Journal of Colloid and Interface Science **221,** 38–49 (2000)
Article ID jcis.1999.6551, available online at http://www.idealibrary.com on IDEAL®

Draining Collars and Lenses in Liquid-Lined Vertical Tubes

Oliver E. Jensen[1]

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, United Kingdom

Received April 8, 1999; accepted September 23, 1999

The speed at which an annular liquid collar drains under gravity g in a vertical tube of radius a, when the tube has an otherwise thin viscous liquid lining on its interior, is determined by a balance between the collar's weight and viscous shear stresses confined to narrow regions in the neighborhood of the collar's effective contact lines. Whether a collar grows or shrinks in volume as it drains depends on the modified Bond number $B = \rho g a^2/(\sigma\epsilon)$, where ρ is the fluid density, σ is its surface tension, and ϵa is the thickness of the thin film immediately ahead of the collar. Asymptotic methods are used here to determine the following nonlinear stability criteria for an individual collar, valid in the limit of small ϵ. For $0 < B < 0.5960$, all draining collars grow in volume and, in sufficiently long tubes, ultimately "snap off" to form stable lenses. For $0.5960 < B < 1.769$, small collars may shrink but in long tubes sufficiently large collars will snap off. For $1.769 < B < 11.235$, both stable collars and lenses may arise, although most collars will shrink. If $B > 11.235$, all collars and lenses shrink in volume as they drain, so that any lens ultimately ruptures, unless stabilizing intermolecular forces allow the formation of a lamella supported by a macroscopic Plateau border. If surfactant immobilizes the liquid's free surface, these critical values of B are reduced by a factor of 2 but the distance a collar must travel before it snaps off is unchanged. Gravitationally driven snap off is therefore most likely to occur in long tubes with radii substantially *less* than the capillary lengthscale $(\sigma/\rho g)^{1/2}$. © 2000 Academic Press

Key Words: **surface tension; thin liquid films; wetting collars; capillary instabilities; airway closure.**

INTRODUCTION

Experiments by Quéré (1, 2) have demonstrated that thin liquid layers coating the exterior of vertical fibers form isolated, macroscopic drops (or collars) via a capillary instability when

$$B \equiv \frac{\rho g a^2}{\sigma\epsilon} < B^*, \qquad [1]$$

where $B^* = 0.72 \pm 0.05$. Here a is the tube radius, ρ is the fluid density, σ is the surface tension (assumed uniform), g is the acceleration due to gravity, and ϵa is the initial liquid lining thickness, where $\epsilon \ll 1$. These experiments, and related studies based on lubrication theory (3–7), have shown that large collars do not arise spontaneously for $B > B^*$ because of nonlinear saturation of the primary capillary instability at $O(\epsilon a)$ amplitudes under a balance between the effects of wave steepening due to gravity and the stabilizing effects of surface tension associated with axial interfacial curvature (8). For $B < B^*$ however, drainage due to gravity has a destabilizing effect on collars generated by capillary instability, since small collars can grow and possibly coalesce to form large ones (1, 5–7). Theoretical studies based on lubrication theory have yielded estimates of B^* that are close to the range of values found by Quéré (e.g., $B^* \approx 2/3$ according to Kerchman and Frenkel (6); $B^* \approx 1/1.68$ according to Chang and co-workers (5, 7)).

Since gravity can cause collars to grow to large amplitudes on the exterior of fibers having radii substantially less than the capillary lengthscale $(\sigma/\rho g)^{1/2}$, it is natural to consider whether weak gravitational effects may similarly cause an initially thin liquid layer lining the interior of a narrow vertical tube to "snap-off" to form a lens that occludes the tube. Determining when snap off occurs is important in understanding "airway closure" in the lung (9), which occurs when the aqueous liquid lining of small airways is perturbed to form lenses (10) or lamellae (11) that occlude the airway and impede gas exchange. Lens and lamella formation is also a significant feature of two-phase flows in porous media (12, 13) and of various manufacturing processes.

According to lubrication theory, the evolution of an initially thin, axisymmetric liquid layer lining either the interior or the exterior of a vertical tube is described by the dimensionless equation (5–7)

$$h_t + Bh^2 h_z + \tfrac{1}{3}\left[h^3(h + h_{zz})_z\right]_z = 0, \qquad [2]$$

where the axial coordinate z is scaled on a, axial speed on $\epsilon^3(\sigma/\mu)$, and time t on $\mu a/(\sigma\epsilon^3)$; the thin film has thickness $\epsilon a h(z,t)$, where $\epsilon \ll 1$ and $h = O(1)$, and subscripts z and t denote derivatives. When $B = 0$, [2] predicts saturation of the primary capillary instability with $h = O(1)$ (14–17), since in any tube of finite length there is only enough fluid to form a series of wetting collars which do not migrate significantly along the tube. This model has been extended for interior collars with $B = 0$ using a regularized model (effectively a composite asymptotic approximation) in which the full curvature terms in the expression for the pressure are retained in the evolution equation (15, 16, 18, 19), so that quasi-steady collars of finite

[1] E-mail: O.E.Jensen@damtp.cam.ac.uk.

0021-9797/00 $35.00

amplitude can be described. The ultimate state of the system is then dictated by the amount of fluid available in a tube of fixed length, relative to the maximum volume (approximately $5.47a^3$) that any single wetting collar can attain, and the minimum volume ($\frac{2}{3}\pi a^3$) above which lens formation is possible (20). The extended model shows that if the initial film thickness in a long tube exceeds roughly $0.1a$, lens formation is inevitable provided one waits a sufficiently long time (15).

The situation is quite different when $B > 0$, however, since individual collars described by [2] can grow to unbounded amplitude in finite time as they drain down the tube, with $h_{\max}$ increasing like $(t_0 - t)^{-2}$ as t increases toward a blow-up time t_0 (7). The blow-up time t_0 is reduced if a collar coalesces with its neighbors as it grows. According to Chang and Demekhin (7), blow-up arises for $B < 1/1.68$ because of the linearization of the Young–Laplace equation that is implicit in [2]: if the full curvature terms are retained in [2], growing collars *exterior* to tubes saturate at $O(a)$ amplitudes, yielding large drops on fibres resembling those observed by Quéré (1). Blow-up of initially thin collars also occurs when the driving force is an external shear flow rather than gravity (21). Indeed, Aul and Olbricht showed experimentally how weak shear from a pressure-driven core flow can cause collars inside a cylindrical tube to grow in volume and ultimately snap-off (22).

Here, asymptotic methods will be used to identify stability criteria analogous to [1] to describe collar, lens, and lamella formation by a liquid layer lining the *interior* of a vertical tube. It will be shown below that the finite-amplitude saturation found for collars exterior to tubes found in (7) does not occur for interior collars, since the relevant bifurcation is subcritical rather than supercritical. Instead, under suitable conditions interior collars in sufficiently long tubes grow until they are so large that they snap off to form lenses. Three critical values of B are identified that dictate whether a lens forms, ruptures, or (in the presence of suitable stabilizing forces) forms a lamella bridging the core of the tube, supported by macroscopic Plateau borders. Simple snap-off criteria dependent on B, the tube length, and the initial lining thickness ϵ will also be presented.

THE MODEL

We consider an open-ended vertical tube of radius a, lined on its interior with a wetting liquid of constant density ρ, viscosity μ and uniform surface tension σ. The remainder of the tube is filled with a dynamically passive gas at zero pressure. Gravity g causes the liquid to drain down the tube. The liquid layer is unstable to capillary instabilities and can form either annular collars (14, 20), lenses that occlude the tube (15) or possibly lamellae supported by macroscopic Plateau borders (23–26), as illustrated in Fig. 1. In the following we formulate the model for a collar of volume $O(a^3)$, adding details relevant to a lens or lamella where appropriate. If a lens or lamella forms, it is assumed that there is no difference in the core gas pressure across it, although small pressure differences could be readily incorporated in the present model; for related studies of pressure-driven lens motion see (27, 28). For perfectly pure liquids, a lens that shrinks in volume as it drains will normally rupture as soon as its two menisci intersect; we treat both this possibility and the alternative, in which a suitable stabilizing disjoining pressure (possibly arising from the presence of surfactants) allows the formation of a lamella supported by Plateau borders.

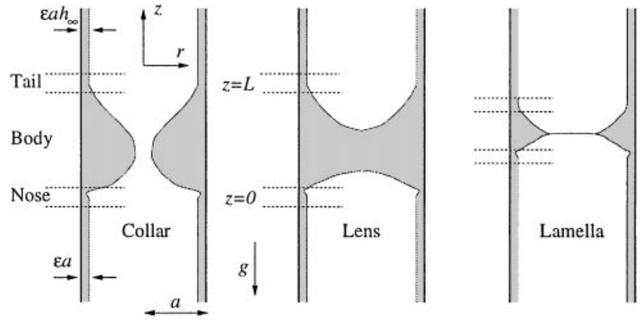

FIG. 1. A collar, lens and lamella (supported by a Plateau border) draining under weak gravity inside a liquid-lined tube. Capillary forces are dominant in each "body" region (gravity is a weaker effect that breaks the up–down symmetry); viscous stresses are significant also in the "nose" and "tail" regions in the neighborhood of the effective contact lines.

