} is a differential subfield field of E.*

Proof. Let $\alpha, \beta \in E_{\text{diff}}$ and suppose $\gamma \in \{\alpha+\beta, \alpha\beta, \alpha-\beta, \frac{\alpha}{\beta}, \delta\alpha\}$. Then γ is differentially separable over $F<\alpha,\beta> = F<\alpha><\beta>$. As β is differentially separable over $F<\alpha>$, by Lemma 3.2, γ is differentially separable over $F<\alpha>$ and as α is differentially separable over F, by another application of Lemma 3.2, γ is differentially separable over F. □

The following proposition and lemmas do not use Kolchin's results, but instead are proofs developed in this dissertation.

Proposition 3.4. *Let $\alpha \in E/F$ where E/F is a differential field extension. Then α is differential separable over F if and only if there exists some $f \in F\{X\}$ such that $f(\alpha) = 0$ and $s_f(\alpha) \neq 0$ where $s_f(x) := \frac{df}{dx_n}$ where $n = ord(f)$.*

Before beginning the proof, we first prove two lemmas:

Lemma 3.5. *Let E/F be a differential field extension. For all $f \in F\{X\}$ with $n = ord(f)$ and for all $0 \leq i \leq n$:*

$$\frac{d(\delta f)}{dx_i} + \frac{df}{dx_{i-1}} = \delta\left(\frac{df}{dx_i}\right)$$

We define $\frac{df}{dx_{-1}} = 0$.

Proof. We proceed via induction on the order of f, denoted by n. When $n = 0$, $f = \sum_{j=0}^{m} a_j x_0^j$. Thus

$$\begin{aligned}\frac{d(\delta f)}{dx_0} + \frac{df}{dx_{-1}} &= \frac{d}{dx_0}\Big(\sum_{j=0}^{n} \delta(a_j)x_0^j + \sum_{j=1}^{n} j a_j x_0^{j-1}\Big)\\ &= \sum_{j=1}^{n} j\delta(a_j)x_0^{j-1} + \sum_{j=2}^{n} j(j-1)a_j x_0^{j-2}\\ &= \delta\left(\frac{df}{dx_0}\right)\end{aligned}$$

Let $n \geq 1 \in \mathbb{N}$ and assume the statement holds for all natural numbers less than n. We can write $f = \sum_{j=0}^{m} g_j x_n^j$ where $g_j \in F\{X\}$ with $\text{ord}(g_j) < n$. We prove the statement in two cases: when $1 < i < n$ and when $i = n$.

$1 < i < n$:

$$\begin{aligned}\frac{d(\delta f)}{dx_i} + \frac{df}{dx_{i-1}} &= \sum_{j=0}^{m} \frac{d(\delta g_j)}{dx_i}x_n^j + \sum_{j=1}^{m} j\frac{dg_j}{dx_j}x_n^{j-1} + \sum_{j=0}^{m} \frac{dg}{dx_{i-1}}x_n^j\\ &= \sum_{j=0}^{m}\left(\frac{d(\delta g_j)}{\delta x_i} + \frac{dg}{dx_{i-1}}\right)x_n^j + \sum_{j=1}^{m} j\frac{dg_j}{dx_i}x_n^{j-1}\\ &= \sum_{j=0}^{m} \delta\left(\frac{dg_j}{dx_i}\right)x_n^j + \sum_{j=1}^{m} j\frac{dg_j}{dx_i}x_n^{j-1}\\ &= \delta\left(\frac{df}{dx_i}\right)\end{aligned}$$

as $\text{ord}(g_j) < n$ for all g_j.

$i = n$:

$$\begin{aligned}\frac{d(\delta f)}{dx_n} + \frac{df}{dx_{n-1}} &= \sum_{j=2}^{m} j(j-1)g_j x_n^{j-2} + \sum_{j=1}^{m} j\delta(g_j)x_n^{j-1} + \sum_{j=0}^{m} \frac{d(\delta g_j)}{dx_n} x_n^j + \sum_{j=0}^{m} \frac{dg_j}{dx_{n-1}} x_n^j \\ &= \sum_{j=0}^{m} x_n^j \delta\left(\frac{dg_j}{dx_n}\right) + \sum_{j=2}^{m} j(j-1)g_j x_n^{j-2} + \sum_{j=1}^{m} j\delta(g_j)x_n^{j-1} \\ &= \delta\left(\frac{df}{dx_n}\right)\end{aligned}$$

□

Lemma 3.6. *Let P be a differential ideal of $F\{X\}$ and let $n \in \mathbb{N}$. Define $P_n := P \cap F[x_0, \ldots, x_n]$ and suppose for all $g \in P_n$, $\frac{dg}{dx_i}, \ldots, \frac{dg}{dx_n} \in P$. Then for all $f \in P_n$, $\frac{df}{dx_{i-1}} \in P$.*

Proof. Assume that for all $g \in P_n$, $\frac{dg}{dx_i}, \ldots, \frac{dg}{dx_n} \in P$. Let $f \in P_n$. Then $\frac{df}{dx_i}, \ldots, \frac{df}{dx_n} \in P$ for some $i \leq n$. Then by Lemma 3.5:

$$\frac{d(\delta f)}{dx_i} + \frac{df}{dx_{i-1}} = \delta\left(\frac{df}{dx_i}\right)$$

To show $\frac{df}{dx_{i-1}} \in P$, it suffices to show $\frac{d(\delta f)}{dx_i} \in P$ as $\delta\left(\frac{df}{dx_i}\right) \in P$ by supposition. Using $f^\delta(x)$ to denote f with δ applied to its coefficients, we have

$$\begin{aligned}\delta f &= f^\delta(x) + \sum_{j=0}^{n} \frac{df}{dx_j} x_{j+1} \\ &= f^\delta(x) + \sum_{j=0}^{n-1} \frac{df}{dx_j} x_{j+1} + \frac{df}{dx_n} x_{n+1}\end{aligned}$$

Letting $h := f^\delta(x) + \sum_{j=0}^{n-1} \frac{df}{dx_j} x_{j+1}$, $h \in P$ as $\delta f \in P$ and $\frac{df}{dx_n} \in P$. Then,

$$\frac{d(\delta f)}{dx_i} = \frac{dh}{dx_i} + \frac{d\left(\frac{df}{dx_n}\right)}{dx_i} x_{n+1} + \frac{df}{dx_n}\frac{dx_{n+1}}{dx_n}$$

As $\frac{df}{dx_n} \in P_n$ by supposition, $\frac{d\left(\frac{df}{dx_n}\right)}{dx_i} x_{n+1} \in P$. Moreover, since $\mathrm{ord}(h) \leq n$, $h \in P_n$ and thus $\frac{dh}{dx_i} \in P_n$. Finally as $\frac{df}{dx_n}\frac{dx_{n+1}}{dx_n} = 0$, $\frac{d(df)}{dx_i} \in P$ and the claim thus follows. □

We now begin the proof of Proposition 3.4

Proof. Suppose α is differentially separable. Then there exists some $g \in F\{X\}$ such that $g(\alpha) = 0$ and $\frac{\delta g}{\delta x_i}(\alpha) \neq 0$ for some $i \in \mathbb{N}$. Let n be the order of g and P the defining ideal. Then $\frac{\delta g}{\delta x_i} \notin P$. By the contrapositive to Lemma 3.6, there exists some $h \in P_n$ such that $\frac{\delta h}{\delta x_n} \notin P_n$. Since $\mathrm{ord}(h) \leq n$ and $\frac{\delta h}{\delta x_n} \neq 0$, $\mathrm{ord}(h) = n$ and thus h is the desired element of $F\{X\}$. The converse follows by definition. □

Lemma 3.7. *Let $(E, \delta)/(F, \delta)$ be a differential field extensions. Then $\alpha \in E$ is differentially separable over F if and only if there exists some $i \in \mathbb{N}$ such that $\delta^i(\alpha) \in F(\alpha, \delta(\alpha), \ldots, \delta^{i-1}(\alpha))$*

Proof. Suppose that α is differentially separable over F. By Lemma 3.4, there exists some $f \in F\{X\}$ such that $f(\alpha) = 0$ and $s_f(\alpha) \neq 0$. Let $n := \operatorname{ord}(f)$ As done in the proof of Lemma 3.6, we can write $\delta f = h + s_f x_{n+1}$ where $\operatorname{ord}(h) \leq n$. Then

$$0 = \delta f(\alpha) = h(\alpha) + s_f(\alpha)\alpha_{n+1}.$$

Since $s_f(\alpha) \neq 0$, $\delta^{n+1}(\alpha) = \frac{-h(\alpha)}{s_f(\alpha)}$. Thus $\delta^{n+1}(\alpha) \in F(\alpha, \ldots, \delta^n(\alpha))$. Conversely, suppose such an i exists and take minimal $i \in \mathbb{N}$ such that $\delta^i(\alpha) \in F(\alpha, \ldots, \delta^{i-1}(\alpha)))$. Thus,

$$\delta^i(\alpha) = \sum_{j=0}^{m} g_j(\alpha_1, \ldots, \alpha_{i-2})\alpha_{i-1}^j$$

for some $g_j \in F[x_1, \ldots, x_{i-2}]$ and $m \in \mathbb{N}$.

