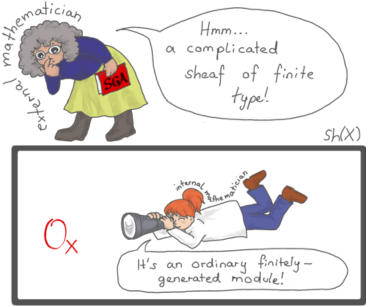

"Without loss of generality, any reduced ring is Noetherian and a field."

Using the internal language of toposes in commutative algebra

Ingo Blechschmidt
University of Augsburg

101st Peripatetic Seminar on Sheaves and Logic (Leeds)
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Quick summary

By employing the internal language of toposes in various ways, you can pretend that:

1. Sheaves of modules are plain modules.
2. Schemes are sets:

$$\mathbb{P}^2_S = \{[x_0 : x_1 : x_2] \mid x_0 \neq 0 \vee x_1 \neq 0 \vee x_2 \neq 0\}.$$

3. Reduced rings are Noetherian and in fact fields.

What is a topos?

Formal definition

A **topos** is a category which has finite limits, is cartesian closed and has a subobject classifier.

Motto

A topos is a category which is sufficiently rich to support an **internal language**.

Examples

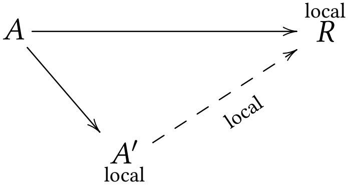
- **Set:** the category of sets
- **$\mathrm{Sh}(X)$:** the category of set-valued sheaves on a space X
- **$\mathrm{Zar}(S)$:** the big Zariski topos of a base scheme S

Universal localisation

Recall

A ring is local iff it has precisely one maximal ideal.
A homomorphism is local iff it reflects invertibility.

Let A be a ring. Is there a **free local ring** $A \to A'$ over A?



No, if we restrict to Set. **Yes,** if we allow a change of topos:
Then $A \to A^\sim$ is the universal localisation.

The internal language of a topos

The internal language of a topos $\mathcal{E}$ allows to

1. construct objects and morphisms of the topos,
2. formulate statements about them and
3. prove such statements

in a **naive element-based** language:

externally	internally to $\mathcal{E}$
object of $\mathcal{E}$	set
morphism in $\mathcal{E}$	map of sets
monomorphism	injective map
epimorphism	surjective map
group object	group

The internal language of $\mathrm{Sh}(X)$

Let X be a topological space. Then we recursively define

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for open subsets $U \subseteq X$ and formulas φ.

The internal language of $\mathrm{Sh}(X)$

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for open subsets $U \subseteq X$ and formulas φ.

$U \models f = g : \mathcal{F}$ $:\Longleftrightarrow$ $f|_U = g|_U \in \mathcal{F}(U)$

$U \models \varphi \wedge \psi$ $:\Longleftrightarrow$ $U \models \varphi$ and $U \models \psi$

$U \models \varphi \vee \psi$ $:\Longleftrightarrow$ ~~$U \models \varphi$ or $U \models \psi$~~

there exists a covering $U = \bigcup_i U_i$ s. th. for all i:

$U_i \models \varphi$ or $U_i \models \psi$

$U \models \varphi \Rightarrow \psi$ $:\Longleftrightarrow$ for all open $V \subseteq U$: $V \models \varphi$ implies $V \models \psi$

$U \models \forall f : \mathcal{F}.\, \varphi(f)$ $:\Longleftrightarrow$ for all sections $f \in \mathcal{F}(V)$, $V \subseteq U$: $V \models \varphi(f)$

$U \models \exists f : \mathcal{F}.\, \varphi(f)$ $:\Longleftrightarrow$ there exists a covering $U = \bigcup_i U_i$ s. th. for all i:

there exists $f_i \in \mathcal{F}(U_i)$ s. th. $U_i \models \varphi(f_i)$

The internal language of Sh(X)

Locality

If $U = \bigcup_i U_i$, then $U \models \varphi$ iff $U_i \models \varphi$ for all i.

Soundness

If $U \models \varphi$ and if φ implies ψ constructively, then $U \models \psi$.