Ahead of the draining collar is a uniform film of thickness ϵa (Fig. 1), where $\epsilon \ll 1$. The thickness of this film influences the rate at which fluid can enter the collar. A film of comparable thickness is deposited behind the collar. The collar drains downwards with respect to the tube at a speed U that we wish to determine. When $\mu U/\sigma \ll 1$, the thickness of the deposited film is of magnitude $a(\mu U/\sigma)^{2/3}$ (29, 30), which we also expect to be $O(\epsilon a)$. We therefore scale lengths on a, speeds on $\epsilon^{3/2}\sigma/\mu$, pressure on σ/a, and time on $\mu a/(\sigma\epsilon^{5/2})$. We are considering speeds a factor $\epsilon^{-3/2}$ larger than those used in deriving [2], and time scales a factor $\epsilon^{1/2}$ shorter, so that in describing the evolution of a collar we can neglect (in a first approximation) relatively slow transient variations of the neighboring film.

We introduce cylindrical polar coordinates (ar, az) with z pointing upward (Fig. 1), so that the tube lies at $r = 1$ and the free surface of the liquid lies at $r = R(z, t)$. The Bond number B defined in [1] is assumed to be $O(1)$. The full governing equations in the liquid are those of Stokes flow, which in dimensionless form are

$$\nabla \cdot \mathbf{u} = 0, \quad \mathbf{0} = -\nabla p - \epsilon B\hat{\mathbf{z}} + \epsilon^{3/2}\nabla^2\mathbf{u} \qquad [3]$$

for $R(z, t) \le r \le 1$. Here $(\sigma/a)p(r, z, t)$ is the dimensional liquid pressure and $(\epsilon^{3/2}\sigma/\mu)\mathbf{u}(r, z, t)$ its velocity. The condition for inertia to be negligible in (3) is that the Suratnam number is sufficiently small, i.e., $\rho\sigma a\epsilon^3/\mu^2 \ll 1$. On the liquid's free surface, at $r = R(z, t)$, there is no tangential stress and the normal stress condition may be written (using the Young–Laplace equation) as

$$-p + O(\epsilon^{3/2}) = \kappa \equiv (N/R) - N^3 R_{zz}, \qquad [4]$$

where $N \equiv (1 + R_z^2)^{-1/2}$ and κ is the interfacial curvature. The

$O(\epsilon^{3/2})$ term in [4] represents viscous stresses that play no role in what follows. Mass conservation in the liquid demands that

$$\epsilon(\pi R^2)_t = q_z, \qquad q(z,t) = 2\pi \int_R^1 r u \, dr, \qquad [5]$$

where $u = \hat{\mathbf{z}} \cdot \mathbf{u}$ is the vertical component of velocity and q is the axial volume flux. Writing

$$U = (\sigma/\mu)\epsilon^{3/2}\mathcal{U}, \qquad [6]$$

we seek the dimensionless collar speed $\mathcal{U}(t)$ as a function of collar volume. In the frame in which the collar is instantaneously stationary, the boundary condition on the tube wall is $\mathbf{u} = \mathcal{U}\hat{\mathbf{z}}$ on $r = 1$.

In the limit $\epsilon \to 0$ with $B = O(1)$, for which the capillary number $\mu U/\sigma = O(\epsilon^{3/2})$ is small, the problem has a three-region asymptotic description, as illustrated in Fig. 1. The majority of the collar (the "body") is controlled by strong capillary forces, but the draining speed $\mathcal{U}$ is controlled by viscous forces acting in short transition regions (the "nose" and "tail" in Fig. 1) where the collar meets the neighboring thin film. By constructing asymptotic expansions in each region, matching them appropriately, and performing integral force and mass balances, the dynamics of a collar, lens, or lamella can be described. Numerous studies using asymptotic techniques similar to those used here may be found elsewhere, e.g., (7, 29–33). For example, solutions of the thin-film equation [2] that are steady in a frame moving at some speed c, governed by

$$c(1-h) + \tfrac{1}{3}B(h^3 - 1) + \tfrac{1}{3}h^3(h + h'')' = 0,$$

also exhibit a three region structure for draining collars when $1 \ll c \ll \epsilon^{-3/2}$ (5, 7), for which $h = O(c^{2/3})$ in a body region of $O(1)$ length (where $h^3(h + h'')' = 0$ at leading order), while in the adjacent nose and tail regions, of length $O(c^{-1/3})$ with $h = O(1)$, $c(1-h) + \frac{1}{3}h^3 h''' = 0$ at leading order. Here we develop an alternative theory for the case in which $c = O(\epsilon^{-3/2})$, allowing the collar thickness to be comparable to the tube radius.

The Body Region

In the main body of the collar, at leading order in ϵ, the pressure is uniform (see [3]), and κ is therefore uniform also (from [4]), so that the draining collar has the same shape as a static wetting collar (up–down asymmetry in the collar shape due to gravity arises at $O(\epsilon)$). Such symmetric collars (a one-parameter family of unduloids, see (20)) were computed numerically, assuming zero contact angle, by solving [4] with $R = 1$ and $R_z = 0$ at $z = 0$ and $z = L$, say, and some examples are shown in Fig. 2a. Note that with κ constant, [4] may be integrated once to relate R to the angle θ between the interface and the tube axis,

$$\tfrac{1}{2}\kappa(R^2 - 1) = RN - 1, \qquad \cos\theta = N, \qquad [7]$$

so that the minimum value of R for a collar is given by $R = (2-\kappa)/\kappa$, where $1 < \kappa < 2$. The collar's length $L(\kappa)$ and volume $V(\kappa)$, given by

$$V(\kappa) = \pi \int_0^L (1 - R^2)\, dz, \qquad [8]$$

are shown in Fig. 2b. As κ increases from 1 to 2, L falls

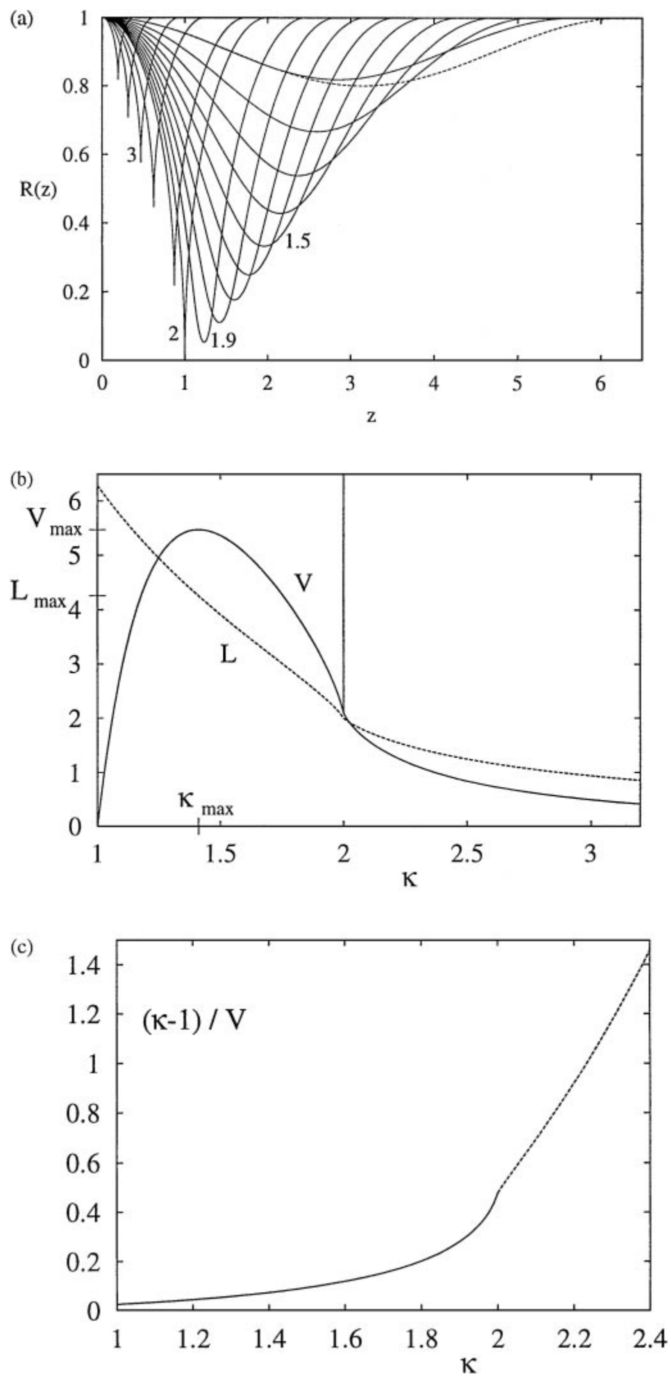

FIG. 2. (a) Numerical solutions of [4] (solid) showing collars in $R < 1$ for which $\kappa = 1.1, 1.2, \ldots, 1.9, 2$. The dashed line shows the approximation [9] for $\kappa = 1.1$. Plateau borders for $\kappa = 2.1, 2.5, 3, 4,$ and 6 are shown also. (b) V (solid) and L (dashed) versus κ; collars have $1 < \kappa < 2$, lenses have $\kappa = 2$, lamellae $\kappa > 2$. (c) $(\kappa - 1)/V$ versus κ for collars (solid) and Plateau borders (dashed).

monotonically from 2π to 2, but V rises from zero to a maximum $V_{\max} \approx 5.4712$ when $\kappa = \kappa_{\max} \approx 1.4114$ and $L = L_{\max} \approx 4.2568$ before falling to $2\pi/3$ as $\kappa \to 2$ (Fig. 2b). We will refer to these two solution branches (for which $dV/d\kappa > 0$ and <0) as the "left" and "right" collar branches respectively. We will later make use of the quantity $(\kappa - 1)/V$, which is shown to be a monotonic function of κ in Fig. 2c.