Letting $f(X) = x_i - \sum_{j=0}^{m} g_j(x_1, \ldots, x_{i-2})x_{i-1}^j$, $f(\alpha) = 0$ and $\frac{\delta f}{\delta x_i} = 1$ and thus $\frac{\delta f}{\delta x_i}(\alpha) \neq 0$ and α is differentially separable over F. □

Theorem 3.8. *Let E be a finitely generated differential extension of F. Then E is a differentially separable extension if and only if E is finitely generated as a field extension of F.*

Proof. By hypothesis, $E = F<\alpha_1, \ldots, \alpha_n>$ for some $\alpha_i \in E$. Suppose E is finitely generated as a field extension of F, then there exists some $\beta_1, \ldots, \beta_r$ such that $E = F(\beta_1, \ldots, \beta_r)$. For $\gamma \in E$, as $F<\gamma> \subset E$ and E/F is finitely generated as a field, $F<\gamma>/F$ is finitely generated as a field as well. Thus for some $n \in \mathbb{N}$, $\delta^n(\gamma) \in F(\gamma, \delta(\gamma), \ldots, \delta^{n-1}(\gamma))$. By Lemma 3.7, γ is differentially separable over F. Conversely suppose that E is differentially separable over F. Then for all $j \leq n$, there exists some k_j such that $\delta^{k_j}(\alpha_j) \in F((\delta^i(\alpha_j)_{i<k_j})$ by Lemma 3.7. For any $l \geq k_j$, $\delta^l(\alpha_j) \in F((\delta^i(\alpha_j)_{i<k_j})$. Then, letting $k := \max_{1 \leq j \leq n}(k_j)$, $E = F<\alpha_1, \ldots, \alpha_n> \subset F((\delta^i(\alpha_j)_{i \leq k_j, j \leq n})$ and thus $E = F((\delta^i(\alpha_j)_{i<k, j \leq n})$. □

The following lemma is adapted from [Rit50]; however in Ritt, it only appeared in the case of characteristic zero:

Lemma 3.9. *Let F be a differential field. Elements, $\eta_1, \ldots \eta_s$, in a differential field F are linearly dependent over F_0 if and only if:*

$$\begin{vmatrix} \eta_1 & \ldots & \eta_s \\ \delta(\eta_1) & \ldots & \delta(\eta_s) \\ & \vdots & \\ \delta^{s-1}(\eta_1) & \ldots & \delta^{s-1}(\eta_s) \end{vmatrix} = 0.$$

Proof. Let $\eta_1, \ldots \eta_s \in F$ such that $\eta_1, \ldots \eta_s$ are linearly dependent over F. Then there exists constants $c_i \in F$ not all zero such that

$$c_1\eta_1 + \ldots + c_s\eta_s = 0$$

Differentiating $s-1$ times yields the following system:

$$\begin{aligned} c_1\eta_1 + \ldots + c_s\eta_s &= 0 \\ c_1\delta(\eta_1) + \ldots + c_s\delta(\eta_s) &= 0 \\ \vdots \qquad\qquad & \quad \vdots \\ c_1\delta^{s-1}(\eta_1) + \ldots + c_s\delta^{s-1}(\eta_s) &= 0 \end{aligned}$$

Thus as this system has a non-trivial solution, it thus follows that the desired determinant is zero.

Conversely, we proceed by induction. The case when $s=1$ follows as since the determinant vanishes, $\eta_1 = 0$. Now suppose the statement holds for all natural numbers less than some $s=r$. If

$$\begin{vmatrix} \eta_1 & \ldots & \eta_{r-1} \\ \delta(\eta_1) & \ldots & \delta(\eta_{r-1}) \\ & \vdots & \\ \delta^{r-2}(\eta_1) & \ldots & \delta^{r-2}(\eta_{r-1}) \end{vmatrix} = 0,$$

then by the inductive hypothesis, there exists some $c_i \in F_0$ not all zero such that $c_1\eta_1 + \ldots + c_{r-1}\eta_{r-1} = 0$ and thus $\eta_1, \ldots \eta_r$ are linearly dependent over F_0.

Thus we may assume that

$$\begin{vmatrix} \eta_1 & \ldots & \eta_{r-1} \\ \delta(\eta_1) & \ldots & \delta(\eta_{r-1}) \\ & \vdots & \\ \delta^{r-2}(\eta_1) & \ldots & \delta^{r-2}(\eta_{r-1}) \end{vmatrix} \neq 0. \tag{3.1}$$

and $c_r \neq 0$.

By supposition, as the determinant vanishes, there exists some $c_i \in F$ such

$$c_1\delta^j(\eta_1) + \ldots + c_r\delta^j(\eta_r) = 0$$

for all $j \leq r-1$. To show that $\eta_1, \ldots \eta_s$ are linearly dependent, it suffices to show that these c_i are in fact constants. Without loss of generality, we can take c_r to be unity and thus c_r is a constant.

$$\begin{aligned} 0 &= \delta((c_1\delta^j(\eta_1) + \ldots + c_r\delta^j(\eta_r)) - c_1\delta^{j+1}(\eta_1) + \ldots + c_r\delta^{j+1}(\eta_r) \\ &= \delta(c_1)\delta^j(\eta_1) + \ldots + \delta(c_r)\delta^j(\eta_r) \end{aligned}$$

for all $j \leq r-2$. By (3.1), for all $i \leq r-1$, $\delta(c_i) = 0$ and thus the c_i are all constants as desired. □

Let E, E' be field extension of F. We say that E, E' are *linearly disjoint* over F, if for every finite $A \subset E$ that is F linearly independent, A is also E' linearly independent.

Lemma 3.10. *Let $(E,\delta)/(F,\delta)$ be a differential field extension where F is a field of positive characteristic p. If E^p and F_0 are linearly disjoint over F^p, then E is separable over F.*

Proof. We first show that E_0 and F are linearly disjoint over F_0. Let $\alpha_1, \dots, \alpha_n \in F$ be E_0 linearly independent. By Lemma 3.9 as $F \subset E$,

$$\begin{vmatrix} \alpha_1 & \dots & \alpha_n \\ \delta(\alpha_1) & \dots & \delta(\alpha_n) \\ & \vdots & \\ \delta^{n-1}(\alpha_1) & \dots & \delta^{n-1}(\alpha_n) \end{vmatrix} = 0.$$

As $\alpha_1, \dots, \alpha_n \in F$, we can apply Lemma 3.9 again and thus $\alpha_1, \dots, \alpha_n \in F$ are linearly dependent over F_0. Thus E_0 and F are linearly disjoint over F_0. As $F_0[E^p] \subset E_0$, $F_0[E^p]$ and F are linearly disjoint over F_0. Thus as E^p and F_0 are linearly disjoint over F^p, it follows that E^p and F are linearly disjoint over F^p. We show this implies separability. Let $(\alpha_i)_{i \in I} \subset E$ be algebraically dependent. We can assume that $|I| = n \leq \infty$. Consider $E' = F(\alpha_1, \dots, \alpha_n)$. As E^p and F are linearly disjoint over F_0 by supposition, it follows that $(E')^p$ and F are linearly disjoint over F_0 as $E' \subset E$. By Theorem 8.37 in [Jac80], we can find $\{\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}\} \subset \{\alpha_1, \dots \alpha_n\}$ such that $\{\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}\}$ is a finite separating transcendence basis for E' over F. Then E is separably algebraic over $F(\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k})$ and $k \neq n$ as $\{\alpha_1, \dots \alpha_n\}$ is algebraically dependent. Thus we can find α_j such that α_j is separably algebraic over $F(\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k})$ and thus $\{\alpha_1, \dots \alpha_n\}$ is separably dependent over F and E/F is thus separable. □

4. Non-degenerate Derivations

When looking at polynomials over a infinite field, we know that every non-zero polynomial has non-solutions. However, with differential fields, this is not guaranteed. For instance, letting F be a field with trivial derivation, the differential polynomial $f(X) = x_1$ does not have a non-solution as $\delta(\alpha) = 0$ for all $\alpha \in F$. A differential field (F, δ) is called **non-degenerate** if for every non-zero $f(X) \in F\{X\}$, there exists some $\alpha \in F$ such that $f(\alpha) \neq 0$. If for some f no such α exists, then (F, δ) is called **degenerate**.

