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An Asymptotic Model of Viscous Flow Limitation in a Highly Collapsed Channel

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A viscous flow through a long two-dimensional channel, one wall of which is formed by a finite-length membrane, experiences flow limitation when the channel is highly collapsed over a narrow region under high external pressure. Simple approximate relations between flow rate and pressure drop are obtained for this configuration by the use of matched asymptotic expansions. Weak inertial effects are also considered.

1 Introduction

There are numerous examples of flows through flexible tubes in the body (notably the veins and urethra) in which a large, negative transmural (internal minus external) pressure causes severe deformation of the tube walls [3]. The strong interaction between the flow and the collapsible tube's geometry imparts unusual but biomedically significant properties to such systems, such as flow limitation [9], pressure-drop limitation, and susceptibility to self-excited oscillations. These properties are conveniently investigated experimentally using a Starling Resistor, and theoretically using a model system with relatively simple wall mechanics and geometry, namely a two-dimensional channel in which a segment of one wall is formed by a thin membrane under longitudinal tension [4–8]. Presently, there is no simple quantitative description of pressure-drop or flow limitation for viscous flows in either channels or tubes; all existing solutions have been obtained by numerical means. Here, the method of matched asymptotic expansions is used to show how approximate but explicit relations (Eq. (11)) between pressure drop and flow rate can be obtained for a channel in a highly nonuniform configuration (Fig. 1). The approximation is valid when (*a*) the channel is long compared to its width, (*b*) the upstream transmural pressure is small compared to the external pressure, (*c*) the downstream transmural pressure is large and negative, and (*d*) the flow is predominantly viscous. Under these conditions, the pressure drop along the channel is largely confined to a very short region (a boundary layer) across which the channel is highly collapsed. This asymptotic structure provides a simple and direct insight into the mechanism of viscous flow limitation.

2 The Model

Consider a two-dimensional channel of width h_0, in which flows a fluid of viscosity μ and density ρ with a steady, uniform flux Q_0. A segment of one channel wall of length L_0 is formed by a membrane under longitudinal tension T_0, assumed constant. We specify the pressure external to the membrane (p_e) and the transmural pressures at the upstream and downstream end of the membrane ($p_u - p_e$ and $p_d - p_e$). We set $p_d = 0$ without loss of generality.

We nondimensionalize the problem by scaling lengths on h_0, pressures on T_0/h_0 and velocities on Q_0/h_0. This yields dimensionless parameters

$$L = \frac{L_0}{h_0}, \quad Q = \frac{\mu Q_0}{h_0 T_0}, \quad R = \frac{\rho Q_0^2}{h_0 T_0},$$

$$P_e = \frac{p_e h_0}{T_0}, \quad P_u = \frac{p_u h_0}{T_0}, \quad P = P_u - P_e. \tag{1}$$

The ratio $\mathrm{Re} = R/Q = \rho Q_0/\mu$ is the Reynolds number of the problem; Q is equivalent to a capillary number. The collapsible segment of channel lies in $0 \le x \le L$, $0 \le y \le h(x)$ (assuming h is single-valued, but see (*ii*) below) where $h(0) = 1$, $h(L) = 1$. The dimensionless velocity field $\mathbf{u}(x, y) = (u, v)$ and pressure $p(x, y)$ satisfy

$$\nabla \cdot \mathbf{u} = 0, \quad R(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + Q\nabla^2 \mathbf{u}, \quad \int_0^{h(x)} u(x, y)dy = 1,$$

$$p(x, h) - P_e = -\kappa, \tag{2a–d}$$

with $u = 0$ on $y = 0$ and $y = h(x)$ and $p(0, 1) = P_u$, $p(L, 1) = 0$. Equations (2*a*, *b*) are the Navier–Stokes equations; Eq. (2*c*) implies uniform flux. In the membrane equation, Eq. (2*d*),

