Two-dimensional Categorical Logic

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Aim

"*May I remind you that the participants at Logic Colloquium cover most areas of logic, and we expect to achieve the goal of increasing the overall level of understanding across logic.* "

Paola D'Aquino, LC 2023 Programme Chair

Outline of the talk

Part I: Two-dimensional Categorical Logic

- Review of Categorical Logic
- Categorification

Part II: The differential λ-calculus

- Syntax
- A 1-dimensional model

Part III: A 2-dimensional model

Based on collaborations with Fiore, Hyland, Winskel.

Part I: Categorical Logic

Key ideas of Categorical Logic (Lawvere)

1. A theory T can have models in categories $\mathcal{E}$, where $\mathcal{E} \neq \mathbf{Set}$, e.g.

$$M \times M \xrightarrow{m} M \qquad 1 \xrightarrow{e} M$$

$$\begin{CD} M \times M \times M @>{m\times 1_M}>> M \times M \\ @V{1_M\times m}VV @VV{m}V \\ M \times M @>>{m}> M \end{CD} \qquad \begin{array}{ccccc} M & \xrightarrow{1_M\times e} & M \times M & \xleftarrow{e\times 1_M} & M \\ & {\scriptstyle 1_M}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle 1_M} & \\ & & M & & \end{array}$$

2. A theory T can be seen as a category $\mathrm{Syn}(T)$, cf. Lindenbaum algebra
3. Models can be seen as (structure-preserving) functors $M : \mathrm{Syn}(T) \to \mathcal{E}$
4. Model homomorphisms / elementary embeddings can be seen as natural transformations

Note: $(3) + (4) \Rightarrow \mathrm{Mod}[T, \mathcal{E}] \simeq [\mathrm{Syn}(T), \mathcal{E}]$

Fundamental theorems

- **Completeness Theorems** (Gödel, Deligne, Joyal)
- **Duality theorems** (Lawvere, Gabriel & Ulmer, Makkai, Awodey & Forssell, Frey, ...)

$$\mathsf{Syn}(T) \simeq \mathsf{mod}(T)^{\mathrm{op}}$$

- **Conceptual Completeness** (Makkai, ...): for $F : T \to T'$

$$\mathsf{Mod}(T', \mathbf{Set}) \xrightarrow[F^*]{\simeq} \mathsf{Mod}(T, \mathbf{Set}) \quad \Rightarrow \quad T \xrightarrow[F]{\simeq} T'$$

- **Characterisations of categories of models**

See: Lurie, Categorical Logic, 2018.

Points of contact

1. Set Theory
   - Forcing and Boolean-valued models as sheaves, Algebraic Set Theory
2. Model Theory
   - Imaginaries & groupoids, AEC, model theory of modules
3. Proof Theory
   - Type theory, identity of proofs
4. Computability Theory
   - Realizability toposes
5. Theoretical Computer Science
   - Denotational semantics
6. Philosophical Logic
   - Constructivism, structuralism

Note: Applications both ways, cf. Kelly & Mac Lane's coherence theorems (1971)

Categorification

The art of replacing set-based structures with category-based structures.

Example

- commutative monoid $(M, \cdot, 1)$
- symmetric monoidal category $(\mathcal{E}, \otimes, I)$.

Why?

- To obtain more powerful invariants (e.g. Khovanov homology)
- Applications in algebra (e.g. Kazhdan-Lusztig conjecture)
- Stacks

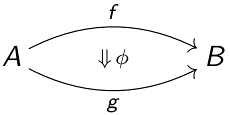
2-categories

We can apply categorification to the notion of a category itself.

Definition. A **2-category** $\mathcal{K}$ consists of:

- A class of objects $\mathrm{Ob}(K)$
- For each $A, B \in \mathrm{Ob}(K)$, a category $\mathcal{K}(A, B)$
- For each $A, B, C \in \mathrm{Ob}(K)$, composition functors $\mathcal{K}(B, C) \times \mathcal{K}(A, B) \to \mathcal{K}(A, C)$
- For each $A \in \mathrm{Ob}(K)$, an object 1_A of $\mathcal{K}(A, A)$
- ...

Idea:

- write $f \in \mathcal{K}(A, B)$ as $f : A \to B$
- write $\phi : f \Rightarrow g$ as a 2-cell 

Examples

Basic examples

- **Cat**: categories, functors, natural transformations
- **Gpd**: groupoids, functors, natural isomorphisms

2-categories of categories with structure

- **FinProd**: categories with finite products, product-preserving functors, natural transformations
- **MonCat**: monoidal categories, lax monoidal functors, monoidal transformations

Standard constructions of new categories from old extend.