Proposition 4.1. *Suppose (F, δ) is non-degenerate. If E/F is a differential field extension, then for non-zero $f(X_1, \dots, X_n) \in E\{X_1, \dots, X_n\}$, there exists an n-tuple $(b_1, \dots, b_n) \in F^n$ such that $f(b_1, \dots, b_n) \neq 0$*

Proof. We prove this by proving that if $g(b_1, \dots, b_n) = 0$ for all $(b_1, \dots, b_n) \in F^n$, then $g \equiv 0$ and we proceed by induction on n.

When $n = 1$, consider $g(X_1) \in E\{X_1\}$ such that g vanishes on F. Since E is a vectorspace over F, we can find a basis $(b_i)_{i \in I}$. Then we can write $g(X_1) = \sum_{i \in I} g_i(X_1) b_i$ with $g_i \in F\{X_1\}$. Since $g(X_1) \equiv 0$ on F, it thus follows that $g_i \equiv 0$ over F. Since $g_i \in F\{X_1\}$, $g_i \equiv 0$ and thus $g \equiv 0$.

Suppose that the statement holds for some $n \in \mathbb{N}$. and consider $g(X_1, \dots, X_{n+1})$ in $E\{X_1, \dots, X_{n+1}\}$ such that g vanishes on F^{n+1}. Fix $X_i = a_i$ for $i \leq n$ and some $a_i \in F$ and consider $\bar{g}(X) \coloneqq g(a_1, \dots, a_n, X)$. Then $\bar{g}$ is a polynomial in $E\{X\}$ that vanishes over F and thus $\bar{g} \equiv 0$. Viewing $g(X_1, \dots, X_{n+1})$ as a polynomial with coefficients $g_i \in E\{X_1, \dots, X_n\}$, then as $\bar{g}(X) \equiv 0$ for all n-tuples in F^n by previous reasoning, it follows from the inductive hypothesis that $g_i(X_1, \dots X_n) \equiv 0$ for all $i \leq n$. Thus, $g \equiv 0$ as desired. □

Proposition 4.1 shows that if (F,δ) is non-degenerate so is any differential extension $(E,\delta)/(F,\delta)$.

Before showing a condition equivalent to a field being non-degenerate, we need the following lemma.

Lemma 4.2. *Let F be an infinite field of positive characteristic. Then F_0 is infinite.*

Proof. As $F \hookrightarrow F^p$ via the map $\varphi(x) = x^p$, F^p is infinite. For any $\alpha^p \in F^p$, $\delta(\alpha^p) = p\alpha^{p-1}\delta(\alpha) = 0$. Thus $F^p \subset F_0$ and the claim follows. □

Proposition 4.3. *A differential field (F,δ) is non-degenerate if and only if $[F : F_0] = \infty$.*

Proof. Suppose F/F_0 is a finite extension and let $s>[F : F_0]$. Define

$$p(x) := \begin{vmatrix} x^{(1)} & \dots & x^{(s)} \\ \delta(x^{(1)}) & \dots & \delta(x^{(s)}) \\ & \vdots & \\ \delta^{s-1}(x^{(1)}) & \dots & \delta^{s-1}(x^{(s)}) \end{vmatrix}.$$

Then $p(x) \neq 0$ but for every $\lambda \in F$, $p(\lambda) = 0$ by Lemma 3.9 since $\{\lambda, \delta(\lambda), \dots, \delta^{s-1}(\lambda)\}$ is a linearly dependent set. Thus (F,δ) is degenerate.

Conversely, suppose let $f(x_0,\dots,x_s) \neq 0$ be a given polynomial and let $\xi_0,\dots,\xi_s$ be a linearly independent set over F_0. We will show that for some $c_i \in F_0$, $\alpha = c_0\xi_0 + \dots c_s\xi_s$, $f(\alpha) \neq 0$. We proceed by induction and assume that the theorem holds for any polynomial in $x_0,\dots,x_s$ of total degree less than the total degree of f. The base case follows trivially when the degree of f is 0.

First, we consider when f were polynomial in $x_0^p,\dots,x_s^p$. Note that $[F : F_p] = \infty$ as $F_p \subset F_0$. Thus, Let $w_1, w_2, \dots$ be a basis of F/F^p Then we can write $f = f_1w_1 + f_2w_2 + \dots$ where $f_i \in F^p[x_0^p,\dots,x_s^p]$. Since $f \neq 0$, there exists some $f_i(X) \neq 0$. By the inductive hypothesis, there exists $c_i \in F_0$, such that $f_i^{\frac{1}{p}}(c_0\xi_0 + \dots c_s\xi_s) \neq 0$. Thus $f_i(c_0\xi_0 + \dots c_s\xi_s) \neq 0$ and thus nor is f.

Now, consider f such that f is not a polynomial in $x_0^p,\dots,x_s^p$. Thus, there exists some i such that $\frac{\partial f}{\partial x_i} \neq 0$. Suppose, toward contradiction that $f(c_0\xi_0 + \dots c_s\xi_s) = 0$ for all $(c_0,\dots c_s) \in F_0^{s+1}$. Let $\mathbb{F}_p$ denote the prime field of F. Suppose, toward contradiction that ξ is algebraic over $\mathbb{F}_p$ and let $g(x)$ be the minimal polynomial of least degree satisfied by ξ. Since $\mathbb{F}_p$ is a perfect field, $g(x)$ is separable and thus $g'(\xi)\xi' = 0$ implies that $\xi' = 0$, which is a contradiction as $\xi \notin F_0$. Let $f(x_0,\dots,x_s) = f(X)$ where x_is are algebraic variables and X is a differential variable. Let $t_0, t_1, \dots, t_s$ be algebraic indeterminates over F and consider $f(t_0\xi_0 + \dots t_s\xi_s) \in F[t_0,\dots,t_s]$. Note that $f \equiv 0$ in $F[t_0,\dots,t_s]$ as if not, by Lemma 4.1, since F_0 is infinite by Lemma 4.2, we can find $(c_0,\dots,c_s) \in F_0^{s+1}$ such that $f(c_0\xi_0 + \dots c_s\xi_s) \neq 0$, which is a contradiction. We can thus take partials of

$f(c_0\xi_0 + \dots c_s\xi_s) = 0$ with respect to each c_i. Thus we have for each c_j:

$$\begin{aligned} 0 &= \frac{\partial f}{\partial c_j} \\ &= \frac{\partial f}{\partial x_0}\frac{\partial x_0}{\partial c_j} + \dots + \frac{\partial f}{\partial x_s}\frac{\partial x_s}{\partial c_j} \\ &= \frac{\partial f}{\partial x_0}\xi_j + \dots + \frac{\partial f}{\partial x_s}\delta^s(\xi) \end{aligned}$$

Note that as $\frac{\partial f}{\partial x_i} \neq 0$, it follows from induction, that there exists some choice of $c_i \in F_0$ such that $\frac{\partial f}{\partial x_i}(c_0\xi_0 + \dots c_s\xi_s) \neq 0$. Thus,

$$\begin{vmatrix} \xi_0 & \dots & \xi_s \\ \delta(\xi_0) & \dots & \delta(\xi_s) \\ & \vdots & \\ \delta^s(\xi_0) & \dots & \delta^s(\xi_s) \end{vmatrix} = 0.$$

and it follows from Lemma 3.9, that $\{\xi_0, \dots, \xi_s\}$ are linearly dependent, which is a contradiction. Thus, there must exists some $c_i \in F_0$ such that $f(c_0\xi_0 + \dots c_s\xi_s) = 0$. □

Before finding an additional condition for a differential field (F, δ) being non-degenerate, we define $\text{Der } F := \{\delta : F \to F \mid \delta \text{ derivation}\}$. We define for derivations $\delta_1, \delta_2 \in \text{Der } F, c \in F$:

$$\begin{aligned} (\delta_1 + \delta_2)(x) &= \delta_1(x) + \delta_2(x) \\ (c\delta_1)(x) &= c\delta_1(x) \end{aligned}$$

