Inverse monoids of partial graph automorphisms

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Partial automorphisms of graphs

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- directed, edge-colored: $\Gamma = (V; E_1, \cdots, E_n)$

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Partial graph automorphism: a partial one-to-one map $\psi \colon V \to V$ which preserves edges (with direction/color/multiplicity) and non-edges.

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The reason this is hard: some partial automorphisms between subgraphs on $n-1$ points don't extend to automorphisms.

The automorphism group

Aut(Γ): the group of all automorpisms of Γ.

Theorem (Frucht, 1939)

For all finite groups G, there exists a graph Γ such that $G \cong \mathrm{Aut}(\Gamma)$ (moreover, Γ can be chosen to be 3-regular).

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A more difficult question:
If $G = \mathrm{Aut}(\Gamma)$, then $G \leq S_V$. Given a permutation group $G \leq S_n$, does there exist a graph Γ on n vertices for which $G = \mathrm{Aut}(\Gamma)$?

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There is a necessary and sufficient condition for edge-colored graphs, in general (to my knowledge) such is not known.

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- For which inverse monoids S does there exist a graph Γ such that $\mathrm{PAut}(\Gamma) \cong S$?
- For which inverse submonoids S of I_V does there exist a graph Γ on V such that $\mathrm{PAut}(\Gamma) = S$?

Overview: the structure of I_X

Idempotents: partial identical maps

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Rank of a map: cardinality of its domain/image

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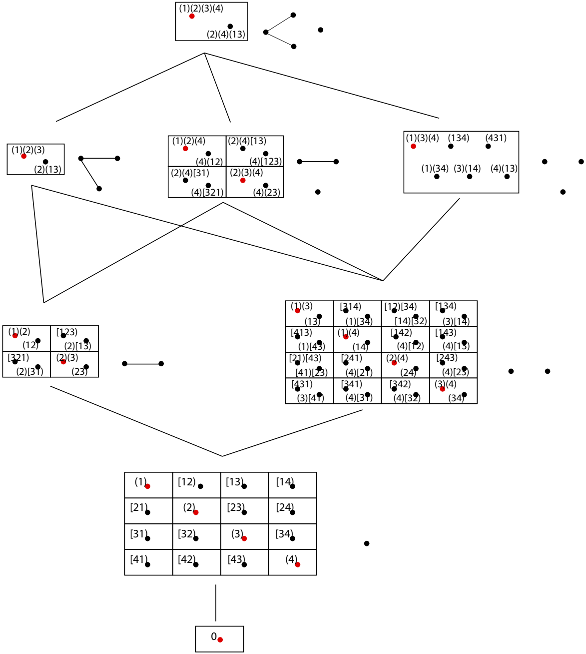
Theorem

If for graphs Γ_1, Γ_2 *we have* $\mathrm{PAut}(\Gamma_1) = \mathrm{PAut}(\Gamma_2)$*, then* $\Gamma_1 = \Gamma_2$ *or* $\Gamma_1 = \overline{\Gamma_2}$.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For all $A \subseteq I_V$ pairwise compatible rank 1 maps, $\bigcup A \in \mathrm{PAut}(\Gamma)$ iff for all $a_1, a_2 \in A$ we have $a_1 \cup a_2 \in \mathrm{PAut}(\Gamma)$.

The characterization of $\mathrm{PAut}(\Gamma)$ in I_V

Theorem

Given an inverse monoid $S \subseteq I_V$ (V is finite), there exists a simple, undirected graph Γ on V such that $\mathrm{PAut}(\Gamma) = S$ if and only if the following hold:

1. $E(I_V) \subseteq S$,
2. *the rank 2 elements of S form at most two $\mathcal{D}$-classes,*
3. *the rank 2 $\mathcal{H}$-classes of S are nontrivial,*
4. *for any compatible subset $A \subseteq \mathcal{S}$ of rank 1 partial permutations, if S contains the join of any two elements of A, then S contains the join of the set A.*

The characterization of $\mathrm{PAut}(\Gamma)$ in I_V

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Given an inverse monoid $S \subseteq I_V$ (V is finite), there exists an edge-colored digraph Γ on V such that $\mathrm{PAut}(\Gamma) = S$ if and only if the following hold:

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Boolean inverse monoids

Definition

An inverse monoid S with zero is called *Boolean* if the semilattice $E(S)$ is the meet semilattice of a Boolean algebra.

