On the graph condition regarding the F-inverse cover problem

Nóra Szakács

University of Szeged

AMS-EMS-SPM meeting, Porto

2015. 06. 13.

Table of contents

Inverse monoids

Recall the following notions:

- **Inverse monoid:** $\forall a \in M \ \exists! a^{-1} \in M$ with $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$

Inverse monoids

Recall the following notions:

- **Inverse monoid:** $\forall a \in M \ \exists! a^{-1} \in M$ with $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$
- natural partial order $\leq$

Inverse monoids

Recall the following notions:

- **Inverse monoid:** $\forall a \in M\ \exists! a^{-1} \in M$ with $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$
- natural partial order $\leq$
- smallest group congruence σ

Inverse monoids

Recall the following notions:

- **Inverse monoid:** $\forall a \in M\ \exists! a^{-1} \in M$ with $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$
- natural partial order $\leq$
- smallest group congruence σ

Definition

An inverse monoid is called F-**inverse** if each σ-class has a greatest element wrt $\leq$.

Inverse monoids

Recall the following notions:

- **Inverse monoid:** $\forall a \in M \ \exists! a^{-1} \in M$ with $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$
- natural partial order $\leq$
- smallest group congruence σ

Definition

An inverse monoid is called F-**inverse** if each σ-class has a greatest element wrt $\leq$.

E.g. free inverse monoids.

F-inverse covers

Definition

An inverse monoid F is an F-**inverse cover** of M if there is an idempotent separating, surjective morphism $F \to M$.

F-inverse covers

Definition

An inverse monoid F is an F-**inverse cover** of M if there is a surjective morphism $F \to M$.

F-inverse covers

Definition

An inverse monoid F is an F-**inverse cover** of M if there is a surjective morphism $F \to M$.

Every inverse monoid has an F-inverse cover — a free inverse monoid.

F-inverse covers

Definition

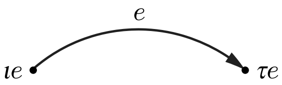
An inverse monoid F is an F-**inverse cover** of M if there is a surjective morphism $F \to M$.

Every inverse monoid has an F-inverse cover — a free inverse monoid.

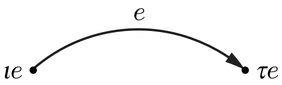
Open problem (Henckell, Rhodes, 1991):
Does every finite inverse monoid have a *finite* F-inverse cover?

Table of contents

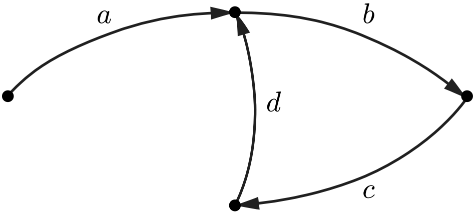
Graphs: finite, directed



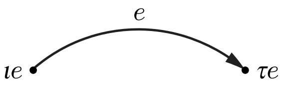
Graphs: finite, directed



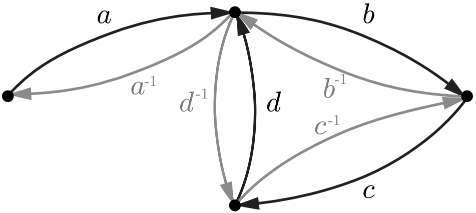
Paths



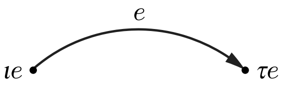
Graphs: finite, directed



Paths can go "both directions", traversing a formal inverse edge when going backwards

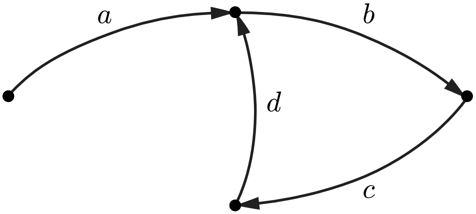


Graphs: finite, directed

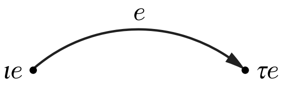


Paths can go "both directions", traversing a formal inverse edge when going backwards

E.g. $p = ad^{-1}c^{-1}$

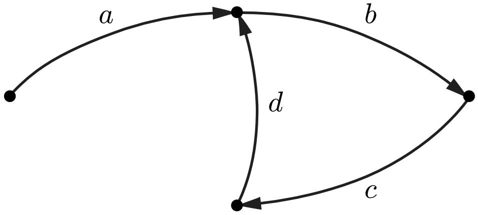


Graphs: finite, directed



Paths can go "both directions", traversing a formal inverse edge when going backwards