Two-dimensional category theory (I)

Theorem. (Kelly, Street, Power, Hyland, Lack, Weber, Garner, Gurski, Shulman, Bourke, . . .)

- All of ordinary category theory carries over to 2-categories

Issues

- More subtle: strict vs weak
- Coherence pervades the subject
- New concepts emerge
- Unavoidable

See: Lack, A 2-categories companion, 2007.

Two-dimensional categorical logic (II)

0-dimensional categorical logic

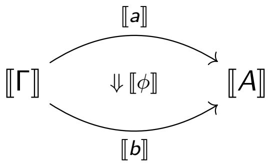
$$[\![\Gamma \vdash A]\!] \quad \text{given by} \quad [\![\Gamma]\!] \leq [\![A]\!]$$

1-dimensional categorical logic

$$[\![\Gamma \vdash a : A]\!] \quad \text{given by} \quad [\![a]\!] : [\![\Gamma]\!] \to [\![A]\!]$$

2-dimensional categorical logic

$$[\![\Gamma \vdash \phi : a \Rightarrow b]\!] \quad \text{given by}$$



Two-dimensional categorical logic (III)

- Regularity and exactness (Bourke & Garner, Lack and Tendas)
- 2-toposes (Street, Weber, Shulman)
- 2-fibrations (Hermida)
- Coherence and rewriting (Gurski & Osorno, ...)
- Computer-assisted formalisation of proofs (Bar & Kissinger & Vicary)

Challenge: What are the key notions?

- Some guidance from HoTT / Univalent Foundations / ∞-category theory

Part II: The differential λ-calculus

Differential λ-calculus (I)

Extension of simply-typed λ-calculus with a differential operator [Ehrhard & Regnier].

Product types

$$\frac{a : A \qquad b : B}{\text{pair}(a, b) : A \times B} \qquad \frac{c : A \times B}{\pi_1(c) : A} \qquad \frac{c : A \times B}{\pi_2(c) : B}$$

Function types

$$\frac{x : A \vdash b : B}{(\lambda x : A) b : B^A} \qquad \frac{f : B^A \quad a : A}{\text{app}(f, a) : B}$$

Differential λ-calculus (II)

Differentiation rule

$$\frac{\Gamma \vdash f : B^A \qquad \Delta \vdash a : A}{\Gamma, \Delta \vdash \mathrm{D}f \cdot a : B^A} \tag{$*$}$$

Idea: Let $f : A \to B$ be differentiable. For $x \in A$, we have a linear map (the Jacobian)

$$\begin{aligned} f'(x) : \quad A &\longrightarrow B \\ a &\longmapsto f'(x) \cdot a \end{aligned}$$

Transposing, for $a \in A$, we have a (generally) non-linear map

$$\begin{aligned} f'(-) \cdot a : \quad A &\longrightarrow B \\ x &\longmapsto f'(x) \cdot a \end{aligned}$$

Rule in $(*)$ corresponds to this.

Differential λ-calculus (III)

β-rule

$$\mathrm{app}((\lambda x : A)b, a) = b[a/x] : B$$

Differential β-rule

$$\mathrm{D}\big((\lambda x : A)b\big) \cdot a = \lambda x \left(\frac{\partial b}{\partial x} \cdot a \right),$$

Here $\frac{\partial b}{\partial x} \cdot a$ is defined by structural induction on b, to express chain rule, product rule, etc..

(Need to fix a commutative rig R and allow linear combinations of λ-terms)

Applications. New tool to study λ-terms: Taylor series expansion!

Differential λ-calculus (IV)

Concrete models

- Köthe spaces (some topological vector spaces) [Ehrhard]
- Finiteness spaces [Ehrhard]
- Relational model [Blute, Cockett, Seely], [Ehrhard], [Hyland]

Categorical axiomatisations. Differential categories and variants

- [Blute, Cockett, Seely]
- [Fiore]
- [Blute, Cockett, Seely and Lemay]
- [Manzonetto]

The category of relations

Define the category **Rel** as follows.

