Simplicity of contracted inverse semigroup algebras

Nóra Szakács

University of York, UK
University of Szeged, Hungary

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Inverse semigroups

Definition

A semigroup S is called an **inverse semigroup** if for any $s \in S$, there exists a unique element $s^{-1} \in S$ for which

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The archetypal example

The set of partial one-to-one maps on a set A under composition and inverse: the symmetric inverse semigroup $\mathcal{I}_A$.

The polycylic monoid

Example

Fix a set $|X| > 1$ (alphabet). The **polycyclic monoid** $P(X)$ on X is

- an inverse semigroup with a zero 0 and an identity 1 generated by X,
- defined by relations

$$x^{-1}y = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y \end{cases}$$

for all $x, y \in X$.

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Elements: $\alpha\beta^{-1}$ with $\alpha, \beta \in X^*$, and 0
Idempotents: $\alpha\alpha^{-1}$ with $\alpha \in X^*$, and 0.

Semigroup algebras

Let S be a semigroup, K a field.

The **semigroup algebra** KS consists of finite linear combinations of elements of S over K. It is

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Question: Suppose S in an inverse semigroup. When is the ring KS simple?

A simple answer

Let S be a nontrivial inverse semigroup, K a field.

Then

$$KS \to K,\ \sum_{s \in S} a_s s \mapsto \sum_{s \in S} a_s$$

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is a homomorphism with a nontrivial, proper kernel

$$\implies KS \text{ is not simple.}$$

The contracted inverse semigroup algebra

Let S be an inverse semigroup **with a zero** z, K a field.

Let $K_0S = KS/(z)$ – this effectively identifies z with 0. We call it the **contracted inverse semigroup algebra**.

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Notice: a congruence $\equiv$ on S induces a surjective homomorphism $K_0S \to K_0[S/\equiv]$, so

$$K_0S \text{ is simple} \implies S \text{ is congruence-free.}$$

The contracted inverse semigroup algebra

But

$$K_0 S \text{ is simple } \not\Leftarrow S \text{ is congruence-free.}$$

$P(x, y)$ is congruence-free, but $K_0[P(x, y)]$ is not:

$$I = (xx^{-1} + yy^{-1} - 1)$$

is a proper ideal, in fact $K_0[P(x, y)]/I$ is the Leavitt algebra $L_K(1, 2)$.

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Problem (Munn, 1978)

Characterize those congruence-free inverse semigroups with zero which have a simple contracted algebra.

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- 0-simple: it has no proper, nonzero ideals,
- fundamental: it has no nontrivial idempotent-separating congruences,
- and $E(S)$ is 0-disjunctive: for all idempotents $0 \neq f < e$, there exists $0 \neq f' < e$ such that $ff' = 0$.

Previous results

Let S be an inverse monoid with zero 0, E its semilattice of idempotents.

Let $e \in E$. We say $F \subseteq (e)^{\downarrow}$ **covers** e if for all $h \in E$

$$hf = 0 \text{ for all } f \in F \Longrightarrow he = 0.$$

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However, $P(X)$ is tight if X is infinite.

Nontrivial finite covers give rise to an ideal of K_0S called the **tight ideal**. If K_0S is simple, then S is tight.

Previous results

S is called **Hausdorff** if for each $s, t \in S$, the set $(s)^{\downarrow} \cap (t)^{\downarrow}$ has finitely many maximal elements.

Remark

E^*-unitary $\Longrightarrow$ Hausdorff

Theorem (Steinberg, 2014)

A Hausdorff inverse semigroup S with a zero has a simple contracted algebra over any field K if and only if S is congruence-free and tight.

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In the general case, congruence-free and tight are necessary conditions, but it was not known if they were sufficient.

A reminder for the experts

Some ideals are better seen in a different model of K_0S:

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$$K_0S \cong K\mathcal{G}(S)$$

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Simplicity of ample groupoid algebras was characterized by

- Brown, Clark, Farthing and Sims (2013) in the Hausdorff case,
- Clark, Exel, Pardo, Sims and Starling (2018) in the non-Hausdorff case.

