Women in Maths Day

The Černý conjecture

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29 June, 2023

A trick and a map

A magic trick

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- Starting from 12 and moving clockwise, spell out your number around the clock
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You are now at number 1.

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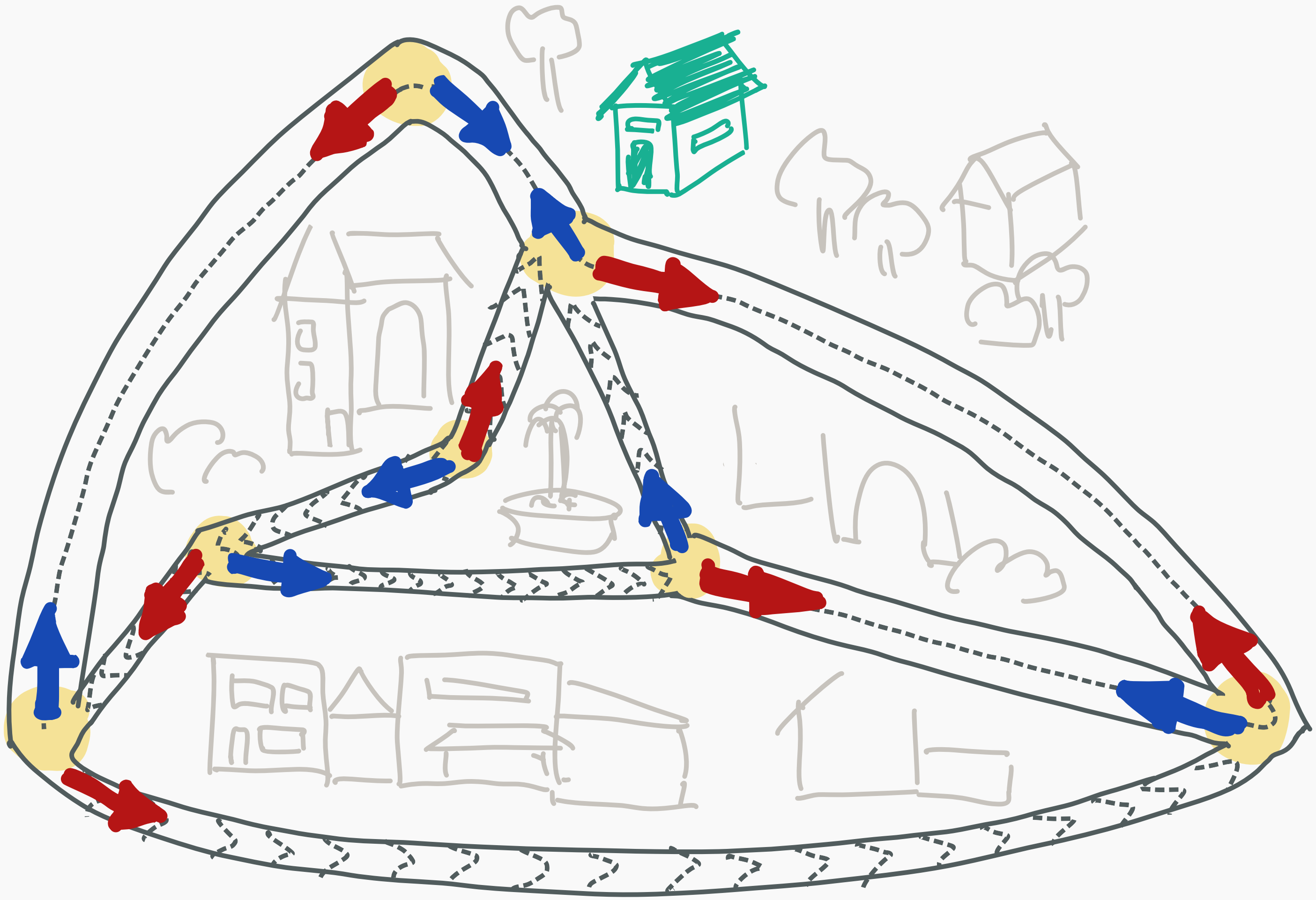

