Aspects of univalence

Nicola Gambino

School of Mathematics, University of Leeds

Homotopy Type Theory and Univalent Foundations
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Overview of the talk

Disclaimer: this is an expository talk, based on work by

- Voevodsky
- Streicher
- Joyal
- Kapulkin, Lumsdaine and Voevodsky
- D.-C. Cisinski

Part I: Univalence

- simplicial sets
- univalent fibrations

Part II: Proof of univalence

- The universe is Kan
- The classifying map is a Kan fibration
- The classifying map is a univalent fibration

Part I: Univalence

Simplicial sets

SSet = category of simplicial sets (presheaf topos)

It has a model structure $(\mathcal{W}, \mathcal{F}, \mathcal{C})$, where

- $\mathcal{W}$ = weak homotopy equivalences
- $\mathcal{F}$ = Kan fibrations
- $\mathcal{C}$ = monomorphisms

The fibrant objects are exactly the Kan complexes.

Note. The rich structure of **SSet** allows us to internalize a lot of constructions. For example, if $p : E \to B$ a fibration, there is a new fibration

$$(s, t) : \mathrm{Weq}(E) \to B \times B$$

such that the fiber over (x, y) is

$$\mathrm{Weq}(E)_{x,y} = \{w : E_x \to E_y \mid w \in \mathcal{W}\}$$

Remark

The model structure is cofibrantly generated.

Futhermore, it is

(1) right proper, i.e. pullback along fibrations preserves weak equivalences

(2) left proper, i.e. pushout along cofibrations preserves weak equivalences

Since pullback along any map preserves cofibrations, we have

(3a) Frobenius, i.e. pullback along fibrations preserves trivial cofibrations

(3b) Π-types, i.e. pushforward along fibrations preserves fibrations

with (3a) $\Leftrightarrow$ (3b)

Note: (3a), (3b) cannot be proved constructively (Bezem, Coquand, Parmann)

Univalent fibrations

Given $p : E \to B$, we have

$$\begin{array}{ccc} B & \longrightarrow & \mathrm{Weq}(E) \\ {\scriptstyle i}\big\downarrow & \overset{j_p}{\nearrow} & \big\downarrow{\scriptstyle (s,t)} \\ \mathrm{Path}(B) & \longrightarrow & B \times B \end{array}$$

Definition. A fibration $p : E \to B$ is said to be **univalent** if

$$j_p : \mathrm{Path}(B) \to \mathrm{Weq}(E)$$

is a weak equivalence.

Idea. Weak equivalences between fibers are 'witnessed' by paths in the base.

A characterisation of univalent fibrations

Proposition (Voevodsky)**.** Let $p : E \to B$ be a fibration. TFAE:

- $p : E \to B$ is univalent
- for every fibration $p' : E' \to B'$, the space of squares

$$\begin{array}{ccc} E' & \xrightarrow{u} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow[v]{} & B \end{array}$$

  such that

$$(u, p') : E' \to B' \times_B E$$

  is a weak equivalence, is either empty or contractible.

Idea. Essential uniqueness of u, v (when they exist).

A classifying map for Kan fibrations

Fix an inaccessible cardinal κ. By Hofmann & Streicher, there exists a map

$$\pi : \tilde{U} \to U$$

that weakly classifies fibrations with fibers of cardinality $< \kappa$, i.e. for every such $p : E \to B$ there exists a pullback diagram

$$\begin{array}{ccc} E & \longrightarrow & \tilde{U} \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ B & \longrightarrow & U\,. \end{array}$$

Note. Given a Kan complex X of cardinality $< \kappa$, there exists a pullback

$$\begin{array}{ccc} X & \longrightarrow & \tilde{U} \\ \downarrow & \lrcorner & \downarrow \\ 1 & \underset{\langle X \rangle}{\longrightarrow} & U \end{array}$$

We think of $\langle X \rangle$ as the ‘name’ of the Kan complex X.

Univalent universes

Theorem (Voevodsky)**.**

(1) The base U is a Kan complex.

(2) The generic map $\pi : \tilde{U} \to U$ is a fibration.

(3) The generic fibration $\pi : \tilde{U} \to U$ is univalent.

Several proofs:

- Voevodsky
- Lumsdaine, Kapulkin, Voevodsky (using simplifications by Joyal)
- Moerdijk (using fiber bundles)
- Cisinski (general setting)

Note. The map $\pi : \tilde{U} \to U$ is therefore

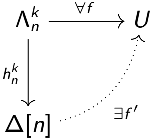
versal & univalent

for the class of κ-small fibrations.

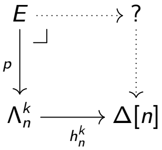
Part II: Proof of the theorem

(1) The base U is a Kan complex

Proof. Show



This reduces to the problem of extending fibrations along horn inclusions:



This can be done using the theory of minimal fibrations.