For $\epsilon \ll \kappa - 1 \ll 1$, the collar is thin (but thicker than the neighboring liquid film), and

$$R \approx 1 - (\kappa - 1)(1 - \cos z), \;\; V \approx 4\pi^2(\kappa - 1) \qquad [9]$$

with $L \approx 2\pi$. This thin-film estimate of collar shape is shown in Fig. 2a to have errors of up to 10% when $\kappa = 1.1$.

Lenses have $\kappa = 2$ and $V = \pi(L - \frac{4}{3})$, for $L > 2$. Depending on the nature of the liquid, either a lens ruptures as soon as L falls beneath 2, or a lamella forms, supported by Plateau borders (23, 25, 26). To model the latter case in the simplest possible way we introduce a disjoining pressure to the Young–Laplace equation [4], so that $-p = (N/R) - N^3 R_{zz} + \Pi$, where Π is negligible where the fluid has finite thickness. Where the two menisci of a lens are separated by a small distance Δ, however, Π becomes significant. A commonly used model takes $\Pi \propto \Delta^{-3}$. A three-region structure then emerges: a lamella in which Δ is large; a Plateau border in which Π is negligible; and a short transition region connecting the two controlled by a balance between capillary and disjoining pressures. Making the assumption that the surface tension of the interfaces in the lamella is independent of Π (a restriction that is relaxed in (26), for example), a force balance across the transition region at $r = R_l$, say, implies that the boundary conditions where the Plateau border meets the lamella are that $\theta \to \pi/2$ as $R \to R_l$. It follows from (7) that

$$\kappa = 2/\left(1 - R_l^2\right), \;\; \kappa > 2.$$

Some Plateau borders of lamellae are shown in Fig. 2a, and the corresponding values of L, V and $(\kappa - 1)/V$ are shown in Figs. 2b and 2c. As κ becomes large, the two interfaces become arcs of circles, and $L \sim 2/\kappa$, $V \sim \pi(4 - \pi)/\kappa^2$. We restrict attention here to the case in which the film ahead of the Plateau border is thin in comparison, requiring that $\epsilon \ll 1/\kappa$.

Combining [5] and [8] implies that, as the collar (or lens) drains down the tube, its volume changes according to

$$\frac{dV}{dt} = Q(0, t) - Q(L, t), \qquad [10]$$

where $q = \epsilon Q$. Differences between the small fluxes entering and leaving the collar can cause it to slowly shrink or grow as it drains. These fluxes are determined by the structure of the nose and tail transition regions at $z = 0$ and $z = L$, as described below. We will shortly use the fact that at each effective contact line, where $R \to 1$ and $R_z \to 0$, $R_{zz} \to \kappa - 1$ (from [4]) and so the local collar thickness may be written

$$R \approx \begin{cases} 1 - \frac{1}{2}(\kappa - 1)z^2 & (z \to 0) \\ 1 - \frac{1}{2}(\kappa - 1)(L - z)^2 & (z \to L). \end{cases} \qquad [11]$$

Details of the leading-order flow in the collar are given in the Appendix; streamlines of the flow in a thin collar are shown in Fig. A1.

The Nose and Tail Regions

The collar matches with the thin film lining the remainder of the tube across much-studied transition regions in the neighbourhood of each contact line (Fig. 1), within which there is dominant viscous dissipation (30). In the nose and tail transition regions, we set $z = \epsilon^{1/2} X$ or $z = L + \epsilon^{1/2} X$, respectively, with $r = 1 - \epsilon y$, $R(z, t) = 1 - \epsilon h(X)$, so that at leading order [3] gives the usual equations of lubrication theory, namely $0 = -p_X + u_{yy} + O(\epsilon)$ and $0 = -p_y + O(\epsilon)$ for $0 \le y \le h$. The pressure is set by the curvature of the interface at $y = h(X)$, so [4] implies $p = -1 - h_{XX} + O(\epsilon)$ for $0 \le y < h$. Since $u_y = 0$ on $y = h$ and $u = \mathcal{U}$ on $y = 0$, it follows that $u = \mathcal{U} - \frac{1}{2} p_X y(2h - y)$ in $0 \le y \le h$, so that the flux through each region (see [5]) is

$$Q = 2\pi \left[-\frac{h^3}{3} p_X + \mathcal{U} h \right], \qquad [12]$$

where $q = \epsilon Q$. According to [5], $Q_X = O(\epsilon^{3/2})$, so that Q is uniform across each region. The dimensionless shear stress on the wall is $\tau(X) = u_y|_{y=0} = -p_X h$. The matching condition where each transition region joins the capillary statics region is, using [11],

$$h \sim \tfrac{1}{2}(\kappa - 1)X^2 \qquad [13]$$

as $X \to \infty$ (in the nose) and as $X \to -\infty$ (in the tail).

At the collar's tail, $h \to h_\infty$ (say) as $X \to \infty$ (Fig. 1), so that $Q = 2\pi \mathcal{U} h_\infty$. We rescale using $h = h_\infty H(\zeta)$ where $\zeta = X(3\mathcal{U})^{1/3}/h_\infty$, so that [12] becomes

$$H^3 H_{\zeta\zeta\zeta} = 1 - H, \quad H \to 1 \text{ as } \zeta \to \infty. \qquad [14]$$

Equation [14] has a unique solution with

$$H \to \tfrac{1}{2} H_2 \zeta^2 + H_0 + \frac{2}{3H_2^2 \zeta} + O(\zeta^{-3}) \qquad [15]$$

as $\zeta \to -\infty$, where $H_2 \approx 0.64304$ and $H_0 \approx 2.89964$ (5, 30, 32). The matching condition [13] tells us the thickness of the deposited film,

$$h_\infty = H_2 (3\mathcal{U})^{2/3}/(\kappa - 1). \qquad [16]$$

The shear stress on the wall in this region is $\tau = (3\mathcal{U}/h_\infty)(1 - H)/H^2$, so that the total dimensionless viscous shear force (per unit circumference) exerted by the wall on the fluid in the tail region is

$$\int_{-\infty}^{\infty} (-\tau)\, dX = (3\mathcal{U})^{2/3} \int_{-\infty}^{\infty} \frac{H-1}{H^2}\, d\zeta$$
$$= (3\mathcal{U})^{2/3} \left[\tfrac{1}{2} H_\zeta^2 - H H_{\zeta\zeta}\right]_{-\infty}^{\infty} = (3\mathcal{U})^{2/3} H_0 H_2. \quad [17]$$

At the collar's nose, $h \to 1$ as $X \to -\infty$ (corresponding to the undisturbed film ahead of the collar), so that $Q = 2\pi\mathcal{U}$. With $h(X) = G(\xi)$, $\xi = (3\mathcal{U})^{1/3} X$, [12] becomes

$$G^3 G_{\xi\xi\xi} = 1 - G, \quad G \to 1 \text{ as } \xi \to -\infty. \quad [18]$$

Equation [18] is satisfied by a one-parameter family of solutions for which $G \sim \frac{1}{2} G_2 \xi^2 + G_0$ as $\xi \to \infty$ (30). The downstream matching condition [13] implies (using [16])

$$G_2 = \frac{\kappa - 1}{(3\mathcal{U})^{2/3}} = \frac{H_2}{h_\infty}. \quad [19]$$

Since H_2 is a constant, h_∞ therefore parameterises the family of solutions of [18]. If $G_2 = H_2$ (so that $h_\infty = 1$), then $G_0 \approx -0.84529$ (32). Whereas H is monotonic, G has a characteristic wavy structure (30), shown schematically as a dimple in the nose region in Fig. 1. The shear stress at the wall in the nose region is $\tau = (3\mathcal{U})(1-G)/G^2$, so the total shear force (per unit circumference) exerted by the wall on the fluid in this region is, as in [17],

$$(3\mathcal{U})^{2/3} \int_{-\infty}^{\infty} \frac{G-1}{G^2}\, d\xi = -(3\mathcal{U})^{2/3} G_0 G_2. \quad [20]$$

