Inverse monoids and immersions of cell complexes

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Immersions

Definition

A continous map $f : \mathcal{Y} \to \mathcal{X}$ between topological spaces is called a (topological) immersion if every point $y \in \mathcal{Y}$ has a neighborhood U that is mapped homeomorphically onto $f(U)$ by f.

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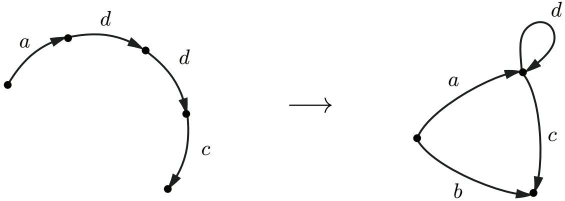
Example: immersions between (directed) graphs



Additional restriction: respect the structure of the graph

Covering spaces

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The idea is to characterize immersions by replacing the fundamental group with an appropriate inverse monoid.

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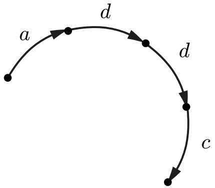
Free inverse monoids exist. (Notation: $\mathrm{FIM}(X)$)

Inverse monoid actions

Definition

An inverse monoid S acts on the set X if there is a homomorphism $S \to \mathrm{SIM}(X)$.

Example: let Γ be a graph edge-labeled in a deterministic and co-deterministic way over a set A, then $\mathrm{FIM}(A)$ acts on $V(\Gamma)$.



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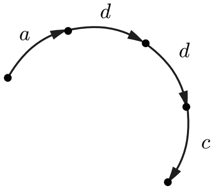

Note: the stabilizer of a set is always an inverse submonoid, and it is **closed upwards** in the natural partial order. Such inverse submonoids are called closed. (Notation: $M \leq^{\omega} S$)

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Note: if Γ be a digraph edge-labeled over the set X in a deterministic and co-deterministic way, then Then **paths starting at v are words over $X \cup X^{-1}$**, hence $L(\Gamma, v) \leq \mathrm{FIM}(X)$.

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A key idea: $L(\Gamma, v)$ is the stabilizer of v under the action of $\mathrm{FIM}(X)$.

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Remarks:

- $H, K \subseteq L(\Gamma_1, v_1)$ correspond to the same immersion iff they are conjugate
- $L(\Gamma, v)$ and $L(\Gamma, v')$ are conjugate, but not necessarily isomorphic (unlike in the case of the fundamental group)

Immersions in higher dimensions

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In a CW-complex $\mathcal{C}$, every cell has an attaching map $\varphi\colon S^n \to \mathcal{C}$ and a characteristic map $\sigma\colon B^n \to \mathcal{C}$.

Immersions in higher dimensions

Δ-complexes: CW-complexes with restricted attaching maps:

Each cell has a distinguished characteristic map $\sigma \colon \Delta^n \to \mathcal{C}$ such that the restriction to a face of Δ^n is also a characteristic map of some cell.

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The standard simplex $\Delta^n = [v_0, \ldots, v_n]$ comes with a natural ordering on its vertices. We call the smallest vertex v_0 the **root** of the simplex, $\sigma(v_0)$ is called the **root** of the cell, denoted by $\alpha(C)$.

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Digraphs: edges are 2-simplices $[v_0, v_1]$; $\alpha(e) = \sigma(v_0)$, $\omega(e) = \sigma(v_1)$.

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Immersion between Δ-complexes: a continous map which is a local homeomorphism onto its image and commutes with the characteristic maps.

The loop monoid of a Δ-complex

Let $\mathcal{C}$ be a Δ-complex.
A **generalized** path in $\mathcal{C}$ is a sequence of **cells** $s_1 \dots s_n$ such that $\omega(s_j) = \alpha(s_{j+1})$.

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- labels of cells of dimension at least 2 always act identically
- labels of closed generalized paths on the boundary of a cell act identically wherever the cell lifts

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- if $n = 2$, let $bw(C)$ be the image of the path (v_0, v_1, v_2, v_0) under σ;
- if $n > 2$, let $C_i = [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]$, and let $bw(C)$ be the image of $C_n C_{n-1} \ldots C_1 (v_0, v_1) C_0 (v_1, v_0)$ under σ.

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Note: $bw(\rho) := \ell(bw(C))$, where $\ell(C) = \rho$, is well-defined.

The loop monoid

Take a Δ-complex labeled over $X \cup P$, and consider the inverse monoid $M_{X,P} = \langle X \cup P \rangle$, defined by the following relations: for any $\rho \in P$,

- $\rho^2 = \rho$, and
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$L(\mathcal{C}, v) :=$ generalized paths around v wrt the above relations $=$ the stablizer of v under this action

Note:

- $L(\mathcal{C}, v) \leq^{\omega} M_{X,P}$;
- the greatest group homomorphic image of $L(\mathcal{C}, v)$ is $\pi_1(\mathcal{C})$.

The main theorem

Theorem (Meakin, Sz.)

1. *Let $\mathcal{C}$ and $\mathcal{D}$ be Δ-complexes labeled over a common set $X \cup P$, and suppose $g\colon \mathcal{D} \to \mathcal{C}$ is an immersion that commutes with the labeling, and let $v \in \mathcal{D}^{(0)}$. Then $L(\mathcal{D}, v)$ is a closed inverse submonoid of $L(\mathcal{C}, g(v))$.*

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2. *Let $\mathcal{C}$ be a Δ-complex labeled over $X \cup P$, let $u \in \mathcal{C}^{(0)}$, and let H be any closed inverse submonoid of $L(\mathcal{C}, u) \subseteq M_{X,P}$. Then there exists a unique complex $\mathcal{C}_H$ and a unique immersion $f : \mathcal{C}_H \to \mathcal{C}$ with $H = L(\mathcal{C}_H, v)$ for some vertex $v \in \mathcal{C}_H$ with $f(v) = u$.*

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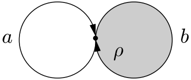
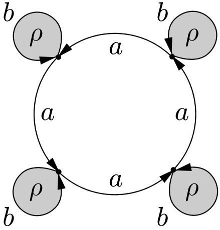
Remark: the above theorem was proven by Meakin and Sz. for arbitrary CW-complexes in the 2-dimensional case.

Examples of immersions over $D \vee S^1$

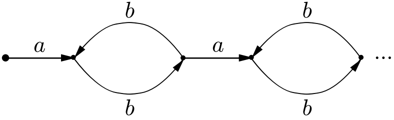
$$M_{X,P} = Inv\langle a, b, \rho \mid \rho^2 = \rho,\ \rho \leq b\rangle$$





$$\langle a^k, a^n \rho a^{-n} : n \in \{1, \ldots, k\}\rangle^\omega$$

$k \in \mathbb{N}$, $(k = 4)$



$$\langle (ab)^n ab^2 a^{-1} (ab)^{-n} : n \in \mathbb{N}\rangle^\omega$$

Thank you for your attention!