First we verify that $\delta_1 + \delta_2, c\delta_1 \in \text{Der } F$:

Let $a, b \in F$

$$\begin{aligned} (\delta_1 + \delta_2)(a + b) &= \delta_1(a + b) + \delta_2(a + b) \\ &= \delta_1(a) + \delta_1(b) + \delta_2(a) + \delta_2(b) \\ &= (\delta_1 + \delta_2)(a) + (\delta_1 + \delta_2)(b) \end{aligned}$$

$$\begin{aligned} (\delta_1 + \delta_2)(ab) &= \delta_1(ab) + \delta_2(ab) \\ &= a\delta_1(b) + b\delta_1(a) + a\delta_2(b) + b\delta_2(a) \\ &= a(\delta_1 + \delta_2)(b) + b(\delta_1 + \delta_2)(a) \end{aligned}$$

$$\begin{aligned} (c\delta_1)(a + b) &= c\delta_1(a + b) \\ &= c\delta_1(a) + c\delta_1(b) + \delta_2(a) + \delta_2(b) \\ &= (c\delta_1)(a) + (c\delta_1)(b) \end{aligned}$$

$$\begin{aligned} (c\delta_1)(ab) &= c\delta_1(ab) \\ &= ca\delta_1(b) + cb\delta_1(a) \\ &= a(c\delta_1)(b) + b(c\delta_1)(a) \end{aligned}$$

It's easy to verify that $\text{Der } F$ is in fact a vector space over F. Thus, it makes sense to talk about linear independence of derivations over a differential field.

Lemma 4.4. *For a differential field (F, δ) such that char $F = p > 0$, δ^{p^i} is a derivation for F for all $i \in \mathbb{N}$.*

Proof. Let $a, b \in F$:

$$\delta^{p^i}(a+b) = \delta^{p^i}(a) + \delta^{p^i}(b)$$

$$\delta^{p^i}(ab) = \sum_{j=0}^{p^i} \binom{p^i}{j} \delta^j(a)\delta^{p^i-j}(b)$$

$$= a\delta^{p^i}(b) + b\delta^{p^i}(a)$$

as $p \mid \binom{p^i}{j}$ for all $0 < j < p^i$. □

Proposition 4.5. *For a differential field, (F, δ), when char $F = 0$, (F, δ) is non-degenerate if and only if δ is linearly independent, namely non-trivial. When char $F = p > 0$, (F, δ) is non-degenerate if and only if $\{\delta^{p^i}\}_{i \in \mathbb{N}}$ is linearly independent.*

Proof. To prove this statement, we utilize Proposition 4.3 and we prove the contrapositive. Suppose char $F = 0$ and suppose that δ is linearly dependent. Then $\delta \equiv 0$. Let $w \in F$. Then $\delta(w) = 0$ and thus $w \in F_0$ and $F = F_0$ and $\{1\}$ is a finite basis for F over F_0.

Conversely, suppose $[F : F_0] < \infty$. Then F/F_0 is a separable algebraic extension and by Corollary 2.6, as $\delta(F_0) \equiv 0$, $\delta(F) = 0$ as well.

Suppose char $F = p > 0$. Suppose $\{\delta^{p^i}\}_{i \in \mathbb{N}}$ is linearly dependent. Then there exists some set $\{\delta^{p_1}, \ldots, \delta^{p_n}\}$ that is linearly dependent. Thus, exists some $c_i \in F$ such that

$$c_1\delta^{p_1} + \ldots c_n\delta^{p_n} = 0$$

over F. Consider the differential polynomial $f(X) = \sum_{i=1}^n c_i\delta^{p_i}$. By construction, $f(\alpha) = 0$ for all $\alpha \in F$. Thus, by definition, (F, δ) is degenerate.

Conversely, suppose $[F : F_0] < \infty$, then there exists some finite basis, $\{v_1, \ldots, v_n\}$, of F over F_0. Consider the following matrix:

$$A = \begin{pmatrix} \delta(v_1) & \ldots & \delta(v_n) \\ \delta^p(v_1) & \ldots & \delta^p(v_n) \\ \vdots & & \vdots \\ \delta^{p^n}(v_1) & \ldots & \delta^{p^n}(v_n) \end{pmatrix}$$

If $\det(A) = 0$, then there exists some $\gamma_i \in F_0$, such that

$$\gamma_1 \begin{pmatrix} \delta(v_1) \\ \vdots \\ \delta(v_n) \end{pmatrix} + \ldots + \gamma_n \begin{pmatrix} \delta^{p^n}(v_1) \\ \vdots \\ \delta^{p^n}(v_n) \end{pmatrix} = 0.$$

Then for all $a \in F$, $\gamma_0\delta(a) + \ldots + \gamma_n\delta^{p^n}(a) = 0$ and the desired linear dependence follows.

If $\det(A) \neq 0$, then there exists some $\beta_i \in F_0$, such that

$$\beta_1 \begin{pmatrix} \delta(v_1) \\ \vdots \\ \delta(v_n) \end{pmatrix} + \ldots + \beta_n \begin{pmatrix} \delta^{p^n}(v_1) \\ \vdots \\ \delta^{p^n}(v_n) \end{pmatrix} = \begin{pmatrix} \delta^{p^{n+1}}(v_1) \\ \vdots \\ \delta^{p^{n+1}}(v_n) \end{pmatrix}.$$

Thus for all $a \in F$, $\delta^{p^{n+1}}(a) = \beta_0\delta(a) + \ldots + \beta_n\delta^{p^n}(a)$ and the desired linear dependence thus follows. $\square$

5. The Differential Primitive Element Theorem

In this section, we prove Kolchin's primitive element theorem [Kol73], whose proof is similar to the algebraic version.

Theorem 5.1. Differential Primitive Element (Kochin's version) *Assume (F, δ) is non-degenerate. Then every finitely generated differentially separable extension of F is generated by a single element.*

Proof. As in the algebraic case, it suffices to show that if α, β are differentially separable over F, then $F<\alpha, \beta> = F<\alpha + \lambda\beta>$ for some $\lambda \in F$.

Let t, y, z be differential indeterminates. Then by Lemma 3.3, $\alpha + t\beta$ is differentially separable over $F<t>$. By Proposition 3.4, there exists some $f \in F<t>\{y\}$ such that $f(\alpha + t\beta) = 0$ and $s_f(\alpha + t\beta) \neq 0$. Let n denote the order of f. By clearing denominators, we can find $g \in F\{y\}\{z\}$ such that

$$g(\alpha + t\beta, t) = 0$$

where the order of g is n. As $s_f(\alpha + t\beta) \neq 0$, $\frac{dg}{d(\delta^n y)}(\alpha + t\beta, t) \neq 0$. Taking the partial of g with respect to $\delta^n t$ yields:

$$\begin{aligned} 0 &= \sum_{i=0}^{n} \frac{dg}{d(\delta^i y)} \frac{d(\delta^i y)}{d(\delta^n t)}(\alpha + t\beta, t) + \sum_{i=0}^{n} \frac{dg}{d(\delta^i z)} \frac{d(\delta^i z)}{d(\delta^n t)}(\alpha + t\beta, t) \\ &= \beta \frac{dg}{d(\delta^n y)}(\alpha + t\beta, t) + \frac{dg}{d(\delta^n z)}(\alpha + t\beta, t) \end{aligned}$$

as $\frac{d(\delta^i y)}{d(\delta^n t)} = \beta$ when $i = n$ and 0 elsewhere and $\frac{d(\delta^i z)}{d(\delta^n t)} = 1$ when $i = n$ and 0 elsewhere. Then as $\frac{dg}{d(\delta^n y)}(\alpha + t\beta, t) \neq 0$ and (F, δ) is non-degenerate, there exists some $\lambda \in F$ such that $\frac{dg}{d(\delta^n y)}(\alpha + \lambda\beta, \lambda) \neq 0$. Thus $\beta \in F<\alpha + \lambda\beta>$ and thus so is α. $\square$

The following example suggest that the condition of (F, δ) is non-degenerate may be relaxed.