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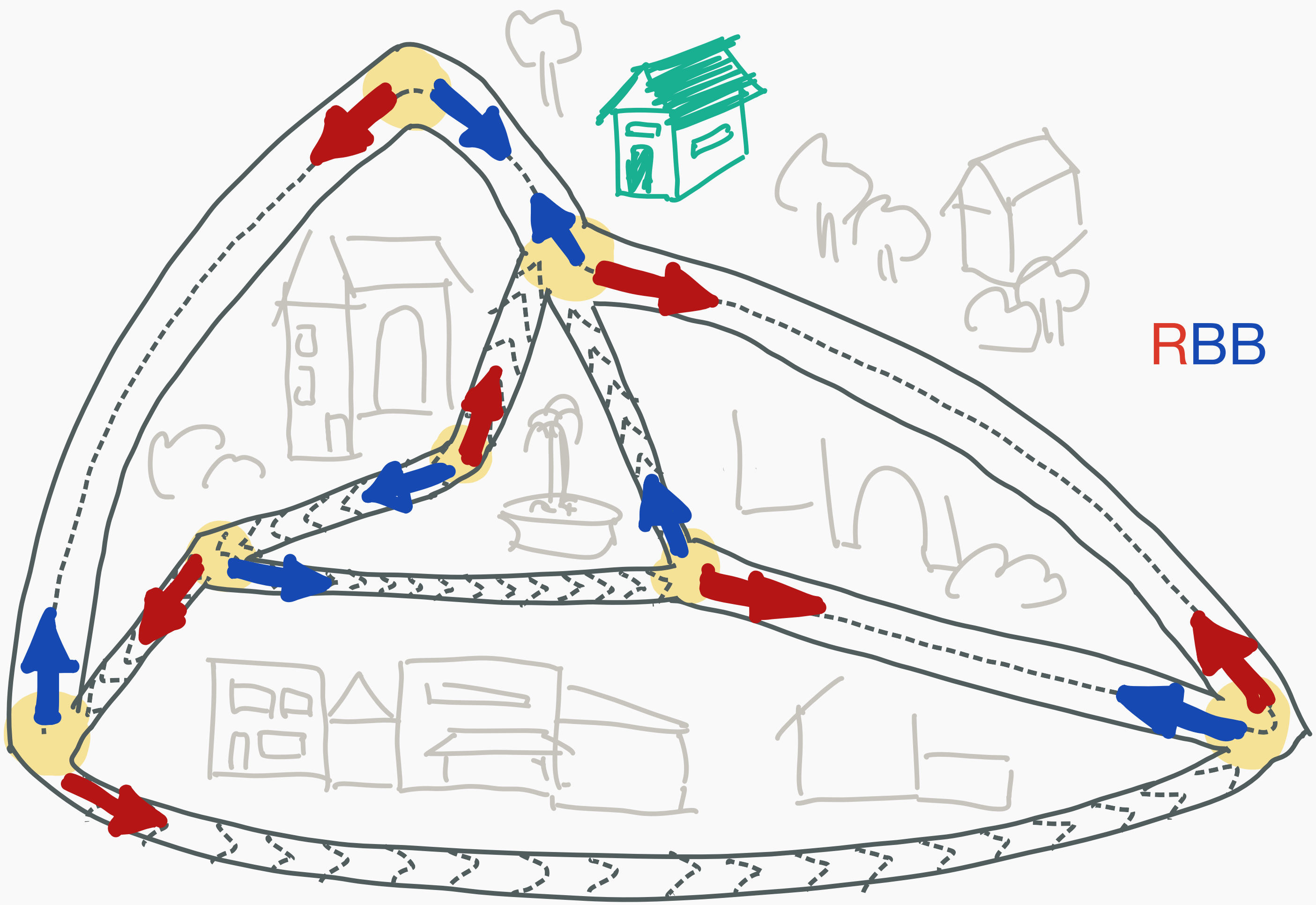
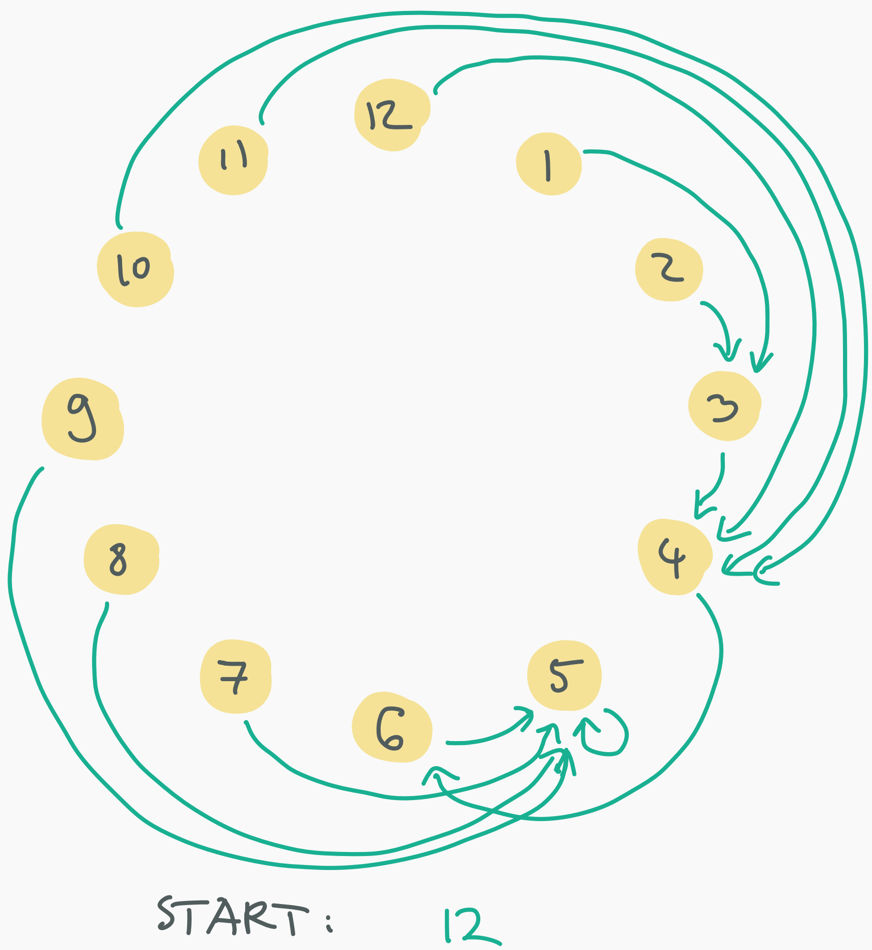