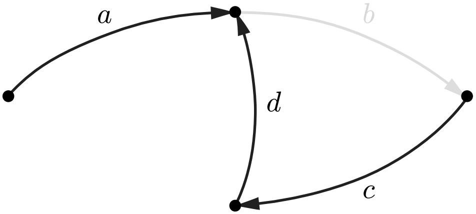
E.g. $p = ad^{-1}c^{-1}$



Note: **paths** of a graph Γ can be regarded as **words** over the alphabet $E(\Gamma) \cup (E(\Gamma))^{-1}$

$\langle p \rangle$: the subgraph **spanned by** the path p

For $p = ad^{-1}c^{-1}$, $\langle p \rangle$ is:



Group varieties

A **group variety** is a class of groups defined by **identities**.

For example

- **Ab**: $xy = yx$

Group varieties

A **group variety** is a class of groups defined by **identities**.

For example

- **Ab**: $xy = yx$
- $\mathbf{Ab}_n$: $xy = yx$ and $x^n = 1$

Group varieties

A **group variety** is a class of groups defined by **identities**.

For example

- **Ab**: $xy = yx$
- **Ab**$_n$: $xy = yx$ and $x^n = 1$

Definition

A group variety is **locally finite** if its finitely generated groups are finite.

Group varieties

A **group variety** is a class of groups defined by **identities**.

For example

- **Ab**: $xy = yx$ $\varnothing$
- $\mathbf{Ab}_n$: $xy = yx$ and $x^n = 1$

Definition

A group variety is **locally finite** if its finitely generated groups are finite.

Group varieties

A **group variety** is a class of groups defined by **identities**.

For example

- **Ab**: $xy = yx$ $\varnothing$
- $\mathbf{Ab}_n$: $xy = yx$ and $x^n = 1$ ✓

Definition

A group variety is **locally finite** if its finitely generated groups are finite.

Two sequences of subgraphs

Fix a graph Γ and a (locally finite) group variety $\mathbf{U}$.
For every path p of Γ, let

$$C_0(p) = \bigcap \{ \langle q \rangle : q \text{ is a path} \qquad \qquad \}$$

Two sequences of subgraphs

Fix a graph Γ and a (locally finite) group variety $\mathbf{U}$.
For every path p of Γ, let

$$C_0(p) = \bigcap \{\langle q \rangle : q \text{ is a path coterminal to } p,\ p \equiv_{\mathbf{U}} q\}$$

Two sequences of subgraphs

Fix a graph Γ and a (locally finite) group variety $\mathbf{U}$.
For every path p of Γ, let

$$C_0(p) = \bigcap \{\langle q \rangle : q \text{ is a path coterminal to } p,\ \ p \equiv_{\mathbf{U}} q\}$$

$\downarrow$

q can be obtained from p using identities of $\mathbf{U}$

Two sequences of subgraphs

Fix a graph Γ and a (locally finite) group variety $\mathbf{U}$.
For every path p of Γ, let

$$C_0(p) = \bigcap \{\langle q \rangle : q \text{ is a path coterminal to } p,\ p \equiv_{\mathbf{U}} q\}$$

$\downarrow$ q can be obtained from p using identities of $\mathbf{U}$

Note: if Γ is a two-edge-connected graph, then $C_0(p)$ is just the graph spanned by the $\mathbf{U}$-content of p together with ιp.

Two sequences of subgraphs

Fix a graph Γ and a (locally finite) group variety $\mathbf{U}$.
For every path p of Γ, let

$$C_0(p) = \bigcap \{ \langle q \rangle : q \text{ is a path coterminal to } p,\ p \equiv_{\mathbf{U}} q \}$$

$\downarrow$ q can be obtained from p using identities of $\mathbf{U}$

Note: if Γ is a two-edge-connected graph, then $C_0(p)$ is just the graph spanned by the $\mathbf{U}$-content of p together with ιp.

Let $P_0(p)$ be the connected component of $C_0(p)$ containing ιp.