Example 5.2. Consider the field F with the trivial derivation and consider the field extension $F(t, s)$ where t, s are algebraic indeterminates. We equip the derivation, $\delta(F) \equiv 0, \delta(t) = 1$ and $\delta(s) = s$, which is a possible derivation by Corollary 2.7. Let $\alpha = t + s$. Note that $t = \alpha - \alpha' + 1$ and $s = \alpha' - 1$. Thus $F<t, s> = F<t + s>$

Note that the primitive element found in the previous example and the primitive element found in Kolchin's primitive element theorem is a linear combination of the generators. The following example shows a differential field where the primitive element is not simply a linear combination of the field extension's generators.

Example 5.3. Consider $\mathbb{Q}(x, y)$ where $\delta(\mathbb{Q}) = 0$, $\delta(x) = 1$, and $\delta(y) = 0$. Then $x = \frac{1}{2}\delta(x^2 + y)$. Thus $x \in \mathbb{Q}<x^2 + y>$ and $y \in \mathbb{Q}<x^2 + y>$. Thus $\mathbb{Q}(x, y) = \mathbb{Q}<x^2 + y>$. We show that there is no primitive element of the form $y + \lambda x$ for

$\lambda \in \mathbb{Q}$. Note $\delta(y + \lambda x) = \lambda$ and $\delta^2(y + \lambda x) = 0$. Thus if $x \in \mathbb{Q}<y + \lambda x>$, for some $a, b \in \mathbb{Q}$,

$$\begin{aligned} x &= a(y + \lambda x) + b\lambda \\ 0 &= ay + (a\lambda - 1)x + b\lambda \end{aligned}$$

which has no solution and $x \notin \mathbb{Q}<y + \lambda x>$ for any $\lambda \in \mathbb{Q}$. Thus no primitive element of the form $y + \lambda x$ exists.

5.1. **Pogudin's Differential Primitive Element.** In this section, the primitive element theorem put forth in [Pog19] is derived, which relaxes the condition presented in Kolchin's differential primitive element theorem, requiring only the field extension to be non-degenerate. The theorem is stated below. Note that in this subsection, **all fields are assumed to have characteristic 0**. Recall that in characteristic zero, (F, δ) being non-degenerate is the same as δ being trivial on F.

Theorem 5.4. *Suppose that $(E, \delta)/(F, \delta)$ is finitely differentially generated and that E is differentially algebraic over F. Then if (E, δ) is non-degenerate, there exists $\alpha \in E$ such that $E = F<\alpha>$.*

Before proving this theorem, we first begin with lemmas. Recall that $E[[x]]$ is defined as the power series ring with coefficients in E, where power series are elements of the form $\sum_{i=0}^{\infty} a_i x^n$ with $a_i \in E$ and x an algebraic indeterminate. In this section, for some $(E, \delta)/(F, \delta)$, we extend δ to $E[[x]]$ in the following way:

$$\delta\left(\sum c_i x^i\right) := \sum \delta(c_i) x^i$$

We also define

$$e^x := \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

and

$$e^{ax} := \sum_{i=0}^{\infty} \frac{(ax)^i}{i!}$$

for any $a \in E$.

Recall that $F[\frac{d}{dx}]$ is the ring of differential operators with constant coefficients in F, namely elements of the form:

$$a_0 + a_1 \frac{d}{dx} + a_2 \frac{d^2}{dx^2} + \ldots + a_n \frac{d^n}{dx^n}$$

for some $a_i \in F$.

Lemma 5.5. *Let $f = \sum c_i x^i \in F[[x]]$. Suppose that for some $D \in F[\frac{d}{dx}]$ with ord$(D) = m$,*

$$Df \equiv 0 \text{ and } [f]_m = 0$$

where $[f]_m = \sum_{i=0}^{m} c_i x^i$. Then $f \equiv 0$.

Proof. As $[f]_m = \sum_{i=0}^{m} c_i x^i = 0$, $c_i = 0$ for $0 \leq i \leq m$. Thus we can write $\sum_{i=m+1}^{\infty} c_i x^i$. As $D \in F[\frac{d}{dx}]$ of order m, we can write $D = \sum_{i=0}^{m} a_i \frac{d^i}{dx^i}$ for some

$a_i \in F$. We prove the remaining coefficients of f are 0 by induction.

$$\begin{aligned} 0 &= Df \\ &= \sum_{i=0}^{m} a_i \frac{d^i}{dx^i}\left(\sum_{i=m+1}^{\infty} c_i x^i\right) \\ &= c_{m+1} a_j \frac{d^j}{dx^j}(x^{m+1}) + \sum_{i=j+1}^{m} a_i \frac{d^i}{dx^i}\left(\sum_{i=m+1}^{\infty} c_i x^i\right) \end{aligned}$$

where a_j is the first non-zero coefficient in D. As $j < m+1$, $\frac{d^j}{dx^j} x^{m+1} \neq 0$. Thus $c_{m+1} = 0$. Suppose $c_{l+m} = 0$ for all $1 \leq l \leq k$. We show $c_{k+m+1} = 0$.

$$\begin{aligned} 0 &= Df \\ &= \sum_{i=0}^{m} a_i \frac{d^i}{dx^i}\left(\sum_{i=m+k1}^{\infty} c_i x^i\right) \\ &= c_{k+m+1} a_j \frac{d^j}{dx^j}(x^{k+m+1}) + \sum_{i=j+1}^{m} a_i \frac{d^i}{dx^i}\left(\sum_{i=m+1}^{\infty} c_i x^i\right) \end{aligned}$$

where a_j is the first non-zero coefficient in D. As $j < k+m+1$, $\frac{d^j}{dx^j} x^{k+m+1} \neq 0$. Thus $c_{k+m+1} = 0$ and $f \equiv 0$ by induction. □

Lemma 5.6. *Let (E, δ) be a non-degenerate differential field. Let $a \in E$ such that $\delta(a) \neq 0$. Then the set*

$$S_m := \{e^{ax}, \delta(e^{ax}), \delta^2(e^{ax}), \ldots, \delta^m(e^{ax})\} \subset E[[x]]$$

is linearly independent over E.

Proof. We proceed by induction where the base case follows immediately as A_1 is a non-zero singleton set. Suppose that S_n is linearly independent for n and consider S_{n+1}. It suffices to show that $\delta^{n+1}(e^{ax}) \notin S_n$ as by the inductive hypothesis, S_n is linearly independent over E. Let $V \subset E[x]$ denote the set of all polynomials of degree at most n. Note that

$$\delta^{n+1}(e^{ax}) \equiv (\delta(a)x)^{n+1} e^{ax} \pmod{V e^{ax}}$$

which is non-zero as $\delta(a) \neq 0$. Thus $\delta^{n+1}(e^{ax}) \notin V e^{ax}$. As $S_n \subset V e^{ax}$, the claim thus follows. □

Lemma 5.7. *Let $(E, \delta)/(F', \delta)$ be a differential field extension. Let $a \in E$ such that a is non constant. Suppose there exists some $c_i \in F'$ not all zero such that*

$$c_0[e^{ax}]_{2(m+1)} + c_1[(e^{ax})^{(1)}]_{2(m+1)} + \ldots + c_m[(e^{ax})^{(m)}]_{2(m+1)} = f$$

for some $f \in F'[x]$. Then $a \in F'$.

Proof. Consider

$$c_0 e^{ax} + c_1 (e^{ax})^{(1)} + \ldots + c_m (e^{ax})^{(m)}. \tag{5.1}$$

Note that this expression evaluates to a non-zero solution by Lemma 5.6 as the c_i are not all zero by supposition. Moreover it is of the form $C(x)e^{ax}$ for some $C(x) \in E[x]$ with $\deg(C) \leq m$ as each term in (5.1) has a factor of e^{ax} and a polynomial of degree at most m. Let $\bar{E}$ be an algebraic closure of E and let $\tau \in \mathrm{Aut}(\bar{E})$ such that $\tau_{|_{F'}} = \mathrm{id}$.