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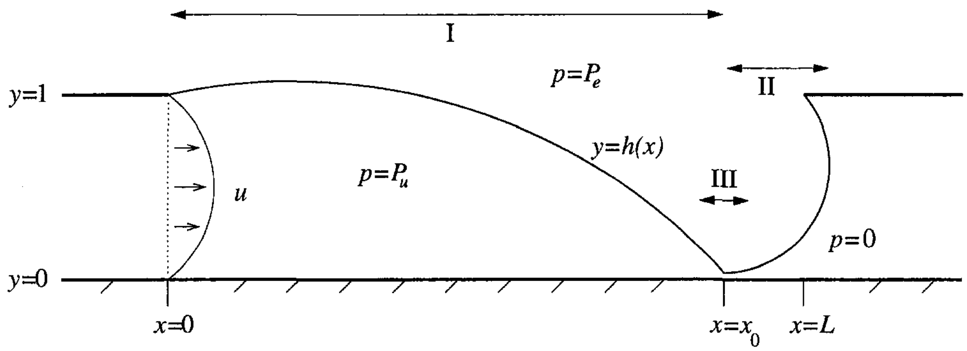

Fig. 1 The three asymptotic flow regimes: a long upstream region in $0 \le x < x_0$ of approximately uniform pressure P_u (I); a downstream region of $O(1)$ length, also of roughly uniform (zero) pressure (II); a short transition region near $x = x_0$ across which the pressure falls abruptly (III)

κ is the membrane's curvature; when h is single-valued, $\kappa = h_{xx}/(1 + h_x^2)^{3/2}$. We assume a Poiseuille entry flow, $u(0, y) = 6y(1 - y)$.

Assuming that the channel is long and severely collapsed over a narrow region near its downstream end, we consider the limit

$$L \gg 1, \quad P_e = O(1), \quad \hat{R} = \tfrac{6}{5}RL^4 = O(1),$$
$$\hat{Q} = 12QL^5 = O(1), \quad \hat{P} = \tfrac{1}{2}PL^2 = O(1), \qquad (3)$$

so that the flow is weak ($Q \ll 1$) although $\mathrm{Re} = O(L)$ may be moderately large. Motivated by computational results [4, 5], we split the flow into the three regions shown in Fig. 1, using $1/L$ as a small parameter in constructing matched asymptotic expansions for each region. In the "outer" regions I and II, the pressure is approximately uniform; the dominant pressure drop occurs across the "inner" region III at $x \approx x_0$. When the solution is expressed in variables appropriate to the long outer length scales, $h(x_0) = 0$ at the constriction; when resolved over short length scales, however, we shall show that the minimum channel width is $O(L^{-2})$. The solution in each region is as follows.

(i) In region I, of approximate length L, the pressure is uniform at leading order, with $O(1)$ value $p = P_u = P_e + P$, but the transmural pressure $P = O(L^{-2})$ is small. From Eq. (2b), pressure variations along this region due to viscous or inertial effects are $O(L^{-4})$ or smaller and are therefore negligible. Setting $x = XL$, $h(x) = \tilde{H}(X)$, Eq. (2d) becomes $-2\hat{P} = \tilde{H}_{XX}$, with error $O(L^{-2})$. With the constriction at $x = x_0 \equiv X_0L$ (see Fig. 1), we impose $\tilde{H}(0) = 1$, $\tilde{H}(X_0) = 0$. The membrane then has a parabolic shape

$$\tilde{H} = (X_0 - X)((1/X_0) + \hat{P}X),$$
$$0 \le X < X_0 < 1. \qquad (4)$$

The channel is inflated at its upstream end if $\hat{P} > X_0^{-2}$. Given Eq. (3), and the fact that $L - x_0 = O(1)$ (see Eq. (5) below), the slope of the membrane at $x = x_0$ is small but nonzero (provided $\hat{P} > -1$), so that $h(x) \sim -(1 + \hat{P})(x - x_0)/L$ as $x \to x_0-$.

(ii) In region II, since $Q \ll 1$ and $R \ll 1$ in Eq. (2b) the pressure is again uniform (zero) at leading order, so (from Eq. (2d)) the membrane forms the arc of a circle with radius $1/P_e$. Assuming $h(x_0) = h_x(x_0) = 0$ and $h(L) = 1$, it follows that

$$x_0 = L - \{(2/P_e) - 1\}^{1/2}. \qquad (5)$$

No solutions of this form exist for $P_e > 2$; for $1 < P_e < 2$ the membrane extends beyond $x = L$ and h is not single valued; for $P_e < 1$ with $P_e \gg O(L^{-2})$, h is well-defined in $x_0 \le x \le 1$. In both cases, however, $h(x) \sim \frac{1}{2}P_e(x - x_0)^2$ for $x \to x_0+$.