Global Force and Mass Balance

To determine the speed at which the collar drains quasi-steadily down the tube we balance forces: its weight is supported by shear forces acting at the tube walls, with the dominant viscous contributions coming from the nose and tail regions. Integrating over a cylindrical control volume extending beyond the nose and tail regions, we find that

$$\frac{V(\kappa)}{2\pi} B = \int_{\text{nose}} -\tau(X)\, dX + \int_{\text{tail}} -\tau(X)\, dX. \quad [21]$$

The dominant error in [21] is $O(\epsilon^{1/2})$, representing shear from the flow in the main body of the collar; this contribution is described in more detail in the Appendix. In terms of the rescaled variables $G(\xi)$ and $H(\zeta)$, using [17] and [20], [21] becomes

$$\frac{V(\kappa)}{2\pi} \frac{B}{(3\mathcal{U})^{2/3}} = \int_{-\infty}^{\infty} \frac{G-1}{G^2}\, d\xi + \int_{-\infty}^{\infty} \frac{H-1}{H^2}\, d\zeta$$
$$\equiv I_{\text{nose}} + I_{\text{tail}}, \quad [22]$$

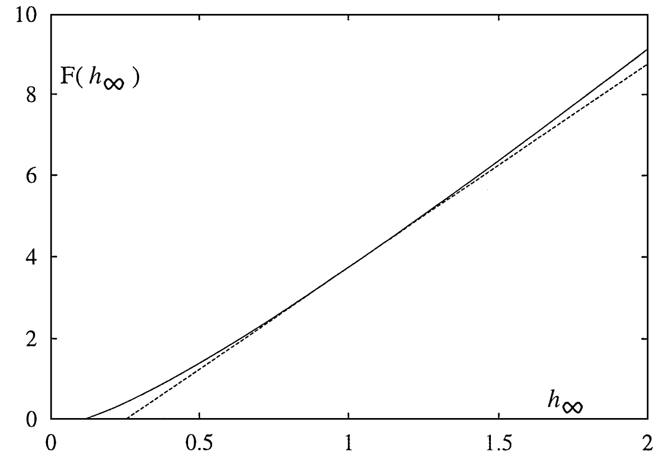

FIG. 3. $F(h_\infty)$ versus h_∞ (solid), showing the linear approximation $3.7449 + 5.02(h_\infty - 1)$ (dashed).

where $I_{\text{tail}} = H_0 H_2 \approx 1.8646$ is a constant but $I_{\text{nose}} = -G_0 G_2$ is a function of H_2/h_∞ (see [19]). Thus [22] may be written using [16] as

$$\frac{B}{2\pi} = \frac{\kappa - 1}{V(\kappa)} F(h_\infty), \quad [23]$$

where $F(h_\infty) \equiv (h_\infty/H_2)[I_{\text{nose}} + I_{\text{tail}}]$. This function was evaluated numerically, using asymptotic approximations for large H and G to improve accuracy when evaluating the integrals (see [A.4a] and [A.4b] below). As shown in Fig. 3, $F(h_\infty)$ is well approximated for $\frac{1}{2} < h_\infty < \frac{3}{2}$ by a straight line with slope $F'(1) \approx 5.02$ passing through $F(1) = H_0 - G_0 \approx 3.7449$.

The quasi-steady evolution of the collar is determined from the mass balance [10], which may be written

$$\frac{dV}{dt} = 2\pi\mathcal{U}(1 - h_\infty). \quad [24]$$

When $h_\infty > 1$ the collar will slowly shrink as it drains; for $h_\infty < 1$ it can slowly grow in volume. Steady solutions exist for $h_\infty = 1$.

RESULTS

The four coupled equations relating V, κ, h_∞, and $\mathcal{U}$, namely [8], [16], [23], and [24], which are parameterized by B, can be solved to determine the evolution of an individual collar (or lens or lamella) of given initial volume. Capillary forces control the relationship between collar volume V and curvature κ through [8] (see also Figs. 2a and 2b). The thickness of the film deposited behind the collar, h_∞, is set by κ and the speed $\mathcal{U}$ at which the collar drains with respect to the tube through [16]. This thickness enters the global force balance [23] and mass balance [24] on the collar.

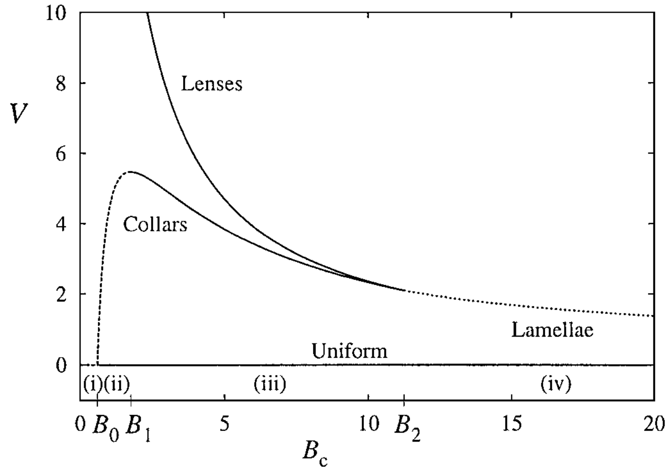

FIG. 4. Solution branches corresponding to steadily translating collars, lenses and lamellae, represented by plotting V versus $B = B_c$ [25]; dashed (solid) curves represent unstable (stable) solutions. The dotted curve indicates stable lamellae, if they exist. The line $V = 0$ represents the uniform state.

Steady Solutions and Their Stability

Steady solutions of [8], [16], [23], and [24], for which $h_\infty = 1$, require that $B = B_c$ where

$$B_c = 2\pi F(1)\frac{\kappa - 1}{V(\kappa)}. \qquad [25]$$

The function $(\kappa - 1)/V$ is monotonic in κ for collars (Fig. 2c) but is a multi-valued function of V; note also that $(\kappa - 1)/V \to 1/(4\pi^2)$ as $\kappa \to 1$ (see [9]). A graph of V versus B_c, parameterized by κ, is shown in Fig. 4. We can identify three significant values of B_c: at $\kappa = 1$, $B_c = B_0 \equiv F(1)/(2\pi) \approx 0.5960$, at which a (dashed) branch of steadily-translating collar solutions bifurcates from the uniform state; at $\kappa = \kappa_{\max}$, $B_c = B_1 \approx 1.7693$, at which the branch of collar solutions has maximum volume (Fig. 2b); and at $\kappa = 2$, $B_c = B_2 \equiv 3F(1) \approx 11.235$, where the (solid) branch of collars terminates and meets a branch of solutions in $B_c < B_2$ corresponding to steadily translating lenses. Along the branch of lens solutions $B_c = 2\pi F(1)/V$, where $V \geq 2\pi/3$. For $B_c > B_2$ there exists a branch of lamellae supported by Plateau borders, shown dotted in Fig. 4.

When h_∞ is close to unity, so that $F(h_\infty) \approx F(1) + (h_\infty - 1)F'(1)$, (24) may be reexpressed as

$$\frac{dV}{dt} = \frac{2\pi\mathcal{U}}{F'(1)}\left[F(1) - \frac{B}{2\pi}\frac{V}{(\kappa - 1)}\right]. \qquad [26]$$

This equation may be used to determine the stability of the steady solutions shown in Fig. 4 to small perturbations. For a collar of given volume and curvature, at fixed B, the sign of the square-bracketed term in [26] determines whether the collar grows or shrinks as it drains down the tube. Since $(\kappa - 1)/V$ is monotonic in κ (Fig. 2c), for collars with κ slightly in excess of the steady-solution value satisfying [25] $dV/dt > 0$; for collars with slightly smaller κ, $dV/dt < 0$. This implies that steady solutions given by [25] lying on the left-hand branch of the $V(\kappa)$ collar curve (in Fig. 2b, for which $dV/d\kappa > 0$) are unstable as t increases, but those on the right-hand collar branch are stable, as are lamellae. Likewise, for lenses with $\kappa = 2$ but V slightly greater than (less than) steady values, $dV/dt < 0$ ($dV/dt > 0$) implying that steadily translating lenses are stable. Four different situations are therefore possible, as illustrated using bifurcation diagrams in Fig. 5.

(i) If $0 < B < B_0$, all collars grow in volume as they drain. An initially small collar grows in magnitude, snapping off rapidly to form a lens when $V = V_{\max}$. The lens lengthens until eventually a steady state is reached. (If a lamella exists, the lamella solution branch is connected smoothly to the branch of lenses, so that the lamella grows until it forms a lens.)

(ii) If $B_0 < B < B_1$, initially small collars (on the left-hand branch of collar solutions) will shrink, but sufficiently large collars on the left branch, and all those on the right branch, grow, and may snap off to form a lens, which grows and stabilizes. A lamella will always grow into a lens.

(iii) If $B_1 < B < B_2$, then all collars on the left branch shrink, while those on the right branch approach a steady state on that branch. A lens, if formed, is also stable. A lamella will always grow into a lens.