In the non-Hausdorff case, a new ideal needs to be considered: the ideal of **singular** functions.

The main theorem

Let

$$I = \{A \in K_0S : \forall\, e \in E \setminus \{0\}\ \exists\, f \leq e, f \neq 0 \text{ such that } Af = 0\}.$$

Theorem (Steinberg, Sz.)

1. *I is an ideal in K_0S.*
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- I contains the tight ideal. In fact it corresponds to the ideal generated by the tight ideal and the ideal of singular functions.
- Simplicity depends on the field K.

A class of congruence-free inverse semigroups

Fix an alphabet X, and consider the polycyclic monoid $P(X)$.

Recall: $P(X)$ is congruence free, and tight whenever X is infinite. We build congruence-free [tight] inverse semigroups from polycyclic monoids and a groups.

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$P(X)$ can be represented by partial one-to-one (right) maps on X^*:

$$\alpha\beta^{-1} \colon \alpha X^* \to \beta X^*$$
$$\alpha w \mapsto \beta w$$

Self-similar groups

Let G be a group with a faithful, length-preserving action on X^*. We call the action **self-similar** if for every $g \in G$, $u \in X^*$ there exists $g|_u \in G$ such that for all $w \in X^*$

$$g(uw) = g(u)g|_u(w).$$

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A very trivial example: $G = C_2 = \{1, a\}$, $X = \{x, y\}$,

$$a(xw) = yw, a(yw) = xw,$$

so for any nonempty word u we have $a(uw) = a(u)1(w)$, that is $a|_u = 1$.

Inverse semigroups from self-similar actions

Let G be a group with a self-similar action on X^*.
Identify G and P_X with their images in the symmetric inverse semigroup $\mathcal{I}_{X^*}$, and let $S = \langle G, P_X \rangle \leq \mathcal{I}_{X^*}$.

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Note: for any $g \in G$, and for any $\alpha, \beta \in A^*(\subseteq P_A)$, $w \in A^*$,

$$(g\alpha)(w) = g(\alpha w) = g(\alpha)g|_\alpha(w) = (g(\alpha)g|_\alpha)(w),$$

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Furthermore we can write elements of S uniquely in the form $\alpha g \beta^{-1}$, where $\alpha, \beta \in X^*(\subseteq P_X)$, $g \in G$.

A congruence-free, tight inverse semigroup S with $I \neq \{0\}$

Let $A = \{x, y\} \dot{\bigcup} Z$ with Z infinite, $G = C_2 = \{1, a\}$, and consider the self-similar action

$$a(xw) = yw, a(yw) = xw, a(zw) = zw$$

for all $z \in Z$, $w \in X^*$.

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Let $S = \langle G, P_A \rangle$.

Recall:

$$I = \{A \in K_0 S : \forall\, e \in E \setminus \{0\}\ \exists\, f \leq e, f \neq 0 \text{ such that } Af = 0\}.$$

Claim:

$$A = (1 - xx^{-1} - yy^{-1}) - (a - axx^{-1} - ayy^{-1}) \in I.$$

A congruence-free, tight inverse semigroup S with $I \neq \{0\}$

$$\begin{aligned} Ax =& ((1 - xx^{-1} - yy^{-1}) - (a - axx^{-1} - ayy^{-1}))x \\ =& (x - xx^{-1}x - yy^{-1}x) - (ax - axx^{-1}x - ayy^{-1}x) \\ =& (x - x) - (ax - ax) = 0. \end{aligned}$$

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$\Longrightarrow$ if $\alpha = x\beta$, then $A\alpha\alpha^{-1} = 0$.

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$\implies$ if $\alpha = z\beta$, then $A\alpha\alpha^{-1} = 0$.

A congruence-free, tight inverse semigroup S with $I \neq \{0\}$

So for all $f \in E \setminus \{1\}$ we have $Af = 0$, so certainly for all $e \in E \setminus \{0\}$ there exists $f \leq e, f \neq 0$ such that $Af = 0$

$$\implies A \in I,$$

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$$\Longrightarrow A \in I,$$

$\Longrightarrow$ S is congruence-free an tight, but K_0S is not simple.