(iii) In region III, if we set $x = x_0 + (\xi/L)$, $y = \eta/L^2$, $h(x) = H(\xi)/L^2$, $u(x, y) = L^2U(\xi, \eta)$ and $v(x, y) = LV(\xi, \eta)$, Eqs. (2a–d) reduce in region III to the boundary-layer equations

$$U_\xi + V_\eta = 0, \quad \frac{5}{6}\hat{R}(UU_\xi + VU_\eta) = H_{\xi\xi\xi} + \frac{1}{12}\hat{Q}U_{\eta\eta},$$
$$\int_0^{H(\xi)} U(\xi, \eta)d\eta = 1 \qquad (6)$$

with error $O(L^{-2})$. Again $(U, V) = (0, 0)$ on $\eta = 0$ and $\eta = H$. If $\hat{R} \ll 1$, we recover the equations of lubrication theory (as in [7]), and the velocity profile is parabolic, $U = 6H^{-3}\eta(H - \eta)$. We assume that the profile is roughly unchanged if $\hat{R} = O(1)$. Then, the momentum integral equation

$$\frac{5}{6}\hat{R}\left(\int_0^H U^2 d\eta\right)_\xi = HH_{\xi\xi\xi} + \frac{1}{12}\hat{Q}[U_\eta]_0^H \qquad (7)$$

yields the following equation for H:

$$\hat{Q} - \hat{R}H_\xi = H^3H_{\xi\xi\xi}, \quad H \sim -(1 + \hat{P})\xi \text{ as } \xi \to -\infty,$$
$$H \sim \tfrac{1}{2}P_e\xi^2 \text{ as } \xi \to \infty, \qquad (8a\text{–}c)$$

which is asymptotically accurate if $\hat{R} \ll 1$ (and well known in this limit from draining-flow problems [1, 2]) and an approximation (presumed accurate) if $\hat{R} = O(1)$. Equations (8b, c) are the boundary conditions on the inner solution that allow matching onto the two outer solutions (see (i) and (ii)). After further rescaling, setting $H(\xi) = \hat{Q}(1 + \hat{P})^{-3}\hat{H}(\hat{\xi})$, $\xi = \hat{Q}(1 + \hat{P})^{-4}\hat{\xi}$ and $r = \hat{R}(1 + \hat{P})/\hat{Q}$, (8) yields a nonlinear eigenvalue problem for a function $C(r)$,

$$1 - r\hat{H}_{\hat{\xi}} = \hat{H}^3\hat{H}_{\hat{\xi}\hat{\xi}\hat{\xi}}, \; \hat{H} \sim -\hat{\xi} + O(1) \text{ as } \hat{\xi} \to -\infty,$$
$$\hat{H} \sim \tfrac{1}{2}C(r)\hat{\xi}^2 + O(1) \text{ as } \hat{\xi} \to \infty. \qquad (9a\text{–}c)$$

Numerical solutions suggest that Eq. (9) has a unique solution for each r; three examples are shown in Fig. 2(a, b). $\hat{H}$ has a single minimum ($\hat{H}_{\min}(r)$, Fig. 2(c)), across which there is an abrupt pressure drop because of the large local viscous resistance. As r (the local Reynolds number) increases from zero, the channel widens (from $\hat{H}_{\min}(0) \approx 1.26$) and the downstream curvature falls monotonically (from $C(0) \approx 1.21$); we also see the development of a weak adverse pressure gradient (Fig. 2(b)) due to flow deceleration, resembling numerical Navier–Stokes solutions in [5]. Finally, the downstream matching condition (linking $C(r)$ in Eq. 9(c) to P_e in Eq. 8(c)) yields the

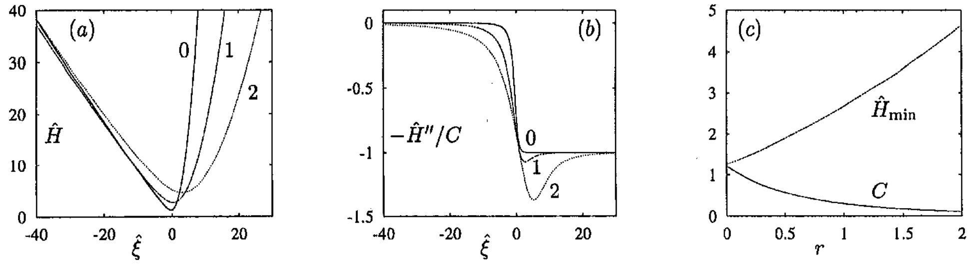

Fig. 2 Solutions of Eq. (9) in region III, showing (a) the scaled channel width $\hat{H}(\hat{\xi})$ and (b) the scaled pressure $-\hat{H}''(\hat{\xi})/C(r)$ for $r = 0$ (solid, purely viscous), $r = 1$ (dashed) and $r = 2$ (dotted); (c) shows $C(r)$ (solid) and $\hat{H}_{\min}(r)$ (dotted)