(iv) If $B_2 < B$, a lens will shorten and either rupture (and the resulting collar will shrink) or will form a stable lamella. All collars shrink in volume.

This behavior can best be understood by remembering that faster moving collars or lenses deposit a thicker film according to [16]. Increasing B, all other parameters remaining fixed, will cause a collar, lens, or lamella to drain faster, deposit a thicker film, and therefore shrink in volume more rapidly than before. It is the nonlinearity in the $V(\kappa)$ relationship that leads to the wide range of possible behaviours illustrated in Fig. 5.

When a collar snaps off and becomes a lens, its length shrinks slightly; a similar length change occurs when a lens ruptures to form a collar. Capturing the details of these aprupt transitions is beyond the scope of the present model; the early stages of snap-off have been described using full Navier–Stokes computations in (16).

Drainage Speed

The linear approximation of $F(h_\infty)$ illustrated in Fig. 3, combined with [16] and [23], implies that

$$(3\mathcal{U})^{2/3} \approx \frac{1}{H_2}\left((\kappa - 1)\left[1 - \frac{F(1)}{F'(1)}\right] + \frac{VB}{2\pi F'(1)}\right). \qquad [27]$$

This shows that the drainage speed $\mathcal{U}$ increases with V for all left-branch collars (for which $dV/d\kappa > 0$), implying that large collars may catch up with small collars ahead of them. Suppose,

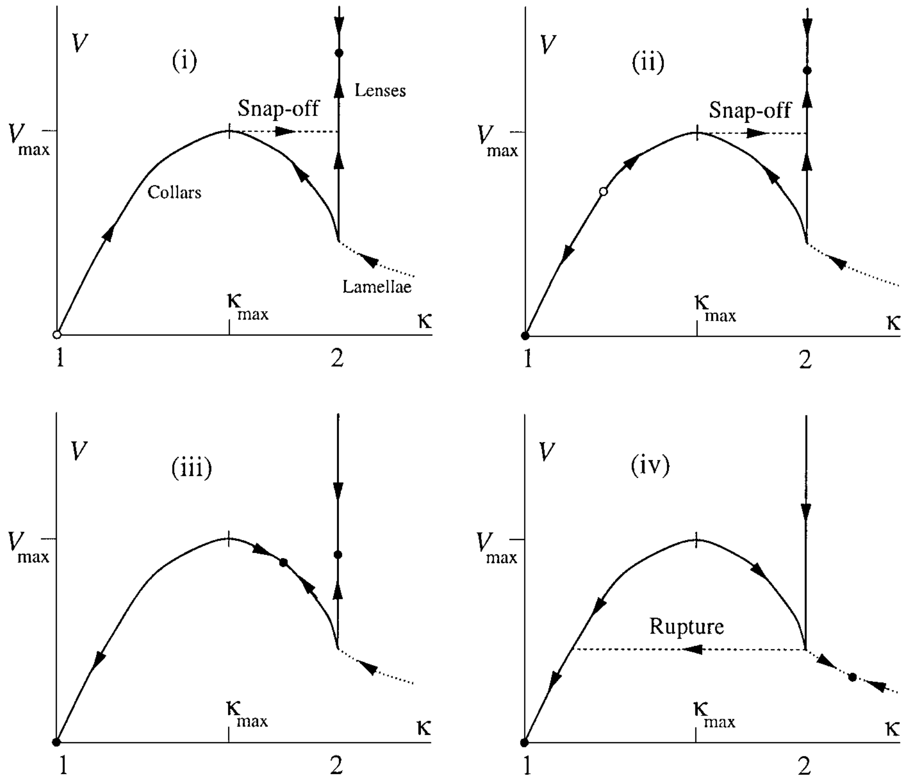

FIG. 5. Bifurcation diagrams showing schematically the static solutions of Fig. 2b, plotting $V(\kappa)$. Arrows indicate the sense in which solutions evolve either quasi-steadily (along solution curves) or abruptly (during snap-off or rupture). (i) If $0 < B < 0.5960$, weak gravity causes collars with $1 < \kappa < \kappa_{\max}$ to grow and ultimately snap off to form a stable lens (with $\kappa = 2$); (ii) if $0.5960 < B < 1.769$, snap-off occurs only for sufficiently large collars; (iii) if $1.769 < B < 11.235$ gravity causes most collars to shrink to small volumes, but lenses may still persist, as do collars with $\kappa_{\max} < \kappa < 2$; (iv) if $11.235 < B$ a lens in a sufficiently long tube shrinks and either ruptures or forms a lamella.

for example, that two small, adjacent collars are draining down a tube. If B is slightly greater than B_0, so that the leading collar shrinks in volume, the film deposited behind it is thicker than that ahead of it ($h_\infty > 1$). The trailing collar advances over film of thickness h_∞, and so its evolution is dependent on the size of $B/h_\infty < B$. Since the trailing collar experiences a smaller value of B, it is more likely to grow in amplitude than the leading collar and hence to coalesce with it. Detailed descriptions of such events have been described extensively elsewhere (5–7) and so are not explored further here. However, we note that it is possible for $\mathcal{U}$ to fall with increasing collar volume for right-branch collars, as it can for lamellae supported by Plateau borders, provided

$$0 < \frac{B}{2\pi} < \frac{F'(1) - F(1)}{-dV/d\kappa}, \; \kappa > \kappa_{\max}. \qquad [28]$$

This indicates that that small right-branch collars can approach and possibly absorb larger ones ahead of them, provided B is sufficiently small. $\mathcal{U}$ always increases with V for lenses, however, although all stable, steadily propagating lenses (for which $h_\infty = 1$, $\kappa = 2$) travel at the same speed (according to [16]).

Growth of a Collar and Snap-Off

The governing equations [8], [16], [23], and [24] can be integrated numerically to determine the evolution of a single collar, given some initial collar volume V_0. Supplementing these with

$$\frac{dZ}{dt} = -\mathcal{U}, \qquad [29]$$

we can determine the distance $Z(t)$ that the collar travels with respect to the tube before snap-off occurs. The corresponding dimensional distance travelled is aZ/ϵ, where $Z = O(1)$; since the liquid lining ahead of the collar is thin, long tubes are typically required for a collar to accumulate sufficient fluid to form a lens. For $0 < B < B_1$, the finite length of a tube will place restrictions on the possible values of V_0 for which snap-off can occur. To establish such constraints it is simplest to integrate the governing equations backward in time from the snap-off point, starting with $Z(0) = 0$, $V = V_{\max}$ and $\kappa = \kappa_{\max}$. Two examples of such integrations are shown in Fig. 6. For $0 < B < B_0$ (e.g., $B = 0.4$ in Fig. 6), a collar can grow from arbitrarily small volumes and will ultimately snap off (as demonstrated in Fig. 5(i)). While this may take a long time, an upper limit can be placed on the length of tube required for this to occur, since $Z \to Z_0$,

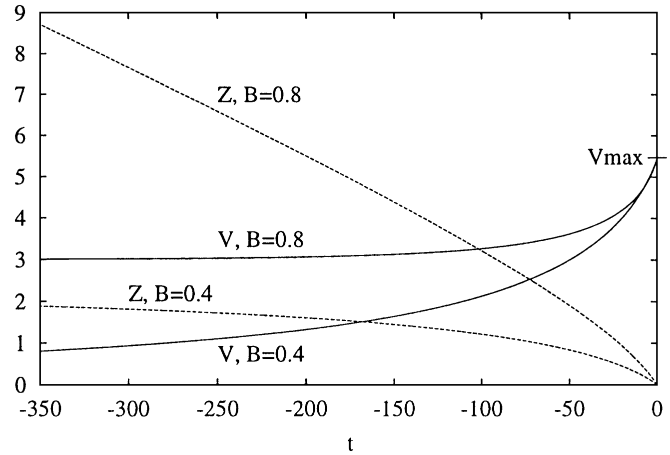

FIG. 6. The growth of collars toward snap-off (at $t = 0$) with $B = 0.4$ and $B = 0.8$, showing collar volume (solid) and the distance travelled before snap-off (dashed).

say, as $t \to -\infty$ and $V \to 0$. This limit may be understood by noting that for $V \to 0$, [9] and [23] imply that h_∞ is dependent on B alone, so then eliminating $\mathcal{U}$ between [24] and [29] and integrating implies that

$$V \approx 2\pi(1 - h_\infty)(Z_0 - Z), \qquad [30]$$

so that $Z \to Z_0$ as $V \to 0$. For $B_0 < B < B_1$, however (e.g., $B = 0.8$ in Fig. 6), only collars with initial volumes in excess of a critical value will proceed to snap off (that value being indicated in Fig. 4; see also Fig. 5(ii)), and a substantial length of tube may be required for snap-off to occur if the initial collar volume is only slightly in excess of the critical value.

