Solving Systems of Equations Involving the j Function

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Motivation

Find conditions that ensure that an algebraic variety $V \subseteq \mathbb{C}^{2n}$ has a (generic) point of the form $(z_1, \ldots, z_n, j(z_1), \ldots, j(z_n))$.

The j function
Definition

Given $\tau \in \mathbb{H}$, it defines the elliptic curve $E_\tau := \mathbb{C}/\langle 1, \tau \rangle_{\mathbb{Z}}$.

E_τ can be realised as an algebraic curve by a polynomial of the form

$$E_\tau = \left\{ Y^2 = 4X^3 - aX - b \right\}, \text{ where } a^3 - 27b^2 \neq 0.$$

The *j-invariant* of this elliptic curve is defined as:

$$j(E_\tau) := 1728 \frac{a^3}{a^3 - 27b^2}.$$

We define the *j function* $j : \mathbb{H} \to \mathbb{C}$ by $j(\tau) = j(E_\tau)$, which is a surjective holomorphic map that classifies isomorphism classes of elliptic curves.

$\mathrm{GL}_2(\mathbb{Q})^+$ acts on $\mathbb{H}$ through Möbius transformations.

$\{\Phi_N(X, Y)\}_{N\in\mathbb{N}^+} \subset \mathbb{Z}[X, Y]$ denotes the family of *modular polynomials*.

$\Phi_1(X, Y) = X - Y$ and $\Phi_N(X, Y)$ is symmetric for $N \geq 2$.

For every $x, y \in \mathbb{H}$ the following statements are equivalent

(M1): $\Phi_N(j(x), j(y)) = 0$,

(M2): $gx = y$ for some $g \in \mathrm{GL}_2(\mathbb{Q})^+$ such that, if $\widetilde{g}$ is obtained by re-scaling g so that the entries of $\widetilde{g}$ are all integers and relatively coprime, then $\det(\widetilde{g}) = N$.

In particular, $j(z) = j(gz)$ if and only if $\widetilde{g} \in \mathrm{SL}_2(\mathbb{Z})$.

Some important examples

Let $E_j^n := \{(z_1, \ldots, z_n, j(z_1), \ldots, j(z_n))\} \subseteq \mathbb{H}^n \times \mathbb{C}^n$. We will always think of varieties $V \subseteq \mathbb{C}^{2n}$ as being defined over the ring of polynomials $\mathbb{C}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$.

Example

Let us look at some examples in which $E_j^n \cap V = \emptyset$.

1. Let $V \subseteq \mathbb{C}^2$ be the plane curve defined by the equation $\{X = r\}$, for some $r \in \mathbb{R}$. As j is not defined over the real line, then $E_j^n \cap V = \emptyset$.
2. Choose $g \in G$, and let $N = \det(\widetilde{g})$. Let $V \subset \mathbb{C}^4$ be defined by $\{X_1 = gX_2; \Phi_N(Y_1, Y_2) + 1 = 0\}$. By the equivalence of (M1) and (M2), V cannot intersect E_j^n.

j-generic varieties

Definition

An algebraic variety $V \subseteq \mathbb{C}^{2n}$ is *j-generic* if it is **not** contained in any hypersurface of one of the following forms:

(a) $X_i = c$ for some $c \in \mathbb{C}$.

(b) $Y_i = c$, for some $c \in \mathbb{C}$.

(c) $X_i = gX_j$ for some $g \in \mathrm{GL}_2(\mathbb{Q})^+$, with $i \neq j$.

(d) $\Phi_N(Y_i, Y_j) = 0$ for some positive integer N, with $i \neq j$.

Motivating Questions

Let $\pi : \mathbb{C}^{2n} \to \mathbb{C}^n$ be the projection onto the first n coordinates.

Conjecture (Existence of Solutions (ES))

For each variety $V \subseteq \mathbb{C}^{2n}$ with $\pi(V)$ Zariski dense in $\mathbb{C}^n$, $E_j^n \cap V \neq \emptyset$.

Conjecture (Existence of Generic Solutions (EGS))

For each j-generic variety $V \subseteq \mathbb{C}^{2n}$ with $\pi(V)$ Zariski dense in $\mathbb{C}^n$, and every finitely generated subfield K of $\mathbb{C}$, there exists $(\overline{z}, j(\overline{z})) \in E_j^n \cap V$ such that $\mathrm{tr.deg.}_K(\overline{z}) = \dim V$.