Let $S \coloneqq Ce^{ax} - \tau(C)e^{\tau(a)x}$. Since $[Ce^{ax}]_{2(m+1)} = f$, then $\tau([Ce^{ax}]_{2(m+1)}) = [\tau(C)e^{\tau(a)x}]_{2(m+1)} = f$. Thus $[S]_{2(m+1)} = 0$. We define $D \coloneqq (\frac{d}{dx} - a)^{m+1}(\frac{d}{dx} - \tau(a))^{m+1}$. At this point, Pogudin claims $DS \equiv 0$ but does not give details. When trying to prove it, the case when $m = 1$ worked out nicely; however when trying to look beyond, there was no elegant way to preform this computation. By Lemma 5.5, $S \equiv 0$. Thus $e^{ax}, e^{\tau(a)x}$ are linearly dependent over $E[x]$ and $a = \tau(a)$. Thus $a \in F'$ as otherwise there would exist some $\tau \in \mathrm{Aut}(\bar{E})$ such that $\tau_{|_{F'}} = \mathrm{id}$ and $\tau(a) \neq a$. □

We now prove Theorem 5.1:

Proof. Assume (F, δ) is degenerate (as if it's not, the desired claim follows from Kolchin's primitive element theorem). Note that it suffices to show this statement holds for $E = F{<}a_0, a_1{>}$ as the general case follows by induction and thus this argument is a minor simplification of that seen in Pogudin. Moreover, we can assume that one of the a_i are non-constant, as if both were, then $E = F(a_0, a_1)$ and thus if both a_i are constants, (E, δ) is degenerate. Without loss of generality, we can assume a_0 is non-constant. Since a_i are differentially algebraic over F, $m \coloneqq \mathrm{trdeg}_F E < \infty$.

Let $\Theta \coloneqq \{\vartheta_j \mid -1 \leq j \leq 2(m+1)\}$ be a set of algebraic indeterminates and extend the derivation on E to $E[\Theta]$ by setting $\delta(\Theta) = 0$.

Let $\alpha \coloneqq \vartheta_{-1}a_1 + \sum_{i=0}^{2(m+1)} \frac{\vartheta_i}{i!} a_0^i$. As $\mathrm{trdeg}_{F(\Theta)} F(\Theta, \alpha, \alpha^{(1)}, \ldots, \alpha^{(m)}) \leq \mathrm{trdeg}_F E = m$, $\{\alpha, \alpha^{(1)}, \ldots, \alpha^{(m)}\}$ are algebraically dependent over $F(\Theta)$. Thus we can find $R \in F(\Theta)\{Z\}$ such that $R(\alpha) = 0$ and $\mathrm{ord}(R) \leq m$. We assume R is of minimal degree.

Let

$$R_i \coloneqq \frac{dR}{dz^{(i)}}(\alpha) \qquad \text{for } 0 \leq i \leq m$$
$$R_{\vartheta_j} \coloneqq \frac{dR}{d\vartheta_j}(\alpha) \qquad \text{for } -1 \leq j \leq 2(m+1)$$

As R was chosen to be minimal, there exists some i such that $R_i \neq 0$. We differentiate $0 = R(\alpha)$ with respect to ϑ_j

$$\begin{aligned} 0 &= \sum_{i=0}^{m} \frac{dR}{dz^{(i)}}(\alpha)\frac{dz^{(i)}}{d\vartheta_j}(\alpha) + \sum_{k=-1}^{m} \frac{dR}{d\vartheta_k}(\alpha)\frac{d\vartheta_k}{d\vartheta_j}(\alpha) \\ &= \left[\sum_{i=0}^{m} R_i \frac{(a_0^j)^{(i)}}{j!}\right] + R_{\vartheta_j} \end{aligned}$$

Thus $\sum_{i=0}^{m} R_i \frac{(a_0^j)^{(i)}}{j!} = -R_{\vartheta_j}$. Summing over all $j \geq 0$ and multiplying each term by x^j yields:

$$\sum_{i=0}^{m} R_i[(e^{a_0 x})^{(i)}]_{2(m+1)} = \sum_{j=0}^{2(m+1)} -R_{\vartheta_j} x^j.$$

As not all R_i are zero, with $F' = F{<}\Theta, \alpha{>}$ and $a = a_0$, by Lemma 5.7, $a_0 \in F{<}\Theta, \alpha{>}$. Thus we can find $p_1, p_2 \in F[\Theta]\{Z\}$ such that

$$a_0 = \frac{p_1(\alpha)}{p_2(\alpha)}.$$

Let $\varphi \in Q^{|\Theta|}$ and regard it as a function from $E[\Theta]$ into E and let

$$U_1 \coloneqq \{\varphi \in \mathbb{Q}^{|\Theta|} \ \mid \ \varphi(p_2(\alpha)) \neq 0 \text{ and } \varphi(\vartheta_{-1}) \neq 0\}.$$

As $F<\Theta, \alpha>$ is non-degenerate (as for instance, $a_0 \in F<\Theta, \alpha>$), by Proposition 4.1, U_1 is non-empty. For any $\varphi \in U_1$, as $\varphi(\vartheta_{-1}) \neq 0$, $a_1 \in F<\varphi(\alpha)>$ and thus $F<\varphi(\alpha)> = F<a_0, a_1>$ and thus a differential primitive element has been found. □

5.2. **Extending to positive characteristic.** We spent many weeks trying to translate Pogudin's argument to positive characteristic; in attempting to do so, we first notice the following lemma, which says for a differential field extension in positive characteristic, if the extension is non-degenerate, so is the base field

Lemma 5.8. *Let $(E, \delta)/(F, \delta)$ be a differential field extension and $\operatorname{char} F = p > 0$. Furthermore suppose (E, δ) is differentially separable over F and is finitely differentially generated. Then if (E, δ) is non-degenerate so is (F, δ).*

Proof. As (E, δ) is differentially separable over F and is finitely differentially generated, by Theorem 3.8, $E = F(\alpha_1, \dots, \alpha_n)$ for some $\alpha_i \in E$. Suppose that (F, δ) is degenerate. Then $[F : F_0] = m$ for some $m \in \mathbb{N}$ by Lemma 4.3. Note that E/E_0 is an algebraic extension as for any $\alpha \in E$, $\alpha^p \in E_0$ and thus α is a root of $x^p - \alpha^p \in E_0[x]$. Since E/F is a finitely generated algebraic extension, by a repeated application of the tower law, it follows that E/F is a finite extension. Thus $[E : F_0] = [E : F][F : F_0]$ is finite as well. As $F_0 \subset E_0$, $[E : F_0] = [E : E_0][E_0 : F_0]$. Since $[E : F_0]$ is finite, it follows that $[E : E_0]$ is finite and (E, δ) is thus degenerate by Lemma 4.3. □

Thus the case of (E, δ) being non-degenerate for positive characteristic is equivalent to Kolchin's differential primitive element theorem. If however, the statement was amended to (E, δ) having a non-constant element (which in the case of zero characteristic is equivalent to (E, δ) being non-degenerate as per Proposition 4.5), once could attempt to translate Pogudin's argument to the case of characteristic $p \neq 0$ when (E, δ) has a non-constant element. In order to do so, one needs to come up with an analog for the exponential $e^x \coloneqq \sum \frac{x^n}{n!}$ in fields with positive characteristic as the traditional definition fails as it requires dividing by factors of p. There is however, no natural obvious way to translate a power series definition in characteristic zero to positive characteristic. When attempting to derive an analog, we looked at defining $e^x \coloneqq \sum_{i=0}^{\infty} \frac{x^{q^i}}{q^i}$ for a prime $q \neq p$ was considered; however, there was no analog of lemma 5.5 as for instance, derivatives could vanish without the coefficients being zero.

The following example shows a case in which a positive characteristic analog of the differential primitive element theorem put forth by Pogudin fails. More precisely, it shows a differential primitive element is not guaranteed if the extension has a non-constant element.

Example 5.9. Consider the field $F = \mathbb{F}_p(x)$ with the derivation $\delta(x) = 1$ and the extension $E = F(y_0, y_1, \dots, y_p)$ with $\delta(y_i) = 0$ for all i. Note that $\operatorname{trdeg}_F E = p + 1$ Assume, toward contradiction, that $E = F<\alpha>$ for some $\alpha \in E$. Note that $\delta^p \equiv 0$ on E as $\delta^p \equiv 0$ on F and $\delta^p(y_i) = 0$ for all i. Thus $E = F(\alpha, \delta(\alpha), \dots, \delta^{p-1}(\alpha))$. Thus $\operatorname{trdeg}_F E \leq p$ which is a contradiction and so there exists no differential primitive element.

6. The Differential Basis Theorem Over Differentially Perfect Fields

In this section, we develop the machinery needed to prove a differential version of the Hilbert basis theorem, specifically for differential fields in arbitrary characteristic.

6.1. **Differentially Perfect Fields.**

In the following subsection, we discuss what it means for a field to be differentially perfect as well as show that for a differentially perfect field F and any ideal $P \subset F\{X\}$, we can find $f \in P$ of non-zero minimal rank that has a non-zero separant, which has importance when proving the differential analog of the Hilbert basis theorem in the subsequent subsection.