- Objects: sets
- Morphisms: relations

$$F : A \to B \quad \text{is} \quad F \subseteq B \times A$$

- Composition: for $A \xrightarrow{F} B \xrightarrow{G} C$ we define

$$(G \circ F)(c, a) = (\exists b \in B)\ G(c, b) \wedge F(b, a)$$

- Identity: define $1_A : A \to A$ by

$$1_A(b, a) = \begin{cases} \top & \text{if } a = b\,, \\ \bot & \text{otherwise.} \end{cases}$$

Structure of **Rel**

- Symmetric monoidal structure: $A \times B$
- Closed structure (internal hom): $A \multimap B = B \times A$, since

$$\begin{array}{rcl} \mathbf{Rel}[X \times A, B] & = & \mathcal{P}(B \times X \times A) \\ & \cong & \mathcal{P}(B \times A \times X) \\ & = & \mathbf{Rel}[X, A \multimap B] \end{array}$$

- Products: $A + B$, since

$$\begin{array}{rcl} \mathbf{Rel}[X, A] \times \mathbf{Rel}[X, B] & = & \mathcal{P}(A \times X) \times \mathcal{P}(B \times X) \\ & \cong & \mathcal{P}((A + B) \times X) \\ & = & \mathbf{Rel}[X, A + B] \end{array}$$

- Terminal object: 0, since $\mathbf{Rel}[X, 0] \cong 1$.

The exponential modality

For $A \in \mathbf{Rel}$, define

$$\begin{aligned} !A &= \text{free commutative monoid on } A \\ &= \text{set of multisets } \alpha = [a_1, \ldots, a_n] \text{ of elements of } A \end{aligned}$$

This is a comonad on **Rel**, with

- $\mathsf{d}_A : !A \to A$, defined by $\mathsf{d}_A(a, \alpha) \Leftrightarrow [a] = \alpha$
- $\mathsf{p}_A : !A \to !!A$, defined by $\mathsf{p}_A([\alpha_1, \ldots, \alpha_n], \alpha) \Leftrightarrow \alpha_1 + \ldots + \alpha_n = \alpha$

Seely equivalences

- $!(A + B) \cong !A \times !B$ and $!0 = 1$

The category **Rel** is a (degenerate) model of classical linear logic.

The Kleisli category

Define the category $\mathbf{Rel}_!$ as follows:

- Objects: sets
- Morphisms: relations $F : !A \to B$
- Composition: given $F : !A \to B$ and $G : !B \to C$, consider

$$!A \xrightarrow{\mathsf{p}_A} !!A \xrightarrow{!F} !B \xrightarrow{G} C$$

- Identity: $\mathsf{d}_A : !A \to A$.

Idea:

- $\mathbf{Rel}$ = sets and linear maps,
- $\mathbf{Rel}_!$ = sets and non-linear maps

Structure of $\mathbf{Rel}_!$

- Products: $A + B$, since

$$\mathbf{Rel}_!\,[X, A] \times \mathbf{Rel}_!\,[X, B] = \mathbf{Rel}\,[!X, A] \times \mathbf{Rel}\,[!X, B] \cong \mathbf{Rel}[!X, A + B]$$

- Exponentials: $B^A =!A \multimap B$, since

$$\mathbf{Rel}_!\,[X + A, B] = \mathbf{Rel}\,[!(X + A), B] \cong \mathbf{Rel}\,[!X \times !A, B] \cong \mathbf{Rel}\,[!X, !A \multimap B] \cong \mathbf{Rel}\,[!X, !A \multimap B]$$

Differential structure (I)

Want:

$$\frac{F : !A \to B}{\mathrm{d}F : !A \times A \to B}$$

Idea: Differential categories [Blute, Cockett, Seely]

- it suffices to have $\partial_A : !A \times A \to !A$. Then $\mathrm{d}F$ is obtained as

$$!A \times A \xrightarrow{\partial_A} !A \xrightarrow{F} B$$

- it suffices to have $\bar{d}_A : A \to !A$. Then $\mathrm{d}F$ is obtained as

$$!A \times A \xrightarrow{1 \times \bar{d}_A} !A \times !A \xrightarrow{\bar{c}_A} !A \xrightarrow{F} B$$

Axioms corresponding to constant rule, product rule, chain rule, . . .

Differential structure (II)

For $F : !A \to B$, define $\mathrm{d}F : !A \times A \to B$ by

$$\mathrm{d}F(b, (\alpha, a)) \Leftrightarrow F(b, \alpha + [a])\,.$$

Note: Shift of one from α to $\alpha + [a]$. This is from $\bar{d}_A : A \to !A$ given by

$$\bar{d}_A(\alpha, a) \Leftrightarrow \alpha = [a]$$

Theorem. [BCS], [Ehrhard], [Hyland]

- $\mathbf{Rel}_!$ is a model of the simply-typed differential λ-calculus.