What is in common?

In both cases, we had a set of possible positions, and possible moves between these:



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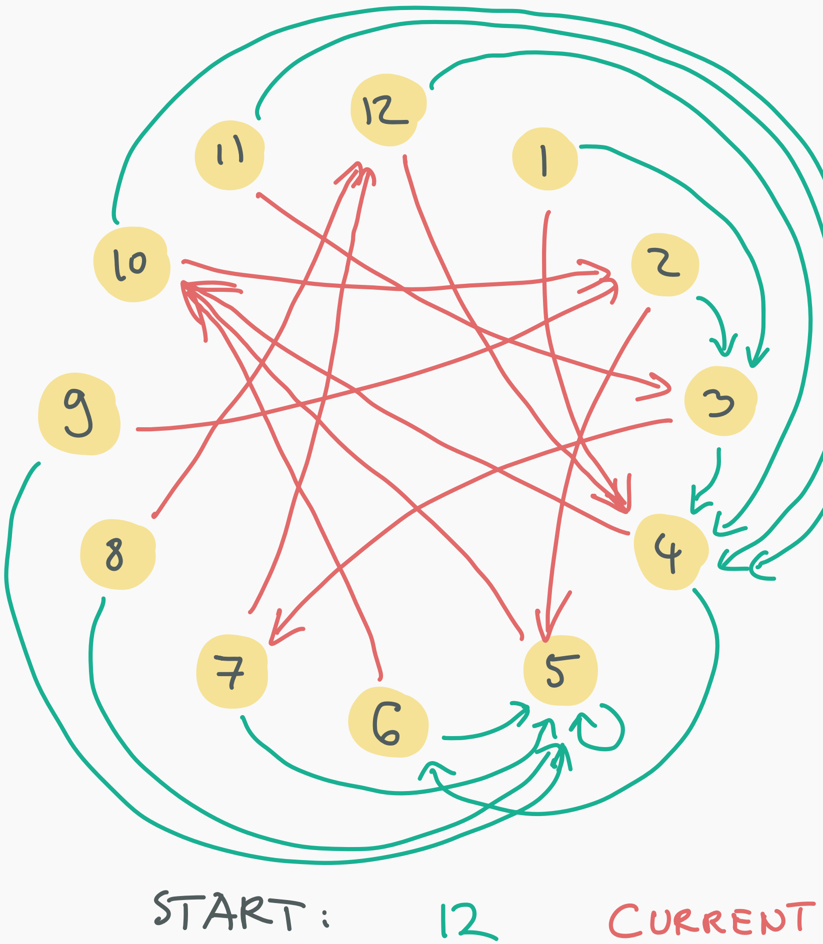

