

On the 2-category of symmetric 2-rigs

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Context and main contributions

Symmetric 2-rigs

- ▶ Categorification of commutative rings
- ▶ Used in algebraic geometry, category theory, enumerative combinatorics, ...

Main contributions

- ▶ New results on symmetric 2-rigs ($\Rightarrow$ Coexponentiability)
- ▶ Study of various sub-2-categories ($\Rightarrow$ Operadic 2-rigs)

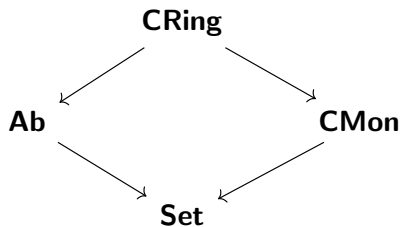
References

- ▶ [GGV 25] = www.arxiv.org/abs/2511.144402
- ▶ [AFG 26a] = www.arxiv.org/abs/2607.12683
- ▶ [AFG 26b] = www.arxiv.org/abs/2607.12705

Plan

1. Symmetric 2-rigs
2. Coexponentiability
3. Operadic 2-rigs
4. Differentiation

Commutative rings



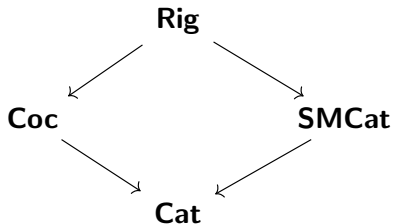
Recall:

- ▶ Distributive law
- ▶ $(\mathbf{Ab}, \otimes, \mathbb{Z})$ symmetric monoidal closed
- ▶ $\mathbf{CRing} = \mathbf{CMon}(\mathbf{Ab}, \otimes, \mathbb{Z})$ has binary coproducts, given by $R \otimes S$

Symmetric 2-rigs

Definition. A **symmetric 2-rig** is a category R such that

- ▶ R cocomplete
- ▶ R is symmetric monoidal
- ▶ the tensor product preserves colimits in each variable.



Note: size issues, presentability.

Examples of symmetric 2-rigs

- ▶ **Free 2-rigs:** $\text{Psh}(S(A))$, for a category A
 - $A = 0$ gives **Set** = the initial 2-rig
 - $A = 1$ gives $[\mathbf{P}, \mathbf{Set}]$ = symmetric sequences (cf. single-colour operads)
- ▶ **Convolution 2-rigs:** $\text{Psh}(M)$, for a symmetric monoidal category M
- ▶ **Operadic 2-rigs:** $\text{Psh}(\text{Env}(P))$, for an operad P

We obtain sub-2-categories of **Rig**:

$$\mathbf{Rig}^{\text{free}} \hookrightarrow \mathbf{Rig}^{\text{opd}} \hookrightarrow \mathbf{Rig}^{\text{conv}}$$

Question: Can we describe these 2-categories intrinsically?

Duality and analytic functors

For a 2-rig R , define its category of points

$$\mathrm{pt}(R) = \mathbf{Rig}[R, \mathbf{Set}]$$

Examples

- ▶ points of free 2-rig on $A \simeq [A^{\mathrm{op}}, \mathbf{Set}]$
- ▶ points of convolution 2-rig on $M \simeq \mathbf{SMCat}[M^{\mathrm{op}}, \mathbf{Set}]$
- ▶ points of operadic 2-rig on $P \simeq P\text{-Alg}$

We obtain

$$\mathrm{pt}: \mathbf{Rig}^{\mathrm{op}} \rightarrow \mathbf{SiftCat}$$

Analytic functors $[\text{Joyal 86}], [\text{FGHW 08}], [\text{GJ 18}] \subseteq \text{functors } \mathrm{pt}(f): \mathrm{pt}(S) \rightarrow \mathrm{pt}(R)$

Coexponentiability for 2-rigs [AFG 26a]

$(\mathbf{Coc}, \otimes, \text{Set})$ symmetric monoidal closed.

Proposition. A cocomplete category is dualizable in $\mathbf{Coc}$ if and only if it is a deformation retract of a presheaf category.