This is demonstrated more clearly in Fig. 7a, where Z is plotted against $V/(4\pi^2)$ as t increases towards zero (the snap-off time), for fixed values of B. This yields a family of curves, each curve parameterized by t, with each approaching the snap-off point $(V_{\max}/(4\pi^2), 0)$ as $t \to 0$. For $0 < B < B_0$, each curve in Fig. 7a intersects the Z-axis, i.e., $V \to 0$ and $Z \to Z_0$ as $t \to -\infty$ (see [30]). The finite distance Z_0 over which an initially small collar must travel before it snaps off is plotted as a function of B for $0 < B < B_0$ in Fig. 7b, showing how this distance increases rapidly as $B \to B_0$. For reference, the horizontal line in Fig. 7b shows the length of tube required $(V_{\max}/(2\pi))$ for an individual collar to snap off assuming that it sweeps up *all* the fluid in the liquid lining ahead of it, leaving none behind; evidently such efficient accumulation is never achieved in practice. For $B_0 < B < B_1$, each curve in Fig. 7a approaches a vertical asymptote, i.e., $V/(4\pi^2) \to \epsilon_0 > 0$, say, and $Z \to \infty$, as $t \to -\infty$ (Fig. 7a). The finite volume $4\pi^2\epsilon_0$ corresponds to the unstable steady solution branch of collars indicated with a dashed line in Fig. 4. For this range of B, a growing collar can potentially travel a very great distance before snap-off occurs.

Figure 7a therefore provides snap-off criteria for individual collars that are dependent on B, the initial collar volume V_0 and the available tube length Z. If, for given B, the point $(V_0/(4\pi^2), Z)$ lies beneath the corresponding curve in Fig. 7a, there is insufficient fluid for the collar to snap off as it drains; above and to the right of each curve snap-off can occur. In more general terms, for a vertical tube of fixed length L (i.e., of dimensional length aL/ϵ), initially lined with a uniform layer of liquid having dimensional thickness $a\epsilon$, we may suppose that a collar forms at the top of the tube, so that the liquid lining occupying the top (dimensional) length $2\pi a$ of the tube readjusts to form a collar of initial volume $V_0 = 4\pi^2\epsilon$. Figure 7a shows whether, for given L $(=Z)$, ϵ $(=V_0/(4\pi^2))$ and B, this collar will still be a collar once it has reached the base of the tube (which is the case if the point (ϵ, L) lies beneath the appropriate curve corresponding to constant B), or whether snap-off occurs. Thus this figure also provides approximate snap-off criteria for a vertical tube in terms of B, L and ϵ alone.

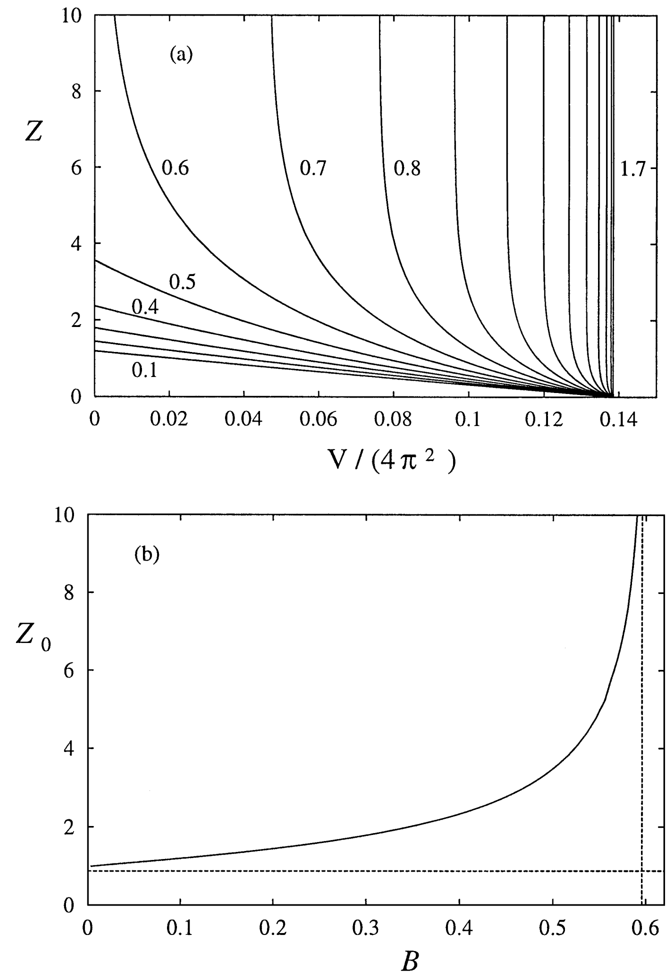

FIG. 7. Stability criteria for snap-off. (a) Distance to snap-off Z versus initial collar volume $V/(4\pi^2)$ for fixed B, with $0.1 < B < 1.7$. Beneath each curve there is insufficient fluid for an individual collar to snap off as it drains; above and to the right of each curve snap-off can occur. (b) The maximum distance to snap-off Z_0 for an individual collar with $0 < B < B_0$ is plotted against B. The dashed vertical line shows B_0, the horizontal line $V_{\max}/(2\pi)$.

Effect of Surfactants

In the presence of small amounts of surfactant in the liquid layer, the boundary condition that the free surface is free of tangential stress must be abandoned, since any flows will redistribute the surfactant and generate surface tension gradients. Only a small surface tension gradient is needed to immobilize the free surface of the film, while at the same time leaving the mean surface tension almost unchanged (2). We use this observation to model the presence of surfactants in the simplest possible way, namely by assuming an immobilized interface. The clean-surface and immobile-surface calculations will put simple bounds on the possible effects of surfactants.

Assuming that $u = 0$ on $y = h$, the velocity profile in the nose and tail regions is instead $u = \mathcal{U} - \frac{1}{2} p_X y(h - y)$, so the flux (see [12]) is $Q/(2\pi) = \mathcal{U}h - p_x h^3/12$. In the tail region, [16] is replaced by $h_\infty = H_2(12\mathcal{U})^{2/3}/(\kappa - 1)$, and the wall shear stress is $\tau = (6\mathcal{U}/h_\infty)(1 - H)/H^2$, so that the total shear force on the wall (see [17]) is $\frac{1}{2}(12\mathcal{U})^{2/3} \int (H - 1)/H^2 \, d\zeta$. In the nose, the matching condition [19] is $G_2 = (\kappa - 1)/(12\mathcal{U})^{2/3} = H_2/h_\infty$. The leading-order force balance [23] therefore becomes

$$\frac{B}{2\pi} = \frac{1}{2}\frac{\kappa - 1}{V} F(h_\infty).$$

Thus the critical values of B in Figs. 4 and 7 are reduced by a factor of 2 by interfacial immobilization.

The model presented for a clean interface can therefore be transformed into that for an immobile interface by replacing $(B, \mathcal{U}, t)$ with $(2B, 4\mathcal{U}, t/4)$, with all the other variables in [8], [16], [23], [24], and [29] remaining unchanged. Thus contaminating the interface with surfactant has the following effects: the time taken for snap-off to occur increases by a factor of four, collar draining speeds are reduced by a factor of 4, but the distance a collar must travel before snap-off occurs is unaltered.

DISCUSSION

We have identified three critical values of the modified Bond number B (defined in [1]) which may be used to predict the quasi-steady evolution of a collar or lens that is draining under weak gravity down a vertical tube having an otherwise thin liquid lining on its interior. The drainage speed U and the collar's stability are determined by balancing the weight of the collar with viscous forces acting at its effective contact lines. The speed U, given by [6] and [27], is an increasing function of the thickness ϵa of the thin liquid layer immediately ahead of the collar, and is also dependent on the volume $a^3 V$ and curvature κ/a of the collar or lens; the relationship between V and κ is shown in Figs. 2a and 2b. The stability properties of draining collars and lenses are summarized in Figs. 4 and 5. For $0 < B < B_0 \approx 0.5960$, weak gravity is strongly destabilizing in the sense that all collars of initially small amplitude will grow and will ultimately snap-off to form lenses in sufficiently long tubes (Fig. 5(i)). Figure 7b shows the distance $Z_0 a/\epsilon$ travelled by a single collar as it grows from an initially small volume $O(\epsilon a^3)$ to a volume sufficiently large for it to snap off to form a lens; this distance increases rapidly as B increases towards B_0. In practice, snap-off may be speeded substantially by coalescence of adjacent collars (5, 6), so this stability boundary should be regarded as a necessary rather than a sufficient condition for snap-off. For $B_0 < B < B_1 \approx 1.769$, although the thin film experiences capillary instabilities, any collar of sufficiently small amplitude that may thereby develop will remain at small amplitude (case (ii) in Fig. 5). The critical collar volume beneath which an individual, finite-amplitude collar will shrink, and above which it will grow, is shown with a dotted line in Fig. 4. Figure 7a shows that the distance aZ/ϵ over which a large collar grows before it snaps off is potentially very great, and that it increases rapidly with B. Defining "clean" criteria for snap-off in this parameter range is therefore difficult, since initial collar amplitude, coalescence, and tube length must all be taken into account, but Fig. 7a provides approximate stability boundaries in terms of initial film thickness $\epsilon(=V/(4\pi^2))$, tube length $L(=Z)$ and B. Increasing the thickness of the liquid lining ahead of a draining collar, i.e., increasing ϵ, increases the volume of fluid available in the tube, therefore making snap-off more likely, but also reduces the value of B, which allows collars to grow more rapidly with distance as they drain. Furthermore, the (dimensional) length of tube required for snap-off, aZ/ϵ, is also diminished. However, the dominant error in this model grows rapidly with ϵ, being $O(\epsilon^{1/2})$ (from the neglected force due to viscous flow in the collar, see [21] and the Appendix), and so the accuracy of the model may only be guaranteed for small values of ϵ. For ultra-thin films, the effects of intermolecular forces and surface roughness on the structure of the transition regions are also likely to be important, but this possibility has not been explored here.