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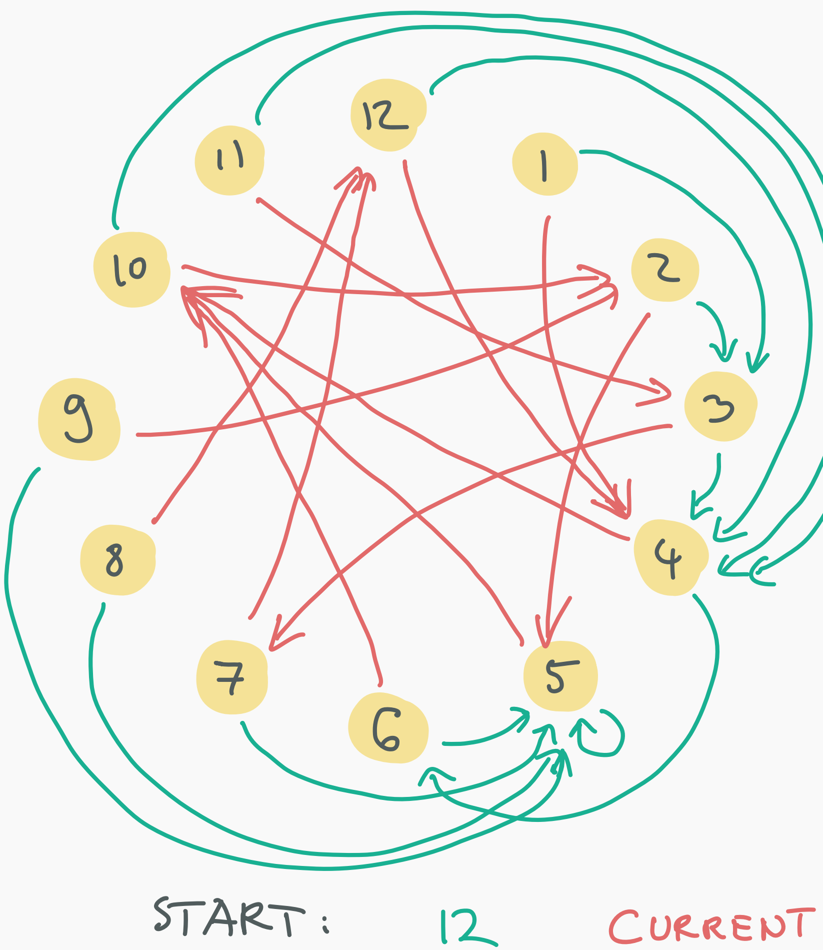