Example

Say $F : !A \to B$ is **constant** if there is $Y \subseteq B$ such that

$$\begin{array}{ccc} !A & \xrightarrow{F} & B \\ & {\scriptstyle w_A}\searrow \quad \nearrow{\scriptstyle Y} & \\ & 1 & \end{array}$$

in **Rel**. This means

$$F(b, \alpha) \quad \Leftrightarrow \quad w_A(*, \alpha) \text{ and } Y(b, *) \quad \Leftrightarrow \quad \alpha = [\,] \text{ and } b \in Y$$

Proposition. If F constant, then $\mathrm{d}F : !A \times A \to B$ is $\emptyset$.

Proof. $\mathrm{d}F(b, (\alpha, a)) \quad \Leftrightarrow \quad F(b, \alpha + [a]) \quad \Leftrightarrow \quad \alpha + [a] = [\,] \text{ and } b \in Y \quad \Leftrightarrow \quad \bot$

Part III: A 2-categorical model

Profunctors

A categorification of relations [Bénabou], [Lawvere].

Definition. Let A, B be small categories. A (B, A)-**profunctor** is a functor

$$F : B^{\mathrm{op}} \times A \to \mathbf{Set}$$

Idea:

- $F(b, a)$ is the set of 'proofs' that b and a are related.
- A matrix of sets $F(b, a)$, together with actions

$$F(b, a) \times A[a, a'] \to F(b, a'), \quad B[b', b] \times F(b, a) \to F(b', a)$$

Example. For a small category A, we have

$$A[-, -] : A^{\mathrm{op}} \times A \to \mathbf{Set}\,.$$

The 2-category of profunctors

Define the 2-category **Prof** as follows.

- Objects: small categories
- Morphisms: profunctors

$$F : A \to B \quad \text{is} \quad F : B^{\mathrm{op}} \times A \to \mathbf{Set}$$

- 2-cells: natural transformations
- Composition: for $A \xrightarrow{F} B \xrightarrow{G} C$ define

$$(G \circ F)(c, a) = \Big(\sum_{b \in B} G(c, b) \times F(b, a) \Big)_{/\sim}$$

- Identity: define $1_A : A \to A$ by

$$1_A(b, a) = A[b, a]$$

The structure of **Prof**

- Symmetric monoidal structure: $A \times B$
- Closed structure (internal hom): $A \multimap B =_{\mathrm{def}} B \times A^{\mathrm{op}}$, since

$$\mathbf{Prof}[X \times A, B] \cong \mathbf{Prof}[X, A \multimap B]$$

- Binary products: $A + B$, since

$$\mathbf{Prof}[X, A] \times \mathbf{Prof}[X, B] \cong \mathbf{Prof}[X, A + B]$$

- Terminal object: 0, since

$$\mathbf{Prof}[X, 0] \cong 1$$

All this is now in a 2-categorical sense.

The exponential modality

For $A \in \mathbf{Prof}$, define $!A =$ free symmetric monoidal category on A as follows.

- Objects: $(a_1, \ldots, a_n)$, where $n \in \mathbb{N}$ and $a_i \in A$,
- Morphisms: $(\sigma, f_1, \ldots f_n) : (a_1, \ldots, a_n) \to (b_1, \ldots, b_m)$, only if $n = m$, with $\sigma \in \mathsf{S}_n$ and $f_i : a_i \to b_{\sigma(i)}$.

This is a pseudocomonad on $\mathbf{Prof}$, with

- $\mathrm{d}_A : !A \to A$ defined by $\mathrm{d}_A(a, \alpha) = !A[\alpha, (a)]$
- $\mathrm{p}_A : !A \to !!A$ defined by $\mathrm{p}_A((\alpha_1, \ldots, \alpha_n), \alpha) = !A[\alpha, \alpha_1 \oplus \ldots \oplus \alpha_n]$

Seely equivalences

- $!(A + B) \simeq !A \times !B$ (equivalences, not isomorphisms) and $!0 \cong 1$

The Kleisli 2-category

Define the 2-category $\mathbf{Prof}_!$ as follows.

- Objects: small categories
- Morphisms: profunctors $F : !A \to B$
- 2-cells: natural transformations
- Composition: for $F : !A \to B$ and $G : !B \to C$, consider

$$!A \xrightarrow{\mathrm{p}_A} !!A \xrightarrow{!F} !B \xrightarrow{G} C$$

- Identity: $\mathrm{d}_A : !A \to A$.