The internal language of Sh(X)

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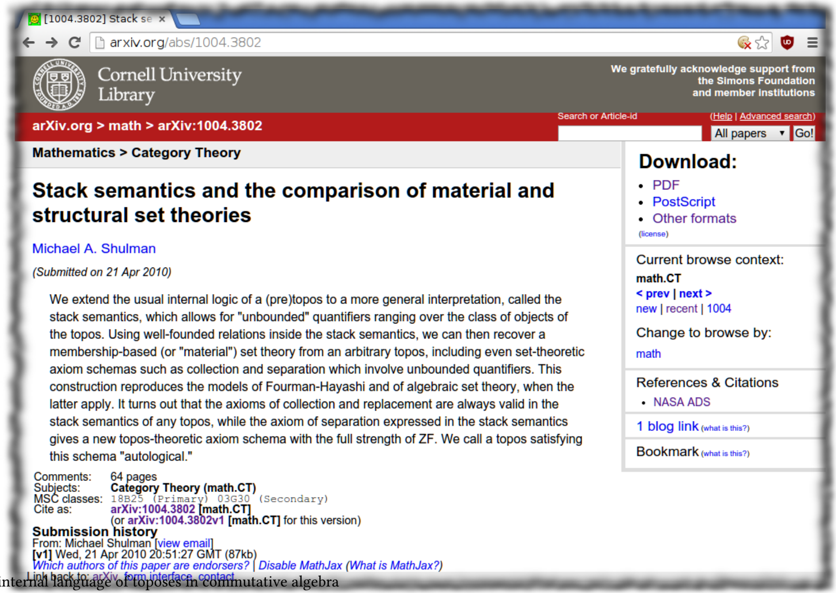
no $\varphi \vee \neg\varphi$, no $\neg\neg\varphi \Rightarrow \varphi$, no axiom of choice

A first glance at the constructive nature

- $U \models f = 0$ iff $f|_U = 0 \in \mathcal{F}(U)$.
- $U \models \neg\neg(f = 0)$ iff $f = 0$ on a dense open subset of U.

Praise for Mike Shulman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Mathematics > Category Theory

Stack semantics and the comparison of material and structural set theories

Michael A. Shulman

(Submitted on 21 Apr 2010)

We extend the usual internal logic of a (pre)topos to a more general interpretation, called the stack semantics, which allows for "unbounded" quantifiers ranging over the class of objects of the topos. Using well-founded relations inside the stack semantics, we can then recover a membership-based (or "material") set theory from an arbitrary topos, including even set-theoretic axiom schemas such as collection and separation which involve unbounded quantifiers. This construction reproduces the models of Fourman-Hayashi and of algebraic set theory, when the latter apply. It turns out that the axioms of collection and replacement are always valid in the stack semantics of any topos, while the axiom of separation expressed in the stack semantics gives a new topos-theoretic axiom schema with the full strength of ZF. We call a topos satisfying this schema "autological."

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The little Zariski topos

Let A be a ring. Its **spectrum $\mathrm{Spec}(A)$** is

- generated by opens $D(f)$ for $f \in A$
- subject to $\mathrm{Spec}(A) = \bigcup_i D(f_i)$ iff $1 = \sum_i g_i f_i$ for some g_i.

The **little Zariski topos** of A is the category $\mathrm{Sh}(\mathrm{Spec}(A))$ of set-valued sheaves on its spectrum. It contains a ring object $A^\sim$ with $A^\sim(D(f)) = A[f^{-1}]$, called the **structure sheaf**.

Motto (to be amended)

The structure sheaf $A^\sim$ is a reification of all of the stalks $A_\mathfrak{p}$.

For instance, all stalks $A_\mathfrak{p}$ are integral domains if and only if

$$\mathrm{Spec}(A) \models \ulcorner A^\sim \text{ is an integral domain} \urcorner.$$

Transfer principles

Theorem

The structure sheaf $A^\sim$ inherits all first-order properties of A which are stable under localisation.

Proof. The structure sheaf $A^\sim$ is the localisation

$$\underline{A}[\mathcal{F}^{-1}]$$

of the constant sheaf $\underline{A}$ at the **generic filter** $\mathcal{F}$, a sheaf with

$$\mathcal{F}_{\mathfrak{p}} = A \setminus \mathfrak{p}.$$

The rings A and $\underline{A}$ share all first-order properties.

Remark. The analogous statement holds for A-modules M and their mirror image $M^\sim = \underline{M}[\mathcal{F}^{-1}]$.

Unique features of the internal world

Internally to $\mathrm{Sh}(\mathrm{Spec}(A))$,

any non-invertible element of $A^\sim$ is nilpotent.

ON THE SPECTRUM OF A RINGED TOPOS 209

For completeness, two further remarks should be added to this treatment of the spectrum. One is that in **E** the canonical map $A \to \Gamma_*(LA)$ is an isomorphism—i.e., the representation of A in the ring of "global sections" of LA is complete. The second, due to Mulvey in the case $\mathbf{E} = \mathbf{S}$, is that in Spec(**E**, A) the formula

$$\neg(x \in U(LA)) \Rightarrow \exists n(x^n = 0)$$

is valid. This is surely important, though its precise significance is still somewhat obscure—as is the case with many such nongeometric formulas. In any case, calculations such as these are easier from the point of view of the Heyting algebra of radical ideals of A, and hence will be omitted here.