Finding Solutions

The crucial step

Proposition

Let $U \subseteq \mathbb{C}^n$ be a (connected) domain such that $U \cap \mathbb{R}^n \neq \emptyset$, and let $p_1, \ldots, p_n : U \to \mathbb{C}$ be holomorphic functions. Then, the system of equations

$$\begin{aligned} j(z_1) &= p_1(z_1, \ldots, z_n) \\ &\vdots \\ j(z_n) &= p_n(z_1, \ldots, z_n) \end{aligned}$$

has infinitely many solutions in $U \cap \mathbb{H}^n$.

In fact, this proposition still hold if we replace j by any meromorphic automorphic functions.

Using the implicit function theorem, one can use the previous Proposition to eventually get:

Proposition

Let $W \subseteq \mathbb{C}^{2n}$ *be an algebraic variety and let* $\pi : \mathbb{C}^{2n} \to \mathbb{C}^{n}$ *be the projection onto the first* n *coordinates. If* $\pi(W)$ *is Zariski dense in* $\mathbb{C}^n$*, then the set* $\pi\left(E_j^n \cap W\right)$ *is Zariski dense in* $\mathbb{C}^n$*.*

Special Points

$z \in \mathbb{H}$ is *special* if there is $g \in \mathrm{GL}_2(\mathbb{Q})^+$ such that z is the unique fixed point of g in $\mathbb{H}$. $\bar{z} \in \mathbb{H}^n$ is *special* if every coordinate is special.

A theorem of Schneider says: $\mathrm{tr.deg.}_{\mathbb{Q}}(z, j(z)) = 0 \iff z$ is special.

$\bar{z} \in \mathbb{H}^n$ is *ordinary*, if no coordinate of $\bar{z}$ is special.

Given $x, y \in \mathbb{H}$ and $g \in G$ be such that $gx = y$, we denote by $g_{x,y}$ any element in G satisfying:

$$\det\left(\widetilde{g}_{x,y}\right) = \min_{g \in G}\left\{\det\left(\widetilde{g}\right) : gx = y\right\}.$$

Varieties of Triangular Form

Definition

An affine algebraic variety $V \subseteq \mathbb{C}^{2n}$ of dimension $d \geq n$ will be called *of triangular form* if it can be defined by polynomials

$$p_1, \ldots, p_{2n-d} \in \mathbb{C}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$$

such that each p_i depends (in a non-trivial way) only on $Y_1, \ldots, Y_i$ among the variables $Y_1, \ldots, Y_n$, so that

$$p_i = p_i(X_1, \ldots, X_n, Y_1, \ldots, Y_i),$$

and also depends (non-trivially) on some X_k.

Proposition

Let $V \subseteq \mathbb{C}^{2n}$ be a variety of triangular form of dimension n. Then there is a Zariski open subset $V_0 \subseteq V$ and a finite set $S \subset \mathbb{H}$ of special points, such that for any special tuple $\bar{z} \in \mathbb{H}^n$ satisfying $(\bar{z}, j(\bar{z})) \in V_0$, the coordinates of $\bar{z}$ are in $\mathrm{SL}_2(\mathbb{Z}) \cdot S$.

Corollary ((EGS) over $\overline{\mathbb{Q}}$ for planar curves)

Let $V \subseteq \mathbb{C}^2$ be an irreducible curve defined over $\overline{\mathbb{Q}}$. Assume that V is not a vertical line nor a horizontal line. Then V has infinitely many points of the form $(z, j(z))$ that are generic over $\overline{\mathbb{Q}}$.

Using known results on isogeny estimates and gonality of modular curves, we can obtain:

Proposition

Let $V \subseteq \mathbb{C}^{2n}$ be a variety of triangular form of dimension n. There exists a Zariski open subset V_0 of V such that for every $\bar{z} \in \mathbb{H}^n$ satisfying $(\bar{z}, j(\bar{z})) \in V_0$, there is a positive integer M such that for every $g_1, \ldots, g_n \in \mathrm{GL}_2(\mathbb{Q})^+$ satisfying

$$(g_1 z_1, \ldots, g_n z_n, j(g_1 z_1), \ldots, j(g_n z_n)) \in V_0$$

we have $\det\left(\widetilde{g}_{z_\ell, g_\ell z_\ell}\right) < M$ *for every* $\ell \in \{1, \ldots, n\}$.