A field (F, δ) is *differentially perfect* if every differential field extension of F is separable.

Proposition 6.1. *Let (F, δ) be a field with characteristic p. F is differentially perfect if and only if $p = 0$ or $p \neq 0$ and $F_0 = F^p$.*

Proof. We may assume $p \neq 0$. Suppose $F \neq F_p$. Then there exists some $\gamma \in F_0$ such that $\gamma \notin F^p$. Consider the prime differential ideal $<x^p - \gamma> \subset F\{y\}$. Then $F\{X\} /_{<x^p - \gamma>}$ is a non-separable extension of F and F is thus not differential perfect. Conversely suppose $F_0 = F^p$ and let E be an extension of F. By Lemma 3.10, E is separable over F. ☐

We now aim to show the following proposition:

Proposition 6.2. *Let (F, δ) be differentially perfect. If $P \neq 0$ is a prime differential ideal of $F\{X\}$ and if $f \in P$ is non-zero of minimal rank, then $s_f \not\equiv 0$.*

Note that the above proposition and its proof, to our knowledge, has not been written up before and is key in proving the differential version of the Hilbert basis theorem present in this dissertation. In order to prove the proposition, we need two lemmas, the first which is Theorem 2.14 in [Jac85].

Lemma 6.3. *Let f and $g \neq 0$ be polynomials in $R[x]$ where R is a ring. Let n be the degree and b_n the leading coefficient of g. Then there exists some $m \in \mathbb{N}$ and polynomials $q, r \in R[x]$ with $\deg r < \deg g$ such that*

$$b_n^m f = qg + r.$$

Lemma 6.4. *Let $P \neq 0$ be a prime differential ideal of $F\{X\}$ and let $f \in P$ be non-zero of minimal rank n. If*

$$\frac{df}{dx_i}, \ldots, \frac{df}{dx_n} \in P,$$

then $\frac{df}{dx_{i-1}} \in P$.

Proof. Recall we denote $s_f := \frac{df}{dx_n}$. Assume

$$\frac{df}{dx_i}, \ldots, \frac{df}{dx_n} = s_f \in p.$$

As $\operatorname{rank}(s_f) < \operatorname{rank}(f)$, $s_f \equiv 0$. By Lemma 3.5,

$$\frac{d(\delta f)}{dx_i} + \frac{df}{dx_{i-1}} = \delta\left(\frac{df}{dx_i}\right).$$

To show $\frac{df}{dx_{i-1}} \in P$, it thus suffices to show $\frac{d(\delta f)}{dx_i} \in P$.

Note $\text{ord}(df) = \text{ord}(f)$ as $df = f^\delta + \sum_{j=0}^{n-1} \frac{df}{dx_j} x_{j+1} + s_f x_{n+1}$ and $s_f = 0$. Let $R = F[x_0, \ldots, x_{n-1}]$. Note that $f, \delta f \in R[x_n]$. Then, by Lemma 6.3, there exists $g, r \in R[x_n]$ with $\deg(r) < \deg(f)$ and $m \in \mathbb{N}$ such that

$$i_f^m \delta f = gf + r$$

where $i_f \in R$ is the leading coefficient of f. As $df, f \in P$, it thus follows that $r \in P$. As f is of minimal rank, $r = 0$. Thus $i_f^m \delta f = gf$. Taking partials yields

$$\frac{di_f^m}{dx_i}\delta f + i_f^m \frac{d(\delta f)}{dx_i} = \frac{dg}{dx_i} f + \frac{df}{dx_i} g$$

As each other term is in P, $i_f^m \frac{d(\delta f)}{dx_i} \in P$. Since P prime and $i_f^m \notin P$ (as $i_f^m \neq 0$ and is of lesser rank than f), $\frac{d(\delta f)}{dx_i} \in P$ □

We now prove Proposition 6.2:

Proof. Let Q denote the field of fractions of ${}^{F\{X\}}/_{P}$. As $(F, \delta) \subset Q$, Q is separable over F. Let $a = x + P \in Q$, then $f(a) = 0$ and a is differentially algebraic. We can thus write $F<a> = Q$. Since $F<a>$ is separable over F, P is equal to the defining differential ideal of a. Let $f \in P$ be non-zero of minimal rank. Since $F<a>$ is separable over F and f is of minimal rank, we know there exists some $i \in \mathbb{N}$ such that $\frac{df}{dx_i}(a) \neq 0$. Thus $\frac{df}{dx_i} \notin P$ and by the contrapositive to Lemma 6.4, $s_f \neq 0$. □

6.2. **The Differential Basis Theorem.** Before beginning this section, recall that a radical ideal I is an ideal such that $I = \sqrt{I}$ where $\sqrt{I} := \{a \mid \exists n \quad a^n \in I\}$. A radical differential ideal is an ideal I such that I is both a radical and a differential ideal.

We begin this section by developing the machinery needed to prove a differential analog of the Hilbert's basis theorem. First we note that for a differential field F, not every ideal of $F\{X\}$is finitely generated; for instance, the ideal $(x, \delta(x), \delta^2(x), \ldots)$ is not. Moreover, not every differential ideal is finitely generated, as seen in the following example:

Example 6.5. Let (F, δ) be a differential field with zero characteristic. Then the differential ideal $<x^2, (\delta(x))^2, \left(\delta^3(x)\right)^2, \ldots>$ is not finitely generated.

When trying to extend this theorem to a differential field context, a natural extension may be "For any field F, every radical ideal of $F\{X\}$ is finitely generated." The following example from [Sei52], however, shows that this is not true.

Example 6.6. Let $F = \mathbb{F}_p(t_0, t_1, t_2, \ldots)$ be equip with trivial derivation and consider the differential ideal $I = <x^p - t_0, (x^{(1)})^p - t_1, (x^{(2)})^p - t_2, \ldots>$, which is not finitely generated.

Note that in the previous example, $F_0 \neq F^p$ and thus by Proposition 6.1, F is not a differentially perfect field.

The following argument has been developed from [Mar96]. In the majority of the following proofs, Proposition 6.2 is implicitly used to guarantee that the separant is non-zero.

Lemma 6.7. *Let (F, δ) be a differentially perfect field and let f be of non-zero minimal rank of some prime differential ideal $P \subset F\{X\}$. Then $f^{(k)} = s_f x_{n+k} + f_k$ for some $f_k \in F\{X\}$ of order at most $n + k - 1$ and $k \geq 1$*

Proof. Let $f = \sum_{i=0}^{m} h_i(x_n)^i$ where h_i have order at most $n - 1$. Then

$$\begin{aligned} f' &= \sum_{i=0}^{m} h_i'(x_n)^i + \sum_{i=1}^{m} i h_i(x_n)^{i-1} x_{n+1} \\ &= s_f x_{n+1} + f_1 \end{aligned}$$

where $f_1 = \sum_{i=0}^{m} h_i'(x_n)^i$. Suppose we have $f^{(k)} = s_f x_{n+k} + f_k$ where f_k has order at most $n + k - 1$. Then

$$\begin{aligned} f^{(k+1)} &= s_f' x_{n+k} + s_f x_{n+k+1} + f_k' \\ &= s_f x_{n+k+1} + f_{k+1} \end{aligned}$$

where $f_{k+1} = f_k' + s_f' x_{n+k}$ and has order at most $n + k$ as desired. □

Lemma 6.8. *Let (F, δ) be a differentially perfect field and let f be of non-zero minimal rank of some prime differential ideal $P \subset F\{X\}$. For any differential polynomial $g \in F\{X\}$, there exists some g_1 of order at most n such that for some $m \in \mathbb{N}$, $s_f^m g \equiv g_1 \bmod {<}f{>}$.*

Proof. Suppose g has order $n + k$ where $k \geq 1$ and suppose the lemma holds for all h of lower rank than g. As per Lemma 6.7, we can write $f^{(k)} = s_f x_{n+k} + f_k$. Let the degree of g be denoted by m. Thus we can write $g = \sum_{i=0}^{m} h_i(x_{n+k})^i$. Let $g_j = s_f^m g - (f^{(k)})^m h_m$. Then $s_f^m g \equiv g_j \bmod {<}f{>}$. Note that g_j is of lower rank than g as if $m = 1$ then g_j is of lower order and when $m \neq 1$, g_j is of lower degree. Thus, by induction it follows that such g_1 exists. □