Miles Tierney. On the spectrum of a ringed topos. 1976.

A sheaf $\mathcal{E}$ of $A^\sim$-modules is quasicoherent if and only if, internally to $\mathrm{Sh}(\mathrm{Spec}(A))$, the set $\mathcal{E}[f^{-1}]$ is a ∇_f-sheaf for any $f : A^\sim$, where $\nabla_f \varphi :\equiv (f \text{ invertible} \Rightarrow \varphi)$.

Unique features of the internal world

Internally to $\mathrm{Sh}(\mathrm{Spec}(A))$,

any non-invertible element of $A^\sim$ is nilpotent.

If A is reduced, then furthermore $A^\sim$ is

- reduced,
- a **field** in that non-invertible elements are zero,
- **anonymously Noetherian** in that any ideal is **not not** finitely generated,
- and has **¬¬-stable equality**: $\neg\neg(f = 0) \Longrightarrow f = 0$

Generic freeness

Let A be a reduced ring and B, M as follows:

$$\begin{array}{ccc} & & M \\ & & \Big| \text{ finitely generated} \\ A & \xrightarrow[\text{of finite type}]{} & B \end{array}$$

Theorem. If $1 \neq 0$ in A, there exists $f \neq 0$ in A such that

1. $B[f^{-1}]$ and $M[f^{-1}]$ are free modules over $A[f^{-1}]$,
2. $A[f^{-1}] \to B[f^{-1}]$ is of finite presentation, and
3. $M[f^{-1}]$ is finitely presented as a module over $B[f^{-1}]$.

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Constructive version. If zero is the only element $f \in A$ such that 1, 2, and 3, then $1 = 0 \in A$.

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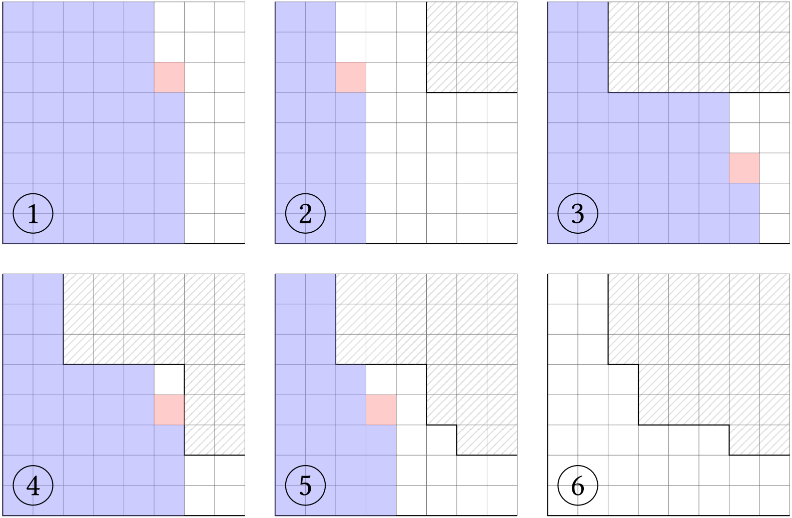
Constructive version. If zero is the only element $f \in A$ such that 1, 2, and 3, then $1 = 0 \in A$.

Proof. The claim is the translation of the fact that, internally in $\mathrm{Sh}(\mathrm{Spec}(A))$, it is **not not** the case that

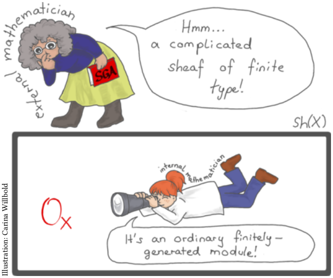
1. $B^\sim$ and $M^\sim$ are free over $A^\sim$,
2. $A^\sim \to B^\sim$ is of finite presentation, and
3. $M^\sim$ is finitely presented as a module over $B^\sim$.

Assume that $B^\sim$ is generated by $(x^i y^j)_{i,j \geq 0}$ as an $A^\sim$-module. It's **not not** the case that either some generator can be expressed as a linear combination of others with smaller index, or not.



Outlook

- Are there general guidelines for when using the internal perspective pays off?
- What about finer topologies (constructible, étale, fppf, …)?
- How should synthetic algebraic geometry be extended?



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