For $B_1 < B < B_2 \approx 11.235$ most collars decay to small amplitude as they drain down the tube, but surprisingly collars on the "right" solution branch (for which $\kappa_{\max} < \kappa < 2$ and $dV/d\kappa < 0$ in Fig. 2b), which are generally unstable, are rendered stable to small-amplitude perturbations by gravity (case (iii) in Fig. 5). Stable lenses are possible too, but spontaneous snap-off cannot occur. Finally, when $B > B_2$ all lenses will shrink and either rupture (in which case the resulting collars will shrink) or (in the presence of suitable stabilizing forces) form stable lamallae, supported by Plateau borders. In the latter case, it is the weight of fluid in the Plateau border that balances viscous stresses at its edges, and the lamella itself plays no active role in the motion.

The present model, which uses lubrication theory to describe the flow in the thin film and the fully nonlinear Young–Laplace equation elsewhere, provides nonlinear stability criteria for draining collars and lenses. An integral force balance [22] and [23] was used to determine the draining speed. As found in (33), the use of such a balance is convenient since there is no need to compute the effects of gravity on the collar shape, and symmetric collar solutions (Figs. 2 and A1) could be used. An alternative approach, in which slightly asymmetric collars are matched directly to transition regions (7, 31), gives equivalent results for thin collars.

The predictions of this model can be compared with those obtained using the thin-collar approximation alone ([2] and [9]), derived using the linearized Young–Laplace equation (5, 7). In this case the branch of collars that bifurcates from the uniform state exists only for $B = B_0$, and for $B < B_0$ the uniform state is unstable (in that solutions of [2] can blow up in finite time), while for $B > B_0$ it is stable (in the sense that disturbances remain of bounded amplitude). The value of B_0 obtained here is in close agreement with the estimate of B^* found by Chang and coworkers in (5, 7). The present theory can readily be used to show that steadily draining finite-amplitude collars exterior to fibres exist for $B < B_0$, and that they are stable, consistent with the finite-amplitude saturation demonstrated by (7). This theory therefore predicts that experimental determination of B^* will be relatively sharp for exterior collars, but amplitude-dependent for interior collars.

One application of this study is to airway closure in the lung. Of the twenty-odd generations of bifurcating airways in the human adult lung, those beyond about the tenth generation have radii smaller than the capillary lengthscale $(\sigma/\rho g)^{1/2} \approx 1$ mm (taking $\sigma \approx 10$ g/s^2). Taking $\epsilon \approx 0.1$ in every airway, B is roughly unity at generation 15 (where the airway radius is about 0.3 mm (34)), and B falls rapidly in the subsequent 5 or so generations. (If $\epsilon = 0.01$, B is roughly unity in the terminal bronchioles. In this case, the drainage speed of thin collars of thickness $O(a\epsilon)$ is $O(\epsilon^3\sigma/\mu) \approx 10^{-3}$ cm/s, taking $\mu \approx 10^{-2}$ g/(cms), whereas collars of thickness $O(a)$ drain significantly quicker, at a speed of $O(\epsilon^{3/2}\sigma/\mu) \approx 1$ cm/s (from [6])). Thus in larger vertical airways for which $B > B_2$, lenses (such as liquid boluses delivered during partial liquid ventilation, or during surfactant replacement therapy (35–37)) will shrink if they drain under gravity alone, and either rupture or form lamellae. Pressure gradients across such lenses and lamellae will undoubtedly be significant here too (28, 35). In these and in smaller vertical airways, for which $B > B_1$, gravity will prevent collars created by capillary instability from growing to form lenses. This stabilizing effect is consistent with observations showing that airway closure occurs predominantly in airways with diameters less than 0.5 mm (10). In the smallest terminal vertical airways, for which $B < B_1$, snap-off may occur (even in the absence of gravitational draining) if the depth of the liquid lining is sufficiently great (15, 16, 18, 19, 38), (the limit $\epsilon = V_{\max}/(4\pi^2) \approx 0.14$ indicated in Fig. 7a approximates the critical threshold), and gravity can induce snap-off of initially thinner liquid linings in sufficiently long airways. However, since most airways are short (only 3 or 4 diameters long), although individual collars may grow as they proceed along an airway, a collar that is initially thin is unlikely to accumulate sufficient fluid in a single airway to snap off (Fig. 7). More significantly, transverse components of gravity in nonvertical airways, or geometrical nonuniformities such as airway curvature (39), ellipticity, taper, and bifurcations may well disrupt collars as they drain, inhibiting snap-off. Other effects (such as airway wall permeability and compliance (40)) may therefore be much more likely than gravity to promote airway closure.

SUMMARY

The theory presented here shows that if the modified Bond number $\rho g a^2/\sigma\epsilon$ is sufficiently large, gravitational effects prevent the liquid lining of a vertical tube from "snapping off" to form lenses via a capillary instability. For vertical tubes with diameters substantially less than the capillary lengthscale $(\sigma/\rho g)^{1/2}$, however, gravity can cause collars created by capillary instability to grow in volume as they drain, so that spontaneous snap-off and lens formation will occur, but this will only occur in tubes that are sufficiently long. Because it is a singular, symmetry-breaking perturbation, weak gravity can therefore have a substantial effect on the liquid-lining distribution of vertical tubes over long distances and times.

APPENDIX

The weight of the collar or lens, which is $O(\rho g a^3)$, is balanced by viscous shear forces from the nose and tail regions, which are $O(\epsilon\sigma a)$, plus a shear force from the body of the collar, which is $O(\epsilon^{3/2}\sigma a)$ (since $U = O(\epsilon^{3/2}\sigma/\mu)$, see [6]). The dominant correction to the force balance [21] will therefore be $O(\epsilon^{1/2})$. We indicate here how such a correction may be determined and compute it explicitly for slender collars.

The leading-order flow in the collar satisfies (from [3])

$$\mathbf{0} = -\nabla\tilde{p} + \nabla^2\tilde{\mathbf{u}}, \quad \nabla\cdot\tilde{\mathbf{u}} = 0 \qquad \text{[A.1]}$$

in $R < r < 1$, where $R(z)$ is the equilibrium collar shape (as shown in Fig. 2a), $\mathcal{U}\epsilon^{3/2}\tilde{p}(r, z)$ is the corresponding contribution to the total pressure and $\mathbf{u} = \mathcal{U}\tilde{\mathbf{u}}(r, z)$ is the leading-order velocity. From [5], the axial flux within the collar $q = O(\epsilon)$, so that $\int_R^1 r\tilde{u}(r, z)\ dr = 0$ to leading order. The boundary conditions on the flow are $\tilde{\mathbf{u}} = \hat{\mathbf{z}}$ on $r = 1$ and zero tangential stress on $r = R(z)$. Since capillary forces control the shape of the collar, the normal stress condition is not needed and hydrostatic effects may be neglected at leading order.

At the outer limit of the nose and tail regions, where $h \gg 1$, [12] implies that $\tilde{p}_z \sim 3\mathcal{U}/(1-R)^2$, and thus the wall shear stress $-\tau \sim 3\mathcal{U}/(1-R)$. Thus, from [11], $u_r(1, z) \sim 6/[(\kappa-1)z^2]$ as $z \to 0$ (and similarly as $z \to L$), so the shear is strongly singular in the neighbourhood of each effective contact line.

The total force balance [21], including the contribution from the main body of the collar, may be written

$$\frac{VB}{2\pi} = \frac{3\mathcal{U}}{\epsilon^{1/2}}\int_{-\infty}^{\alpha}\frac{(h-1)}{h^2}\,dz + \mathcal{U}\epsilon^{1/2}\int_{\alpha}^{L-\beta}\tilde{u}_r(1, z)\,dz + \frac{3\mathcal{U}}{\epsilon^{1/2}}\int_{L-\beta}^{\infty}\frac{(h-h_\infty)}{h^2}\,dz + O(\epsilon), \qquad \text{[A.2]}$$

where the interface lies at $R = 1 - \epsilon h$ in the nose and tail regions.