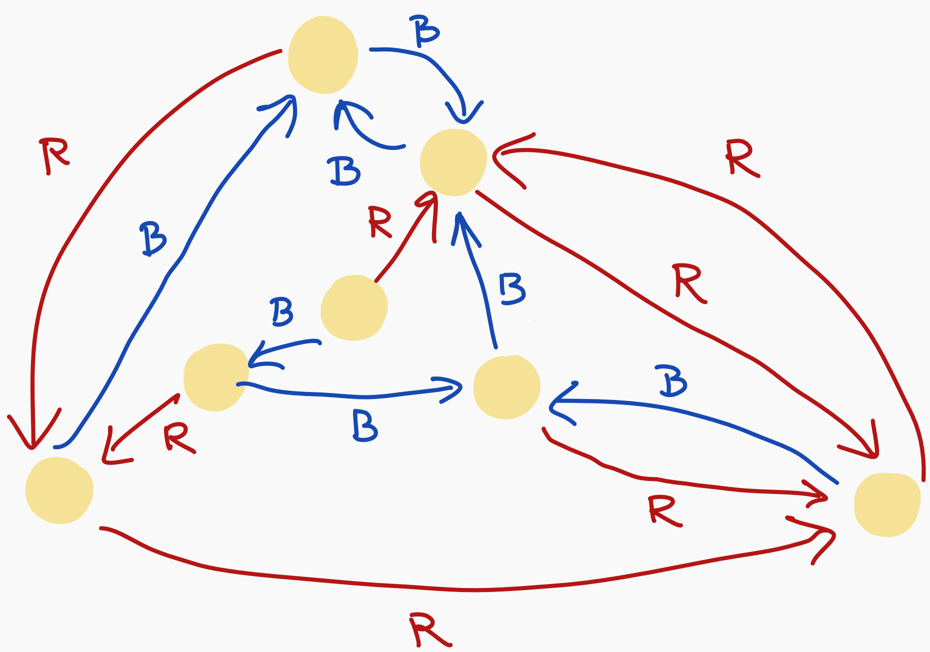

Finite state automata

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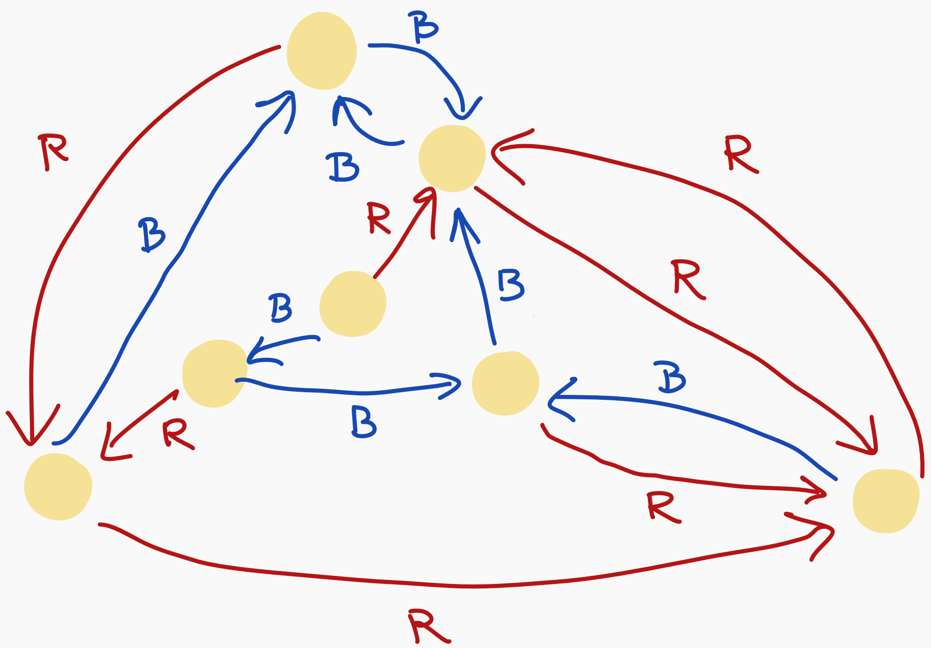