Minimal fibrations extend along trivial cofibrations

Lemma 1.

For every minimal fibration $m : X \to A$ and trivial cofibration $i : A \to A'$, there exists

$$\begin{array}{ccc} X & \xrightarrow{j} & X' \\ {\scriptstyle m}\downarrow & \lrcorner & \downarrow{\scriptstyle m'} \\ A & \xrightarrow[i]{} & A' \end{array}$$

with $m' : X' \to A'$ a minimal fibration.

Trivial fibrations can be extended along cofibrations

Lemma 2. (Joyal)

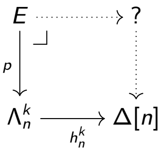
For every trivial fibration $t : E \to X$ and every cofibration $j : X \to X'$ there exists square

$$\begin{array}{ccc} E & \longrightarrow & E' \\ {\scriptstyle t}\downarrow & \lrcorner & \downarrow{\scriptstyle t'} \\ X & \underset{j}{\longrightarrow} & X' \end{array}$$

with $t' : E' \to X'$ a trivial fibration.

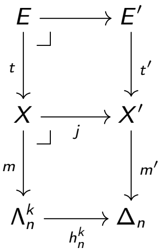
End of proof of (1)

Recall that we need to complete the diagram



It suffices to

- factor p as a trivial fibration t followed by a minimal fibration m
- apply Lemma 1 and Lemma 2 so as to get



(2) The map $\pi : \tilde{U} \to U$ is a fibration

This essentially amounts to showing that a map $p : E \to B$ with κ-small fibers is a Kan fibration **if and only if** it can be obtained as a pullback

$$\begin{array}{ccc} E & \longrightarrow & \tilde{U} \\ {\scriptstyle p}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \pi} \\ B & \longrightarrow & U\,. \end{array}$$

(3) The fibration $\pi : \tilde{U} \to U$ is univalent

We consider the diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad w \quad} & \mathrm{Weq}(U) \\ & {\scriptstyle \Delta} \searrow \quad \swarrow {\scriptstyle (s,t)} & \\ & U \times U & \end{array}$$

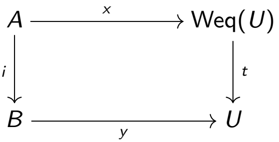
and show $w \in \mathcal{W}$. By composing with $\pi_2 : U \times U \to U$, we get

$$\begin{array}{ccc} U & \xrightarrow{\quad w \quad} & \mathrm{Weq}(U) \\ & {\scriptstyle 1_U} \searrow \quad \swarrow {\scriptstyle t} & \\ & U & \end{array}$$

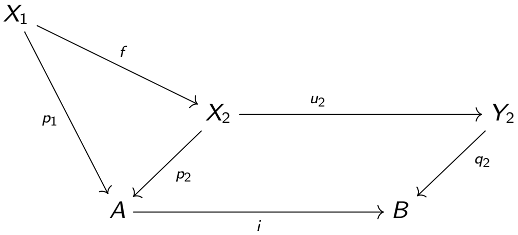
Hence, it suffices to show that $t \in \mathcal{W}$.

Since $t \in \mathcal{F}$, we show that $t \in \mathcal{W} \cap \mathcal{F} = {}^{\pitchfork}\mathcal{C}$

So, we need to prove the existence of a diagonal filler in a diagram of the form

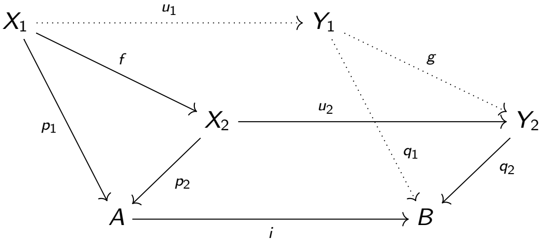


where $i \in \mathcal{C}$. By the definition of t, such a square amounts to a diagram



where $p_1 , p_2 , q_2 \in \mathcal{F}$, $f \in \mathcal{W}$ and $i \in \mathcal{C}$.

The required diagonal filler amounts to a diagram of the form



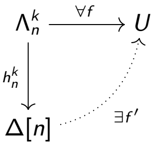
where q_1 is a fibration, g is weak equivalence and all squares are pullbacks.

This is also established via the theory of minimal fibrations.

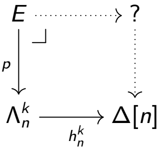
Note. The property that weak equivalences between fibrations can be extended along cofibrations may be regarded as a 'global' form of univalence, since it is a property of the model category.

Concluding remarks

- Moving between, say,



and



  is more subtle than one imagines.
- Cisinski shows that the argument above carries over to a wide class of presheaf categories (elegant Reedy presheaves with an additional property)
- For this, he generalises the theory of minimal fibrations
- Moerdijk and Nuiten have generalized this theory even further.
- Can the cubical set model be presented (classically) in this style?