Two sequences of subgraphs

If C_n, P_n are definied for all paths of Γ, let

$$C_{n+1}(p) = \bigcap \{P_n(q_1) \cup \cdots \cup P_n(q_k) : k \in \mathbb{N}, \\ q_1 \ldots q_k \text{ is a path coterminal to } p,\ p \equiv_{\mathbf{U}} q_1 \ldots q_k\}$$

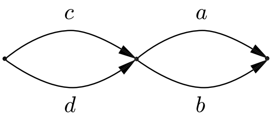
Two sequences of subgraphs

If C_n, P_n are definied for all paths of Γ, let

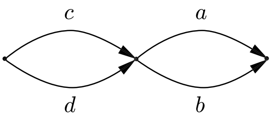
$$C_{n+1}(p) = \bigcap \{P_n(q_1) \cup \cdots \cup P_n(q_k) : k \in \mathbb{N}, \\ q_1 \ldots q_k \text{ is a path coterminal to } p,\ p \equiv_{\mathbf{U}} q_1 \ldots q_k\}$$

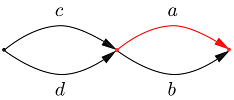
And let again $P_{n+1}(p)$ be the connected component of $C_{n+1}(p)$ containing ιp.

An example

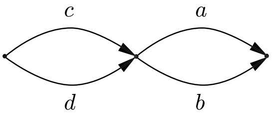
Put $\mathbf{U} = \mathbf{Ab}$ and $p = a$ in the graph 

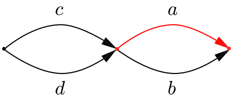
An example

Put $\mathbf{U} = \mathbf{Ab}$ and $p = a$ in the graph 

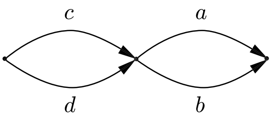
$C_0(a) =$ 

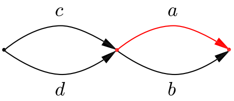
An example

Put $\mathbf{U} = \mathbf{Ab}$ and $p = a$ in the graph 

$C_0(a) = P_0(a) =$ 

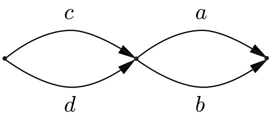
An example

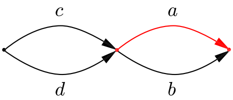
Put $\mathbf{U} = \mathbf{Ab}$ and $p = a$ in the graph 

$C_0(a) = P_0(a) =$ 

$C_1(a)$

An example

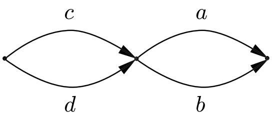
Put $\mathbf{U} = \mathbf{Ab}$ and $p = a$ in the graph 

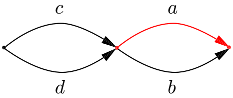
$C_0(a) = P_0(a) =$ 

$$a \equiv_{\mathbf{Ab}} c^{-1} \cdot cab^{-1}c^{-1} \cdot cb$$

$C_1(a)$

An example

Put $\mathbf{U} = \mathbf{Ab}$ and $p = a$ in the graph 

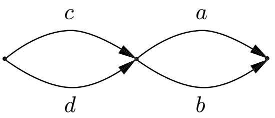
$C_0(a) = P_0(a) =$ 

$$a \equiv_{\mathbf{Ab}} c^{-1} \cdot cab^{-1}c^{-1} \cdot cb$$

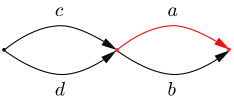
$$C_1(a) \subseteq P_0(a) \cap (P_0(c^{-1}) \cup P_0(cab^{-1}c^{-1}) \cup P_0(cb))$$

An example

Put $\mathbf{U} = \mathbf{Ab}$ and $p = a$ in the graph



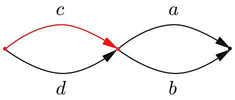
$C_0(a) = P_0(a) =$



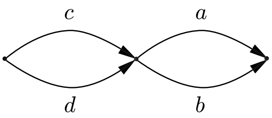
$$a \equiv_{\mathbf{Ab}} c^{-1} \cdot cab^{-1}c^{-1} \cdot cb$$