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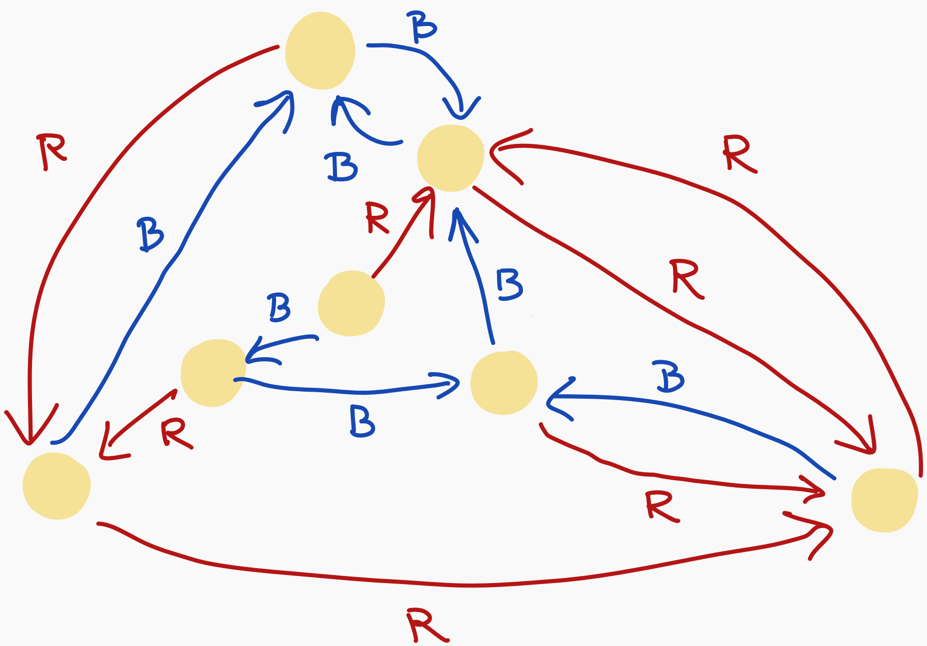