Lemma 6.9. *Let (F, δ) be a differentially perfect field and let f be of non-zero minimal rank of some prime differential ideal $P \subset F\{X\}$. Denote the order of f by n. Then $f = \sum_{i=0}^{m} h_i(x_n)^i$ where h_i have order at most $n - 1$. For any $g \in F\{X\}$, there exists some $g_2 \in F\{X\}$ of lower rank than f and some $k, t \in \mathbb{N}$ such that*

$$h_i^k s_f^t g \equiv g_2 \bmod {<}f{>}.$$

Proof. By Lemma 6.8, there exists some $g_1 \in F\{X\}$ such that $s_f^m g \equiv g_1 \bmod {<}f{>}$. It follows from Lemma 6.3 that there exists some $g_2 \in F\{X\}$ with degree less than m such that $h_i^k g_1 = g_2 \bmod {<}f{>}$ and the claim thus follows. □

Lemma 6.10. *Suppose I is a radical differential ideal. Let $ab \in I$. Then $a\delta(b), \delta(a)b \in I$.*

Proof. As $ab \in I$, then $\delta(ab) = a\delta(b) + \delta(a)b \in I$. Thus $\delta(a)\delta(b)ab + (\delta(a)b)^2 \in I$. As I is radical, $\delta(a)b \in I$. Likewise $(a\delta(b))^2 + \delta(a)\delta(b)ab \in I$ and thus $a\delta(b) \in I$. □

Lemma 6.11. *Let I be a radical differential ideal and let $S \subset F$ be multiplicatively closed. Let $T \coloneqq \{x \in F \mid xS \subset I\}$. Then T is a radical differential ideal.*

Proof. T is clearly an ideal and by Lemma 6.10, T is a differential ideal. Suppose $x^n \in T$. Let $s \in S$. As S is multiplicatively closed, $s^n \in S$. Thus $x^n s^n \in I$. Since I is radical, $xs \in I$. As this holds for all $s \in S$, $x \in T$ and T is thus radical. □

Lemma 6.12. *For $S \subset F$, let $\{S\}$ denote the smallest radical differential ideal containing S. Then for $a \in F$, $a\{S\} \subset \{aS\}$.*

Proof. Let $W := \{a^i \mid i \in \mathbb{N}\}$. Let $T := \{x \in F \mid xW \subset \{aS\}\}$. By Lemma 6.11, T is a radical differential ideal. As $S \subset T$, $\{S\} \subset T$ and the claim follows. ☐

Lemma 6.13. *Let $S, T \subset F$. Then $\{S\}\{T\} \subset \{ST\}$.*

Proof. Let $U := \{x \in F \mid x\{T\} \subset \{ST\}\}$. By Lemma 6.11, U is a radical differential ideal. As S and thus $\{S\} \subset U$, it follows from Lemma 6.12 that $\{S\}\{T\} \subset \{ST\}$. ☐

Lemma 6.14. *Let (F, δ) be a differential field and consider some $S \subset F\{X\}$. For $\alpha \in \{S\}$, there exists some $T \subset S$ such that T is finite and $\alpha \in \{T\}$.*

Proof. Let $S_0 = S$ and $S_{i+1} = \sqrt{<S_i>}$ for $i \geq 1$. Then $\{S\} = \cup S_i$. Let $\alpha \in \{S\}$. Then for some i, $\alpha \in S_i$. Thus it suffices to show that for all $\beta \in S_i$, there exists some finite $T \subset S$ such that $\beta \in \{T\}$. We proceed by induction on i. When $i = 0$, $\beta \in S_0$ and $T = \{\beta\}$. Now suppose that such T exists for all $\beta \in S_i$ for all $i \leq k$ and consider $\beta \in S_{k+1} = \sqrt{<S_k>}$. As $\sqrt{<S_k>}$ is the radical of $<S_k>$, there exists some $n \in \mathbb{N}$ such that $\beta^n \in <S_k> = \{\delta^j s_k |\ s_k \in S_k, j \in \mathbb{N}\}$. Thus $\beta^n = \sum h_{jk}\delta^j(s_k)$ for some $h_{jk} \in S_k$. By the inductive hypothesis as $s_k \in S_k$, there exists finite $T_k \subset S_k$ such that $s_k \in \{T_j\}$. Taking $T = \cup T_k$, it follows that $\beta^n \in \{T\}$. As $\{T\}$ is radical, $\beta \in \{T\}$ ☐

Remark 6.15. In characteristic zero, $\{S\} = S_1$, which is seen in Lemma 1.15 in [Mar96].

Theorem 6.16. *Let F be a differentially perfect field such that every radical differential ideal is finitely generated. Then every radical differential ideal in $F\{X\}$ is finitely generated.*

Proof. Suppose not, then by Zorn's Lemma, there exists a radical differential ideal I which is not finitely generated and is maximal among the not finitely generated radical differential ideals. We first show that I is prime. Let $ab \in I$ and suppose $a, b \notin I$. Then $\{a, I\}$ and $\{b, I\}$ are larger radical differential ideals than I and thus are finitely generated. By Lemma 6.14, we can find $c_1, \ldots, c_r, d_1, \ldots, d_s \in I$ such that $\{a, I\} = \{a, c_1, \ldots, c_r\}$ and $\{b, I\} = \{b, d_1, \ldots, d_s\}$. By Lemma 6.13, $\{a, I\}\{b, I\} \subset \{ab, c_1d_1, \ldots, c_rd_s\} \subset I$ as $ab \in I$. Let $z \in I$. Then $z^2 \in \{a, I\}\{b, I\} \subset \{ab, c_1d_1, \ldots, c_rd_s\}$. As the latter is a radical ideal, $z \in \{ab, c_1d_1, \ldots, c_rd_s\}$. Thus $I = \{ab, c_1d_1, \ldots, c_rd_s\}$. As I is not finitely generated, this is a contradiction. Thus I is prime.

Let J be the finitely generated radical differential ideal of $F\{X\}$ generated by $I \cap F$ and let $f \in I - J$ be of non-zero minimal rank. As $I - J$ is prime and differential, $s_f \neq 0$ by Proposition 6.2. We can write $f = h_m(x_n)^m + f_0(X)$ where f_0 is of lower rank than f. Note $h_m \notin I$ as otherwise $f_0 \in I$, contradicting the choice of f. Moreover, $s_f \notin I$ as otherwise $s_f \in J$ (as it is of lesser rank than f) and thus $f - \frac{1}{m}x_n s_f \in I - J$ which would contradict the minimality of f. As I is prime, $h_m s_f \notin I$. Thus $\{h_m s_f, I\}$ is a larger radical differential ideal than I and thus is finitely generated. We can thus find $c_1, \ldots, c_d \in I$ such that $\{h_m s_f, I\} = \{h_m s_f, c_1, \ldots, c_d\}$ by Lemma 6.14.

Let $g \in I$. By Lemma 6.8, there exists $k, t \in \mathbb{N}$ such that $h_m^k s_f^t g \equiv g_1 \bmod <f>$ where g_1 is of lesser rank than f. Thus $g_1 \in I$ and as g_1 is of lesser rank than f, it thus follows that $g_1 \in J$. Then $h_m^k s_f^t g \in \{J, f\}$. As $(h_m s_f g)^{\max(k,t)} \in \{J, f\}$ and

$\{J, f\}$ is radical, $h_m s_f g \in \{J, f\}$. Thus $(h_m s_f)I \subset \{J, f\}$. Thus

$$\begin{aligned} I &\subset I\{(h_m s_f g), I\} \\ &\subset \{(h_m s_f g)I, Ic_1, \ldots, Ic_d\} \\ &\subset \{J, f, c_1, \ldots, c_d\} \end{aligned}$$

Let $z \in I$. Then, by above, $z \in \{J, fc_1, \ldots, c_d\}$. Thus $I = \{J, f, c_1, \ldots, c_d\}$ and as J is finitely generated, I is finitely generated, which is a contradiction. Thus every radical differential ideal in $F\{X\}$ is finitely generated. □

In the presented version of the differential basis theorem, we imposed conditions on the differential field, requiring it to be differentially perfect. It is worth noting that Kolchin gives a necessary and sufficient condition for the differential basis theorem with respect to all radical differential ideals.

Theorem 6.17. *Let F be a differential field. Then $[F_0 : F^p] = n < \infty$ if and only if every radical differential ideal in $F\{X\}$ is finitely generated.*

Notice that the differential basis theorem presented in this dissertation falls out as an immediate corollary to Kolchin's as when F is differentially perfect, $F^p = F_0$ and thus $[F_0 : F^p] = 1$.

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