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Note: PAut(Γ) is Boolean for any graph, and $V \longleftrightarrow$ the atoms of $E(\mathrm{PAut}(\Gamma))$.

$\Longrightarrow$ PAut(Γ) has a faithful representation on the atoms of $E(\mathrm{PAut}(\Gamma))$, and it is exactly what one gets restricting the (in this case, faithful) Munn representation.

Restricted Munn representation

Let S be a finite inverse monoid, and let X denote the set of atoms of $E(S)$. The *restricted Munn representation* of S is the Munn representation of S restricted to the atoms $\mathcal{A}$ of $E(S)$:
$\alpha_S \colon s \mapsto \hat{m}_s$, $\hat{m}_s \colon \langle ss^{-1} \rangle \cap \mathcal{A} \to \langle s^{-1}s \rangle \cap \mathcal{A}$, $e \mapsto s^{-1}es$.

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If $S = \mathrm{PAut}(\Gamma)$, then $\alpha_S(S) = S$ (under the identification of id_v and v).

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We call $a, b \in S$ compatible if $ab^{-1}, a^{-1}b$ are idempotents.

Fact: If $A \subseteq S$ has a join, then elements of A are pairwise compatible (the converse is not true, however).

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Note: in $\mathrm{PAut}(\Gamma)$, height = rank.

The characterization of $\mathrm{PAut}(\Gamma)$ in the abstract case

Theorem

Given a finite inverse monoid S there exists a simple, undirected graph Γ such that $\mathrm{PAut}(\Gamma) \cong S$ if and only if the following hold:

1. *S is Boolean (hence it has a 0),*
2. *S is fundamental,*
3. *the elements of height 1 form a single $\mathcal{D}$-class,*
4. *elements of height 2 form at most two $\mathcal{D}$-classes, with two-element $\mathcal{H}$-classes,*
5. *if $X \subseteq S$ is a set of pairwise compatible elements of height 1, and for all $a_1, a_2 \in A$, $a_1 \vee a_2$ exists, then $\bigvee A$ exists.*

The characterization of PAut(Γ) in the abstract case

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Given a finite inverse monoid S there exists an edge-colored digraph Γ such that $\mathrm{PAut}(\Gamma) \cong S$ if and only if the following hold:

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If G is a group, and $\Gamma(G)$ is its Cayley graph, then the (edge-colored, directed) automorphism group $\mathrm{Aut}(\Gamma(G)) \cong G$.

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Theorem (Sieben, 2008)

The map $\rho\colon S \to I_S, s \mapsto \rho_s$, where $\rho_s\colon s^{-1}S \to sS, t \mapsto st$ embeds S into $\mathrm{PAut}(\Gamma(G))$.

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The subgraphs arising as domains and images are exactly the ones of the form

$$tail(s) = \{t \in V : t \text{ can be reached by a finite path from } s\}.$$

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Let $\widetilde{\rho(S)}$ be the smallest inverse semigroup in I_S containing $\rho(S)$ which satisfies the properties of inverse monoids of partial graph automorphisms:

1. $E(I_S) \subseteq \widetilde{\rho(S)}$,
2. for any compatible subset $A \subseteq S$ of rank 1 partial permutations, if $\widetilde{\rho(S)}$ contains the join of any two elements of A, then $\widetilde{\rho(S)}$ contains the join of the set A.

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That is, a map φ

- of rank 1 is in $\widetilde{\rho(S)} \Longleftrightarrow$ it is a restriction of a map in ρ_S;
- of rank ≥ 2 is in $\widetilde{\rho(S)} \Longleftrightarrow$ all 2-rank restrictions of φ are restrictions of maps in ρ_S.

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The best we can hope for: $\widetilde{\rho(S)} = \mathrm{PAut}(\Gamma(S))$.

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The best we can hope for now: if the system of generators is large enough, then $\widetilde{\rho(S)} = \mathrm{PAut}(\Gamma(S))$.

Bad news: even wrt to the generating system S, $\widetilde{\rho(S)} \neq \mathrm{PAut}(\Gamma(S))$.

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Happy birthday!