Idea:

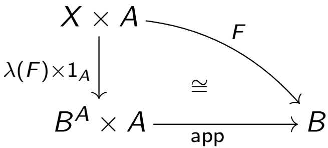
- $\mathbf{Prof}$ = categories and linear maps
- $\mathbf{Prof}_!$ = categories and non-linear maps

Structure of **Prof**$_!$

In analogy with the relational model, we have:

Theorem. The 2-category **Prof**$_!$ is cartesian closed.

This means that, for $F : X \times A \to B$, there is $\lambda(F) : X \to B^A$ and a 2-cell



Note. This 2-cell witnesses the β-rule of the λ-calculus:

$$\mathrm{app}\big((\lambda x : A)F, x\big) \cong F$$

Towards differentiation: Joyal's analytic functors

Consider $A = B = 1$. Then

$$\begin{aligned} F : 1 \to 1 \text{ in } \mathbf{Prof}_! \quad &= \quad F : !1 \to 1 \quad \text{profunctor} \\ &= \quad F : 1^{\mathrm{op}} \times !1 \to \mathbf{Set} \quad \text{functor} \\ &= \quad F : \mathbf{P} \to \mathbf{Set} \quad \text{functor} \end{aligned}$$

where $\mathbf{P}$ is the category of natural numbers and permutations.

The **analytic functor** associated to F is the functor $\widehat{F} : \mathbf{Set} \to \mathbf{Set}$ defined by

$$\widehat{F}(X) = \sum_{n \in \mathbb{N}} \frac{F(n) \times X^n}{S_n}$$

A categorification of exponential power series.

Differentiation of analytic functors

Let $F : \mathbf{P} \to \mathbf{Set}$ be a symmetric sequence. Then $F'(n) = F(n+1)$. So

$$\widehat{F'}(X) = \sum_{n \in \mathbb{N}} \frac{F(n+1) \times X^n}{S_n}$$

Compare with

$$f(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \qquad \rightsquigarrow \qquad f'(x) = \sum_{n=0}^{\infty} f_{n+1} \frac{x^n}{n!}$$

We will generalise this to any $F : !A \to B$ in $\mathbf{Prof}_!$

Differential structure

Differentiation. For $F : !A \to B$, define $\mathrm{d}F : !A \times A \to B$ by

$$\mathrm{d}F(b, (\alpha, a)) = F(b, \alpha \oplus [a])$$

Note: Shift by one.

For $a \in A$, define $\frac{\partial}{\partial a} F : !A \to B$ by

$$\left(\frac{\partial}{\partial a} F\right)(b, \alpha) = F(b, \alpha \oplus [a])$$

Differential Calculus

Theorem.

1. [Symmetry rule] $\frac{\partial}{\partial a'}\frac{\partial}{\partial a}F \cong \frac{\partial}{\partial a}\frac{\partial}{\partial a'}F$
2. [Sum rule] $\frac{\partial}{\partial a}(F + G) \cong \frac{\partial}{\partial a}(F) + \frac{\partial}{\partial a}(G)$
3. [Product rule] $\frac{\partial}{\partial a}(F \cdot G) \cong \left(\frac{\partial}{\partial a}F\right) \cdot G + F \cdot \left(\frac{\partial}{\partial a}G\right)$
4. [Chain rule] $\frac{\partial}{\partial a}(G \circ F) \cong \left(\sum_{b \in B}\left(\frac{\partial}{\partial b}(G)\right) \circ F \cdot \frac{\partial}{\partial a}(F)\right)_{/\simeq}$

The 2-categorical model

Theorem

- $\mathbf{Prof}_!$ is a 2-categorical model of the simply-typed differential λ-calculus.

Note: This comes from properties of $\bar{d}_A : A \to !A$, defined by

$$\bar{d}_A(\alpha, a) = !A[\alpha, (a)]$$

Challenges of categorification

1. Distributivity vs pseudo-distributivity
   - Model is **Rel** is based on interaction between $\mathcal{P}$ and !
   - Model in **Prof** is based on interaction between Psh and !
2. Kleisli construction for pseudomonads
3. Pseudonaturality of equivalences for cartesian closure
4. Some foundational results still not in place

Related and ongoing work

Part of wider investigations on 2-dimensional models of linear logic:

- Coherence theorems (Fiore & Saville, Olimpieri)
- Connections with intersection type systems (Olimpieri)
- Fixpoint operators (Galal)
- Variants (Fiore & Galal & Paquet)
- Pseudocommutativity (Slattery)
- Foundations of 2-categorical models of linear logic (Miranda)