The bicategory of operads is cartesian closed

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References

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Bicategories

A **bicategory** $\mathcal{E}$ consists of

- objects $(X, Y, Z, \dots)$
- morphisms $(M : X \to Y,\ N : Y \to Z,\ \dots)$
- 2-cells $(\alpha : M \Rightarrow M',\ \dots)$

together with

- composition operations (e.g. $N \circ M : X \to Z$)
- identity morphisms and 2-cells (e.g. $1_X : X \to X$)
- associativity and unit isomorphisms

$$\alpha_{M,N,P} : (P \circ N) \circ M \Rightarrow P \circ (N \circ M)\,,$$

$$\lambda_M : 1_Y \circ M \Rightarrow M\,, \quad \rho_M : M \circ 1_X \Rightarrow M\,,$$

subject to axioms.

Examples

1. The bicategory **Cat** of small categories:
   - objects = categories
   - morphisms = functors
   - 2-cells = natural transformations

2. For every monoidal category $(\mathbb{C}, \otimes, I)$, we have a bicategory:
   - objects = $\{*\}$
   - morphisms = objects of $\mathbb{C}$
   - 2-cells = arrows of $\mathbb{C}$

   Example: $(\mathbf{Ab}, \otimes, \mathbb{Z})$

Fix $\mathcal{V}$ symmetric monoidal closed cocomplete category.

3. The bicategory $\mathcal{V}$-**Mat** of $\mathcal{V}$-matrices:

- objects = sets
- morphisms = functors

$$M\colon A \times B \to \mathcal{V}$$

- 2-cells = natural transformations.

The composite of $M : A \times B \to \mathcal{V}\,,\; N : B \times C \to \mathcal{V}$ is

$$N \circ M(a,c) =_{\text{def}} \sum_{b \in B} M(a,b) \otimes N(b,c)\,.$$

Idea. Generalised relations.

4. The bicategory $\mathcal{V}$-**Sym** of symmetric $\mathcal{V}$-sequences:

- objects = sets
- morphisms = functors

$$M\colon \Sigma_*(A) \times B \to \mathcal{V}$$

- 2-cells = natural transformations

Here, the category $\Sigma_*(A)$ has

- objects: sequences $(a_1, \ldots, a_n)$ with $a_i \in A$
- morphisms:

$$(a_1, \ldots, a_n) \to (a'_1, \ldots, a'_n)$$

  given by $\sigma \in \Sigma_n$ such that $a'_i = a_{\sigma(i)}$.

A symmetric sequence $M\colon \Sigma_*(A) \times B \to \mathcal{V}$ determines a functor

$$M^\sharp : \mathcal{V}^A \to \mathcal{V}^B$$

defined by

$$M^\sharp(X, b) = \int^{(a_1,\ldots,a_n)\in\Sigma_*(A)} M(a_1, \ldots, a_n; b) \otimes X(a_1) \otimes \ldots X(a_n)$$

for $X \in \mathcal{V}^A$, $b \in B$.

Composition of morphisms in $\mathcal{V}$-**Sym** is defined so that

$$(N \circ M)^\sharp \cong N^\sharp \cdot M^\sharp$$

Monads and bimodules

Let $\mathcal{E}$ be a bicategory.

Definition. A **monad** in $\mathcal{E}$ consists of

- $X \in \mathcal{E}$
- $A : X \to X$
- $\mu : A \circ A \Rightarrow A$, $\eta : 1_X \Rightarrow A$

subject to associativity and unit axioms.

Examples:

- monads in $\mathbf{Ab}$ = rings
- monads in $\mathcal{V}$-$\mathbf{Mat}$ = small $\mathcal{V}$-categories
- monads in $\mathcal{V}$-$\mathbf{Sym}$ = (symmetric, coloured) $\mathcal{V}$-operads

Bimodules

Let (X, A), (Y, B) be monads in $\mathcal{E}$.

Definition. An (A, B)**-bimodule** consists of

- $M : X \to Y$
- $\rho : M \circ A \Rightarrow M$
- $\lambda : B \circ M \Rightarrow M$

subject to the axioms for a right A-action, a left B-action and a commutation condition.