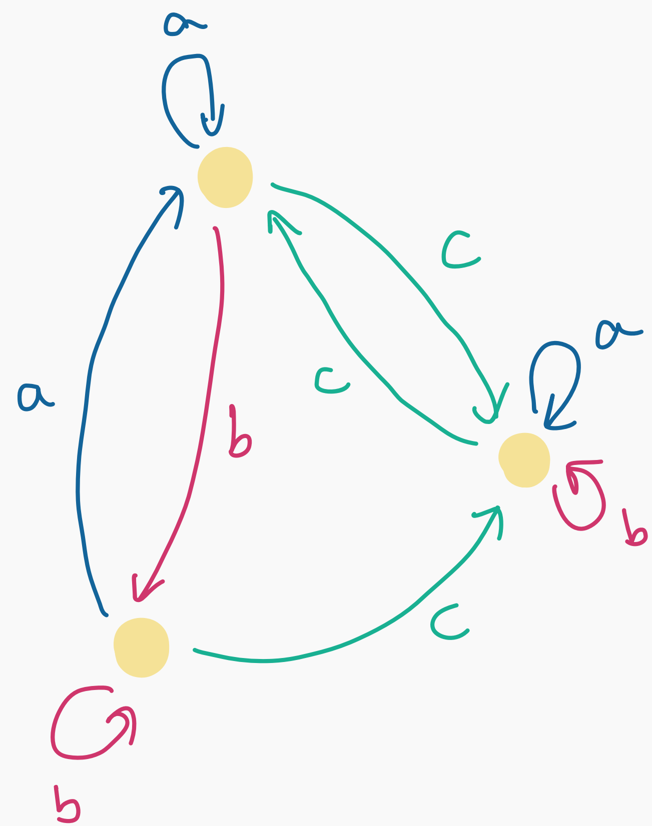

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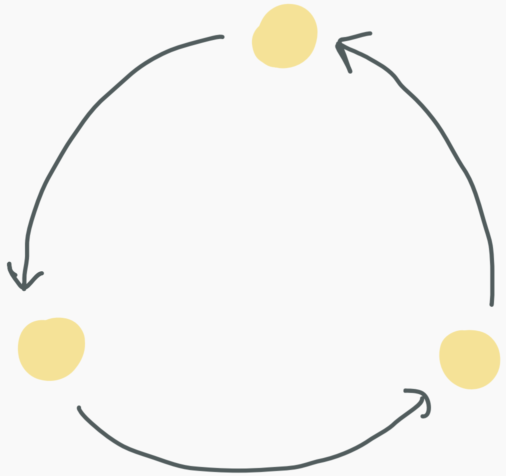
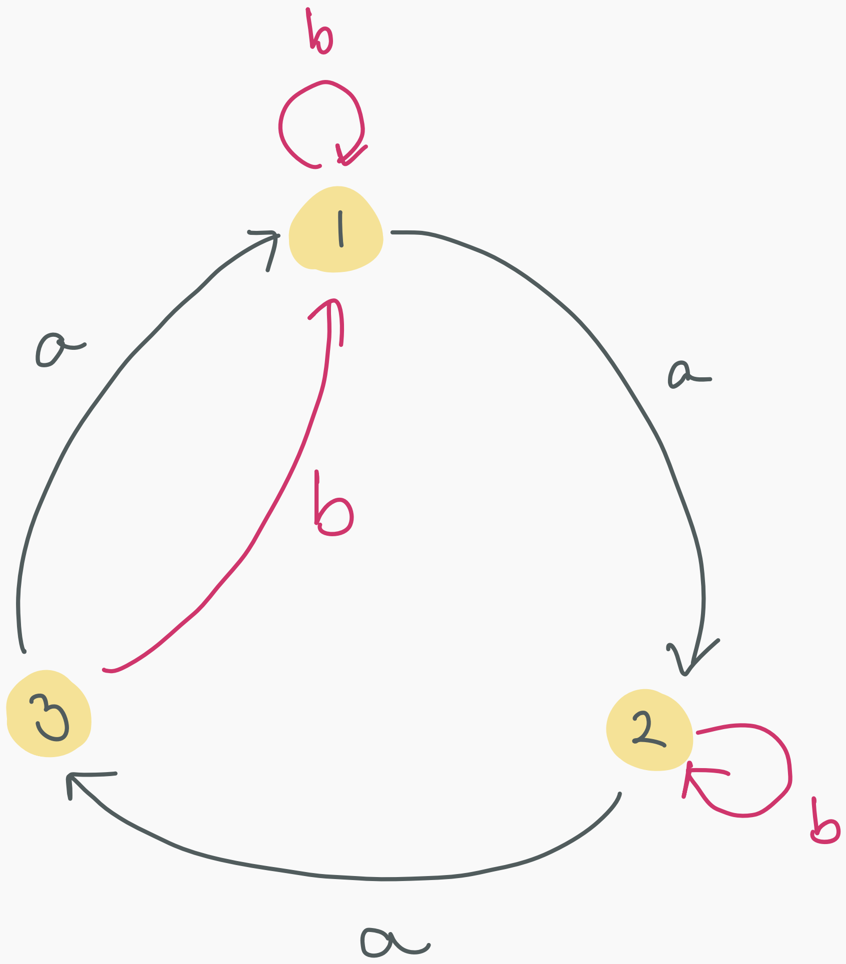
Not all automata have synchronizing words!



How would you find a synchronizing word?



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If $w = a_1 a_2 \cdots a_k$ is a shortest synchronizing word, then the instructions $a_1,\ a_1 a_2,\ a_1 a_2 a_3,\ \ldots,\ a_1 a_2 \cdots a_{k-1}$ must

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So k is at most the number of different nonsingleton (and nonempty) subsets: $k \leq 2^n - n - 1$.

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So k is at most the number of different nonsingleton (and nonempty) subsets: $k \leq 2^n - n - 1$.

However, for every synchronizing automaton anyone's ever tried, the length of the shortest synchronizing word is at most $(n-1)^2$.

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So far, nobody could **prove** that this is true, and it is an open research question in mathematics.

Things we do know

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- The bound $(n-1)^2$ cannot be improved: for every n, there are synchronizing automata on n states where this is the length of the shortest synchronizing word.
- The current best (proven) bound is cubic ($\sim 1/6n^3$).
- the Černý conjecture is true for *almost all* finite state automata: for $n \in \mathbb{N}$, let p_n be the probability that a randomly chosen synchronizing automata on n states satisfies the Černý conjecture. Then
$$\lim_{n \to \infty} p_n = 1.$$
- In fact, *almost all* finite state automata have a synchrozing word whose length is linear in the number of states.