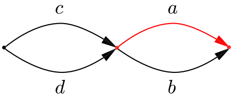
$$C_1(a) \subseteq P_0(a) \cap (P_0(c^{-1}) \cup P_0(cab^{-1}c^{-1}) \cup P_0(cb))$$

$$C_0(c^{-1}) =$$



An example

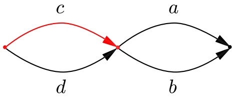
Put $\mathbf{U} = \mathbf{Ab}$ and $p = a$ in the graph 

$C_0(a) = P_0(a) =$ 

$$a \equiv_{\mathbf{Ab}} c^{-1} \cdot cab^{-1}c^{-1} \cdot cb$$

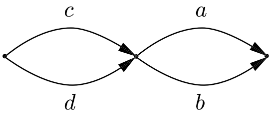
$$C_1(a) \subseteq P_0(a) \cap (P_0(c^{-1}) \cup P_0(cab^{-1}c^{-1}) \cup P_0(cb))$$

$\swarrow$

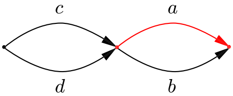


An example

Put $\mathbf{U} = \mathbf{Ab}$ and $p = a$ in the graph



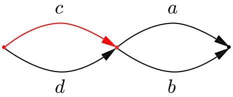
$C_0(a) = P_0(a) =$

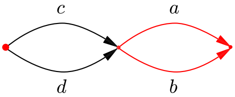


$$a \equiv_{\mathbf{Ab}} c^{-1} \cdot cab^{-1}c^{-1} \cdot cb$$

$$C_1(a) \subseteq P_0(a) \cap (P_0(c^{-1}) \cup P_0(cab^{-1}c^{-1}) \cup P_0(cb))$$

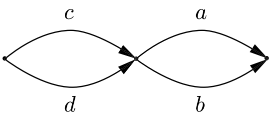
$\swarrow$ $\qquad C_0(cab^{-1}c^{-1}) =$



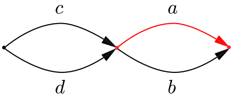


An example

Put $\mathbf{U} = \mathbf{Ab}$ and $p = a$ in the graph



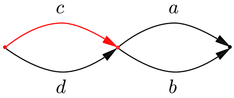
$C_0(a) = P_0(a) =$

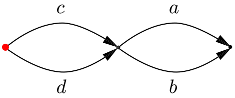


$$a \equiv_{\mathbf{Ab}} c^{-1} \cdot cab^{-1}c^{-1} \cdot cb$$

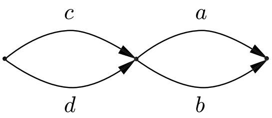
$$C_1(a) \subseteq P_0(a) \cap (P_0(c^{-1}) \cup P_0(cab^{-1}c^{-1}) \cup P_0(cb))$$

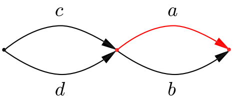
$\swarrow$ $\downarrow$





An example

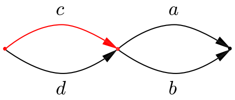
Put $\mathbf{U} = \mathbf{Ab}$ and $p = a$ in the graph 

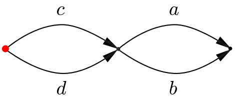
$C_0(a) = P_0(a) =$ 

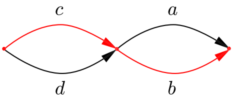
$$a \equiv_{\mathbf{Ab}} c^{-1} \cdot cab^{-1}c^{-1} \cdot cb$$

$$C_1(a) \subseteq P_0(a) \cap (P_0(c^{-1}) \cup P_0(cab^{-1}c^{-1}) \cup P_0(cb))$$

$\swarrow$ $\downarrow$ $C_0(cb) =$

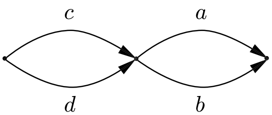




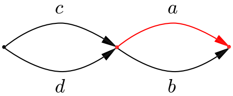


An example

Put $\mathbf{U} = \mathbf{Ab}$ and $p = a$ in the graph

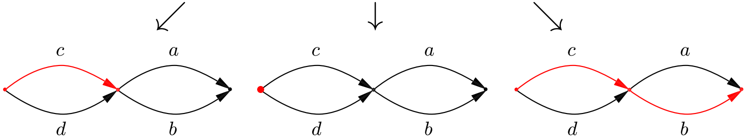


$C_0(a) = P_0(a) =$



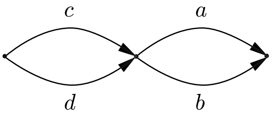
$$a \equiv_{\mathbf{Ab}} c^{-1} \cdot cab^{-1}c^{-1} \cdot cb$$

$$C_1(a) \subseteq P_0(a) \cap \left(P_0(c^{-1}) \cup P_0(cab^{-1}c^{-1}) \cup P_0(cb)\right)$$