Examples:

- bimodules in $\mathbf{Ab}$ = ring bimodules
- bimodules in $\mathcal{V}$-$\mathbf{Mat}$ = functors $\mathbb{A}^{\mathrm{op}} \times \mathbb{B} \to \mathcal{V}$
- bimodules in $\mathcal{V}$-$\mathbf{Sym}$ = operad bimodules

Bicategories of bimodules

Let $\mathcal{E}$ be a bicategory with stable local reflexive coequalizers.

The bicategory $\mathbf{Bim}(\mathcal{E})$ of bimodules:

- objects = monads in $\mathcal{E}$
- morphisms = bimodules
- 2-cells = bimodule morphisms

The composite of $M : (X, A) \to (Y, B)\,,\ N : (Y, B) \to (Z, C)$,

$$N \circ_B M : (X, A) \to (Z, C)\,,$$

is

$$N \circ B \circ M \underset{\rho \circ M}{\overset{N \circ \lambda}{\rightrightarrows}} N \circ M \longrightarrow N \circ_B M$$

Note. Generalisation of the tensor product of bimodules.

Examples

The bicategory of distributors $\mathcal{V}$-**Dist** = **Bim**($\mathcal{V}$-**Mat**) has:

- objects = small $\mathcal{V}$-categories
- morphisms = distributors, i.e. $\mathcal{V}$-functors $\mathbb{A}^{\mathrm{op}} \otimes \mathbb{B} \to \mathcal{V}$
- 2-cells = natural transformations.

The bicategory of operads $\mathcal{V}$-**Opd** $=_{\mathrm{def}}$ **Bim**($\mathcal{V}$-**Sym**) has:

- objects = $\mathcal{V}$-operads
- morphisms = operad bimodules
- 2-cells = operad bimodule morphisms.

Cartesian closed bicategories

A bicategory $\mathcal{E}$ is **cartesian** if it has

- a terminal object 1, characterised by:

$$\mathrm{Hom}_{\mathcal{E}}(X, 1) \simeq \mathbf{1}$$

- binary products $Y_1 \times Y_2$, characterised by

$$\mathrm{Hom}_{\mathcal{E}}(X, Y_1) \times \mathrm{Hom}_{\mathcal{E}}(X, Y_2) \simeq \mathrm{Hom}_{\mathcal{E}}(X, Y_1 \times Y_2)$$

A bicategory $\mathcal{E}$ is **cartesian closed** if it also has

- exponentials $[Y, Z]$, characterised by

$$\mathrm{Hom}_{\mathcal{E}}\big(X \times Y, Z\big) \simeq \mathrm{Hom}_{\mathcal{E}}\big(X, [Y, Z]\big)\,.$$

General result

Theorem. Let $\mathcal{E}$ be a bicategory with stable local reflexive coequalizers. If $\mathcal{E}$ is cartesian closed, then so is $\mathbf{Bim}(\mathcal{E})$.

Idea.

- Products

$$(Y_1, B_1) \times (Y_2, B_2) = (Y_1 \times Y_2, B_1 \times B_2)$$

- Exponentials

$$\big[(X, A), (Y, B)\big] = \big([X, Y], [A, B]\big)$$

Application

Theorem. The bicategory $\mathcal{V}$-**Opd** is cartesian closed.

Proof.

Let $\mathcal{E}$ be just as $\mathcal{V}$-**Sym** but objects are small $\mathcal{V}$-categories.

$$\begin{array}{ccc} \mathcal{V}\text{-}\mathbf{Sym} & \rightarrowtail & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{V}\text{-}\mathbf{Opd} & \simeq & \mathbf{Bim}(\mathcal{E}) \end{array}$$

Observe:

1. $\mathcal{E}$ is cartesian closed, by an extension of [FGHW]
2. $\mathbf{Bim}(\mathcal{E})$ is cartesian closed by general theorem
3. $\mathbf{Opd} = \mathbf{Bim}(\mathcal{V}\text{-}\mathbf{Sym}) \simeq \mathbf{Bim}(\mathcal{E})$.