Here α and β satisfy $\epsilon^{1/2} \ll \alpha \ll 1$, $\epsilon^{1/2} \ll \beta \ll 1$ as $\epsilon \to 0$, so that the nose lies in $z < \alpha$, the tail in $z > L - \beta$ and the body in $\alpha < z < \beta$. The $O(\epsilon)$ term represents the leading-order error in this approximation. In terms of $H(\zeta)$ and $G(\xi)$, which satisfy [14] and [18], respectively, [A.2] becomes

$$\frac{VB}{2\pi} = (3\mathcal{U})^{2/3} \int_{-\infty}^{\alpha(3\mathcal{U})^{1/3}/\epsilon^{1/2}} \frac{G-1}{G^2}\, d\xi + \mathcal{U}\epsilon^{1/2} \int_{\alpha}^{L-\beta} \tilde{u}_r(1,z)\, dz + (3\mathcal{U})^{2/3} \int_{-\beta(3\mathcal{U})^{1/3}/(h_\infty \epsilon^{1/2})}^{\infty} \frac{H-1}{H^2}\, d\zeta \qquad \text{[A.3]}$$

with error $O(\epsilon)$. Given [15], the tail-region integral in [A.3] becomes (see [17])

$$\int_Y^{\infty} \frac{H-1}{H^2}\, d\zeta = H_0 H_2 + \frac{2}{H_2 Y} + O(Y^{-3}) \qquad \text{[A.4a]}$$

as $Y \to -\infty$, while the nose-region contribution (see [20]) is

$$\int_{-\infty}^{Y} \frac{G-1}{G^2}\, d\xi = -G_0 G_2 - \frac{2}{G_2 Y} + O(Y^{-3}) \qquad \text{[A.4b]}$$

as $Y \to \infty$. Using [16], [9], and [23], the total force balance [A.3] is therefore

$$\frac{VB}{2\pi} = (\kappa - 1)F(h_\infty) + 2\mathcal{U}\epsilon^{1/2}\mathcal{I} + O(\epsilon), \qquad \text{[A.5]}$$

where

$$\mathcal{I} = \frac{1}{2}\left[\int_{\alpha}^{L-\beta} \tilde{u}_r(1,z)\, dz - \frac{6}{\kappa-1}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right], \qquad \text{[A.6]}$$

with additional errors in [A.5] of $O(\epsilon^{3/2}/\alpha^3, \epsilon^{3/2}/\beta^3)$ which can be made arbitrarily small. Since the leading-order collar flow is symmetric about $z = L/2$, [A.6] may be written

$$\mathcal{I} = \lim_{\alpha \to 0}\left[\int_{\alpha}^{L/2} \tilde{u}_r(1,z)\, dz - \frac{6}{\kappa-1}\frac{1}{\alpha}\right]. \qquad \text{[A.7]}$$

The singularity in the integrand as $z \to 0$ may be removed by writing [A.7] as

$$\mathcal{I} = \int_0^{L/2} \left\{\tilde{u}_r(1,z) - \frac{6}{(\kappa-1)z^2}\right\} dz - \frac{12}{(\kappa-1)L}. \qquad \text{[A.8]}$$

Equations [A.5] and [A.8] demonstrate explicitly the nature of the $O(\epsilon^{1/2})$ correction to the force balance from shear along the body of the collar. This correction must in general be determined numerically. When the collar is thin, however, the correction can be found explicitly, as follows.

When $\epsilon \ll \delta \equiv \kappa - 1 \ll 1$, so that the collar is thin but is still thicker than the liquid lining elsewhere (the same limit used in deriving [9]), the Young–Laplace equation gives the shape, length L and volume V of the collar as $R = 1 - \delta R_0 - \delta^2 R_1 + O(\delta^3)$, for $0 < z < L = 2\pi[1 - \delta + O(\delta^2)]$, where

$$R_0 = 1 - \cos z, \qquad \text{[A.9a]}$$

$$R_1 = z \sin z + \tfrac{1}{3}\cos 2z + \tfrac{2}{3}\cos z - 1, \qquad \text{[A.9b]}$$

and $V = 4\pi^2\delta - 11\pi^2\delta^2 + O(\delta^3)$. The flow in the collar may be determined using lubrication theory. Setting $r = 1 - \delta y$, $\tilde{u}(r,z) = U(y,z)$, and $\tilde{p}(r,z) = \delta^{-2}P(y,z)$, the governing equations [A.1], with error $O(\delta^2)$, reduce to $0 = -P_z + U_{yy} - \delta U_y$, $0 = -P_y$, where $U_y(R_0 + \delta R_1, z) = 0$, $U(1,0) = 1$, and

$$\int_0^{R_0+\delta R_1} (1 - \delta y) U\, dy = 0. \qquad \text{[A.10]}$$

The $O(\delta)$ terms that appear here arise from the cylindrical geometry. A regular expansion in powers of δ, with $U = U_0 + \delta U_1$, $P_z = P_{0z} + \delta P_{1z}$, gives

$$U_0 = 1 + \frac{3y^2}{2R_0^2} - \frac{3y}{R_0},$$

$$U_1 = \frac{3(R_0^2 - 4R_1)}{2R_0^3}\left(\frac{y^2}{2} - R_0 y\right) + \frac{y^3}{2R_0^3} - \frac{3y^2}{2R_0} - \frac{3R_1 y}{R_0^2} + \frac{3y}{2}$$

with $P_{0z} = 3/R_0^2$ and $P_{1z} = 3(R_0^2 - 4R_1)/(2R_0^3)$. The leading-order streamfunction $\psi = y[1 - (3y/2R_0) + (y^2/2R_0^2)]$ is shown in Fig. A1. The motion of the wall drives fluid from nose to tail, and a pressure gradient is induced that drives it in the reverse direction. The shear along the wall is

$$\tilde{u}_r(1,z) = \frac{3}{\delta R_0} - \frac{3R_1}{R_0^2} + O(\delta). \qquad \text{[A.11]}$$

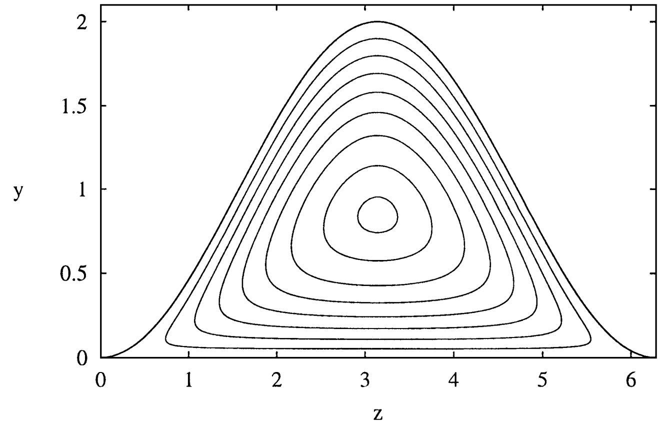

FIG. A.1. A graph of $\psi(z, y)$, showing streamlines in the frame of the collar; the flow is anticlockwise.

Since $\int R_0{}^{-1}\, dz = C - \cot(z/2)$ (for some constant C) and $\int_0^{\pi}(3R_1/R_0^2)\, dz = \pi/2$, [A.8] becomes (using [A.9a] and [A.10])

$$\mathcal{I} = \frac{3}{\delta}\left[\frac{2}{z} - \frac{1}{\tan(z/2)}\right]_0^{\pi(1-\delta)} - \frac{\pi}{2} - \frac{6}{\pi(1-\delta)} \qquad [A.12]$$

with error $O(\delta)$, so (A.5) reduces to

$$\frac{VB}{2\pi} = \delta F(h_\infty) - 4\pi\mathcal{U}\epsilon^{1/2} + O(\epsilon, \epsilon^{1/2}\delta). \qquad [A.13]$$

Surprisingly, the expected term of order $\epsilon^{1/2}/\delta$ vanishes, which is why higher order lubrication theory was required to compute the first correction due to stresses in the collar. Using the collar volume V as the small parameter (instead of $\delta = \kappa - 1$), [A.13] and assuming that $h_\infty = 1$, [A.13] may therefore be reexpressed (using [16]) as

$$B = \frac{F(1)}{2\pi}\left(1 + \frac{11V}{16\pi^2} + O(V^2)\right) - \frac{(\epsilon V)^{1/2}}{3\pi}\left[\frac{h_\infty}{H_2}\right]^{1/2} + O\left(\frac{\epsilon}{V}, V^3, \epsilon^{1/2}V^{3/2}\right), \qquad [A.14]$$

which is asymptotic for $\epsilon^{1/3} \ll V \ll \epsilon^{1/5} \ll 1$. Equation [A.14] gives the correction to the stability boundary shown in Fig. 4 near $B = B_0$. The correction depends not on $\epsilon^{1/2}$ but $(\epsilon V)^{1/2}$, and so is very modest for thin collars. The estimate of B_0 is therefore likely to be more accurate than those for B_1 and B_2. This result also shows how, as expected for fixed V, B is reduced slightly by including this extra shear contribution, and so the draining speed $\mathcal{U}$ falls also (from [27]).

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