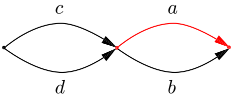


An example

Put $\mathbf{U} = \mathbf{Ab}$ and $p = a$ in the graph

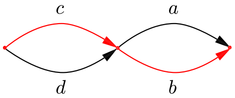


$$C_0(a) = P_0(a) =$$



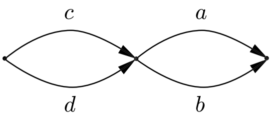
$$a \equiv_{\mathbf{Ab}} c^{-1} \cdot cab^{-1}c^{-1} \cdot cb$$

$$C_1(a) \subseteq P_0(a) \cap \underbrace{\left(P_0(c^{-1}) \cup P_0(cab^{-1}c^{-1}) \cup P_0(cb)\right)}$$

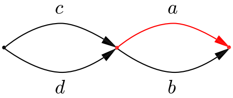


An example

Put $\mathbf{U} = \mathbf{Ab}$ and $p = a$ in the graph

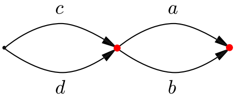


$$C_0(a) = P_0(a) =$$



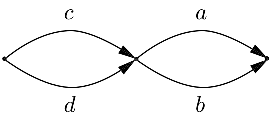
$$a \equiv_{\mathbf{Ab}} c^{-1} \cdot cab^{-1}c^{-1} \cdot cb$$

$$C_1(a) \subseteq \underbrace{P_0(a) \cap (P_0(c^{-1}) \cup P_0(cab^{-1}c^{-1}) \cup P_0(cb))}$$

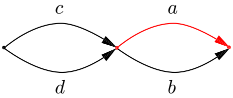


An example

Put $\mathbf{U} = \mathbf{Ab}$ and $p = a$ in the graph



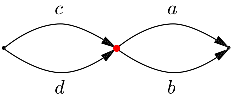
$C_0(a) = P_0(a) =$



$$a \equiv_{\mathbf{Ab}} c^{-1} \cdot cab^{-1}c^{-1} \cdot cb$$

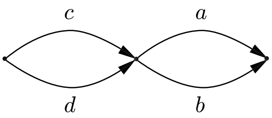
$$C_1(a) \subseteq P_0(a) \cap (P_0(c^{-1}) \cup P_0(cab^{-1}c^{-1}) \cup P_0(cb))$$

$P_1(a) \subseteq$

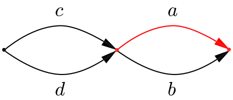


An example

Put $\mathbf{U} = \mathbf{Ab}$ and $p = a$ in the graph



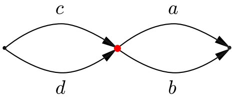
$C_0(a) = P_0(a) =$



$$a \equiv_{\mathbf{Ab}} c^{-1} \cdot cab^{-1}c^{-1} \cdot cb$$

$$C_1(a) \subseteq P_0(a) \cap (P_0(c^{-1}) \cup P_0(cab^{-1}c^{-1}) \cup P_0(cb))$$

$P_1(a) =$



The graph condition

Theorem (Auinger, Szendrei, 2006)

All finite inverse monoids admit a finite F-inverse cover if and only if for each graph Γ there is a locally finite group variety $\mathbf{U}$ such that $\tau p \in P_n(p)$ for all n and all paths p of Γ.

The graph condition

Theorem (Auinger, Szendrei, 2006)

All finite inverse monoids admit a finite F-inverse cover if and only if for each graph Γ there is a locally finite group variety $\mathbf{U}$ such that $\tau p \in P_n(p)$ for all n and all paths p of Γ.

A path p with $\tau p \notin P_n(p)$ for some n: **breaking path**

The graph condition

Theorem (Auinger, Szendrei, 2006)

All finite inverse monoids admit a finite F-inverse cover if and only if for each graph Γ there is a locally finite group variety $\mathbf{U}$ such that $\tau p \in P_n(p)$ for all n and all paths p of Γ.