pressure-flow relation and minimum channel width for a range of Reynolds numbers,

$$(1 + \hat{P})^5 C(r) = P_e \hat{Q}, \quad h_{\min} = \frac{\hat{Q}\hat{H}_{\min}(r)}{L^2(1 + \hat{P})^3}. \quad (10a, b)$$

3 Results and Discussion

We can obtain two explicit pressure-flow relations from Eq. (10a) and Eq. (3):

$$P_u = P_e + \frac{2}{L^2}\left[\left(\frac{12P_eL^5Q}{C(r)}\right)^{1/5} - 1\right], \quad (11a)$$

$$P_u = \frac{1}{2}PL^2 + \frac{(1 + \frac{1}{2}PL^2)^5 C(r)}{12L^5Q}, \quad (11b)$$

which can be put into dimensional form using Eq. (1). These expressions underestimate the total pressure drop along the channel because of the neglect of small viscous losses in regions I and II, but are asymptotically accurate provided Eq. (3) is satisfied and $r \ll 1$.

If the downstream transmural pressure is fixed (P_e = constant), then the pressure drop down the channel P_u satisfies Eq. (11a), so that for $r \ll 1$ it increases weakly with Q (like $Q^{1/5}$), characteristic of pressure-drop limitation [3]. As P_u rises, $\hat{P}$ increases in Eq. (4) and the tube inflates at its upstream end but is largely unaltered downstream (see Eq. (5)), except that the minimum channel width increases slowly, like $Q^{2/5}$ (see Eq. (10a–b)). The approximation fails once $\hat{P}$ becomes large, and the tube becomes greatly inflated at its upstream end.

If the upstream transmural pressure $P = P_u - P_e$ is fixed with $r \ll 1$, then the pressure drop P_u satisfies Eq. (11b). Q falls as P_u and P_e rise, representing flow limitation (or negative effort dependence). The constriction moves closer to the downstream end of the tube (see Eq. (5)) and the membrane may be sucked into the downstream rigid segment; the channel narrows ($H_{\min} \propto Q$ from Eq. (10b)), but the degree of upstream inflation increases only marginally (by $O(L^{-1})$ as P_e varies by $O(1)$, from Eq. (4)).

Weak inertial effects are represented by the parameter r in Eq. (11); although Eq. (11) is then implicit, setting $C = C(0)$ is a reasonable first approximation. As r increases, the integral approximation used to derive Eq. (8) loses accuracy since the velocity profile will not remain parabolic, and the full elliptic boundary layer problem Eq. (6) should be solved. For sufficiently large r, a strong adverse pressure gradient (Fig. 2(b)) will ultimately lead to flow separation, which may be accompanied by flow instability [6]. Thereafter the asymptotic structure of the flow will be fundamentally changed. At high values of the Reynolds number the viscous mechanism of flow limitation described above will operate in addition to the familiar inertial mechanism in which long-wavelength disturbances cannot propagate upstream against a rapid flow [9].

Significant wall shear stress is confined to region III; its maximum dimensional value is $O(T_0/L_0)$. Since the constriction has length $O(h_0^2/L_0)$, the net relative drop in membrane tension due to viscous stresses $\Delta T/T_0 = O(L^{-2})$ and is therefore negligible.

The asymptotic solutions presented here are valuable because they represent a limit in which existing numerical methods fail to converge, even at low Reynolds numbers [4, 5]. There is also scope for extending the approach given above to include more sophisticated membrane mechanics. At present, however, it is not clear the extent to which the asymptotic structure described here for channel flows will be shared by physiologically realistic three-dimensional collapsible tubes, although this model may have some relevance to the sheet-like pulmonary circulation in zone II of the lung.

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