Given $A \subseteq \mathbb{H}$, let $\mathsf{dim}^g(A)$ denote the number of distinct $\mathrm{GL}_2(\mathbb{Q})^+$-orbits generated by the elements of A.

Conjecture (Modular Schanuel Conjecture (MSC))

If $z_1, \ldots, z_n \in \mathbb{H}$ are non-special points, then:

$$\mathrm{tr.deg.}_{\mathbb{Q}}\,(z_1, \ldots, z_n, j(z_1), \ldots, j(z_n)) \geq \mathsf{dim}^g(z_1, \ldots, z_n).$$

The results that follow, which speak of the existence of generic solutions, will be conditional upon this conjecture.

Lemma

Let $V \subseteq \mathbb{C}^{2n}$ be a variety of simple form of dimension n, let K be a finitely generated subfield of $\mathbb{C}$, and let B the set of coordinates of ordinary points $\overline{z} \in \mathbb{H}^n \cap \overline{K}^n$ such that $(\overline{z}, j(\overline{z})) \in V$. Then MSC implies that $\dim^g(B)$ is finite.

Corollary (MSC implies (EGS) for plane curves)

Let $p(X, Y) \in \mathbb{C}[X, Y]$ be irreducible and depending on both X and Y. Then MSC implies that for every finitely generated subfield $K \subset \mathbb{C}$, there is $z \in \mathbb{H}$ such that $p(z, j(z)) = 0$ and $\mathrm{tr.deg.}_{\overline{K}}(z, j(z)) = 1$.

Definition

A subfield $K \subset \mathbb{C}$ will be called a *j-finite field* if there exist $\tau_1, \ldots, \tau_m \in \mathbb{H}$ (m is allowed to be zero) satisfying the following conditions:

1. $\overline{K} \subseteq \overline{\mathbb{Q}(\tau_1, \ldots, \tau_m, j(\tau_1), \ldots, j(\tau_m))}$, and
2. Equality for MSC: $\mathrm{tr.deg.}_{\mathbb{Q}}(\overline{\tau}, j(\overline{\tau})) = \dim^{g}(\overline{\tau})$.

Example

1. Finitely generated subfields of $\overline{\mathbb{Q}}$.
2. Set $M_0 = \overline{\mathbb{Q}}$. For $n \geq 1$ define inductively $M_n := \overline{M_{n-1}\left(j(\tau) : \tau \in M_{n-1} \cap \mathbb{H}\right)}$. Finally, let $M = \bigcup_{n \in \mathbb{N}} M_n$. Finitely generated subfields of M are j-finite.

Theorem

Let K be a j-finite field, and let $V \subseteq \mathbb{C}^{2n}$ be a j-generic variety of triangular form defined over $\overline{K}$. Then MSC implies that V has a point of the form $(\bar{z}, j(\bar{z}))$ which is generic over $\overline{K}$.

Remark

Most finitely generated fields are not j-finite.

Let $C \subset \mathbb{C}$ be the closure of $\emptyset$ under j-derivations.

Definition

We will say that a subfield $K \subset \mathbb{C}$ is *finitely generated by j* if there exist $\tau_1, \ldots, \tau_k \in \mathbb{H} \cap C$ and $t_1, \ldots, t_m \in \mathbb{H} \setminus C$ such that:

1. $\overline{K} \subseteq \overline{\mathbb{Q}\left(\overline{\tau}, \overline{t}, j\left(\overline{\tau}\right), j\left(\overline{t}\right)\right)}$,
2. Equality for MSC: $\mathrm{tr.deg.}_{\mathbb{Q}}\left(\overline{\tau}, j(\overline{\tau})\right) = \dim^{g}\left(\overline{\tau}\right)$,
3. Equality for Ax-Schanuel + something else:

$$\mathrm{tr.deg.}_{C}\left(\overline{t}, j\left(\overline{t}\right)\right) = \mathrm{tr.deg.}_{\mathbb{Q}(\overline{\tau}, j(\overline{\tau}))}\left(\overline{t}, j\left(\overline{t}\right)\right) = \dim^{g}\left(\overline{t}/C\right) + \dim^{j}\left(\overline{t}/C\right)$$

.

Theorem

Let $V \subseteq \mathbb{C}^{2n}$ be a j-generic variety of triangular form defined over a field K which is finitely generated by j. Then MSC implies that there exists $(\bar{z}, j(\bar{z})) \in V \cap E_j^n$ which is generic over K.