A path p with $\tau p \notin P_n(p)$ for some n: **breaking path**

How to solve the F-inverse cover problem: find "good" varieties for graphs

Table of contents

The property $(S_{\mathbf{U}})$

The set of graphs for which $\mathbf{U}$ is "good" (they contain no breaking paths over $\mathbf{U}$): $(S_{\mathbf{U}})$

The property $(S_{\mathbf{U}})$

The set of graphs for which $\mathbf{U}$ is "good" (they contain no breaking paths over $\mathbf{U}$): $(S_{\mathbf{U}})$

Proposition

If $\mathbf{U} \subseteq \mathbf{V}$, *then* $(S_{\mathbf{U}}) \subseteq (S_{\mathbf{V}})$.

Bigger varieties are "good" for more graphs.

Minors

Definition

We call Δ a **minor** of Γ if it can be obtained from Γ by edge-contraction, omitting vertices and edges, and redirecting edges.

Minors

Definition

We call Δ a **minor** of Γ if it can be obtained from Γ by edge-contraction, omitting vertices and edges, and redirecting edges.

Theorem

If Δ is a minor of Γ and Δ contains a breaking path over $\mathbf{U}$, then so does Γ.

Minors

Definition

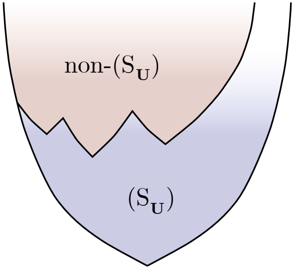
We call Δ a **minor** of Γ if it can be obtained from Γ by edge-contraction, omitting vertices and edges, and redirecting edges.

Theorem

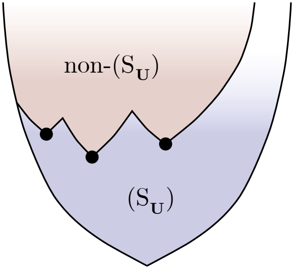
If Δ is a minor of Γ and Δ contains a breaking path over $\mathbf{U}$, then so does Γ.

Proof: the breaking path in the small graph can be lifted.

Another way of putting this: the set of **non-($S_{\mathbf{U}}$)** graphs (=those containing breaking paths over **U**) are **closed upwards** in the minor ordering:



Another way of putting this: the set of **non-**($S_{\mathbf{U}}$) graphs (=those containing breaking paths over **U**) are **closed upwards** in the minor ordering:

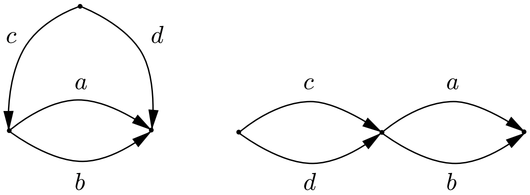


Hence the set of non-($S_{\mathbf{U}}$) graphs can be described by their minimal elements called **forbidden minors**.

Abelian varieties

Theorem

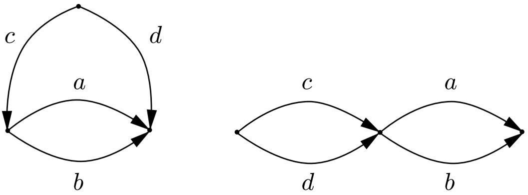
Let **U** *be any nontrivial Abelian group variety. The forbidden minors for* **U** *are the graphs below:*



Abelian varieties

Theorem

Let **U** *be any nontrivial Abelian group variety. The forbidden minors for* **U** *are the graphs below:*



Proof (part 1): the path "a" is a breaking one in both cases.

Other varieties?

$\mathbf{Ab}_n\mathbf{Ab}_m$:

- locally finite
- larger than Ab_n
- the relatively free group has a polynomial word problem

Other varieties?

$\mathbf{Ab}_n\mathbf{Ab}_m$:

- locally finite
- larger than Ab_n
- the relatively free group has a polynomial word problem
- ...and is double exponential in rank

Other varieties?

$\mathbf{Ab}_n\mathbf{Ab}_m$:

- locally finite
- larger than Ab_n
- the relatively free group has a polynomial word problem
- ...and is double exponential in rank

All we know is that if Γ conatins a breaking path, then it has

- at least one of the Abelian minors
- and 

as minors.

Thank you for your attention!