$\mathbf{Rig}$ has binary coproducts, given by $R \otimes S$.

Theorem. A 2-rig is coexponentiable in $\mathbf{Rig}$ if and only if its underlying cocomplete category is dualizable in $\mathbf{Coc}$.

Prop + **Thm** $\Rightarrow$ Simple criterion for coexponentiability (cf. examples)

Cf. [Niefield–Wood 17]

Free 2-rigs

$\mathbf{Rig}^{free}$ = full sub-2-category of $\mathbf{Rig}$ spanned by free 2-rigs.

$$\begin{aligned}\mathbf{Rig}^{free}[\mathrm{Psh}(S(A)), \mathrm{Psh}(S(B))] &\simeq \mathbf{SMCat}[S(A), \mathrm{Psh}(S(B))] \\ &\simeq \mathbf{Cat}[A, \mathrm{Psh}(S(B))] \\ &\simeq \mathbf{Prof}[A, S(B)] \\ &= \mathbf{CatSym}[A, B]\end{aligned}$$

Proposition. $\mathbf{Rig}^{free} \simeq \mathbf{CatSym}$ = bicategory of categories and symmetric sequences.

$\rightsquigarrow \mathbf{CatSym}^{\mathrm{op}}$ is cartesian closed [FGHW 08].

Note: Monad (A, P) in $\mathbf{CatSym}$ = A -coloured operad P .

Operadic 2-rigs [AFG 26b]

$\mathbf{Rig}^{opd}$ = full sub-2-category of $\mathbf{Rig}$ spanned by operadic 2-rigs.

Theorem. $\mathbf{Rig}^{opd} \simeq \mathbf{OpdBim}$ = bicategory of operads and bimodules.

Key ideas of proof.

- ▶ $\mathbf{OpdBim}$ is Eilenberg–Moore–Kleisli completion of $\mathbf{CatSym}$
- ▶ $\mathbf{Rig}^{opd}$ is Eilenberg–Moore–Kleisli completion of $\mathbf{Rig}^{free}$
- ▶ $\mathbf{Psh}(\mathbf{Env}(P))$ is Eilenberg–Moore & Kleisli object of P . □

$\rightsquigarrow \mathbf{OpdBim}^{op}$ is cartesian closed [GJ 18]

Boardman–Vogt tensor product of operad bimodules [GGV 24], [GGV 25]

CatSym and **OpdBim** embed in *double* categories $\mathbb{C}\text{atSym}$ and $\mathbb{O}\text{pdBim}$.

Theorem.

1. $\mathbb{C}\text{atSym}$ admits a normal oplax monoidal structure
 $\rightsquigarrow$ arithmetic (Dirichlet) tensor product of symmetric sequences
2. $\mathbb{O}\text{pdBim}$ admits an oplax monoidal structure
 $\rightsquigarrow$ Boardman–Vogt tensor product of operad bimodules.

Note: extends [Maia & Mendez 08], [Dwyer & Hess 14], [Lopez-Franco & Garner 16]

Differentiation

The embeddings

$$\mathbf{CatSym} \hookrightarrow \mathbf{Rig} \quad \mathbf{OpdBim} \hookrightarrow \mathbf{Rig}$$

allow us to exploit analogy with commutative algebra: Kähler differentials $\Omega_R, \dots$

For $\mathbf{Rig}^{free} \simeq \mathbf{CatSym}$, one recovers the theory of differentiation of [FGH 24]:

$$\Omega_{\mathbf{Psh}(S(A))} = \mathbf{Psh}(A \otimes S(A))$$

The differential $\mathbf{Psh}(S(A)) \rightarrow \mathbf{Psh}(A \otimes S(A))$ induced by the profunctor

$$d_A: S(A) \rightarrow A \otimes S(A) \quad d_A((a, \alpha), \beta) =_{\text{def}} S(A)[a \oplus \alpha, \beta]$$

Cf. [Blute & Lucyshyn-Wright & O'Neill 16]

Summary

- ▶ Characterisation of coexponentiable 2-rigs
- ▶ Sub-2-category of **Rig** spanned by operadic 2-rigs
- ▶ Boardman–Vogt tensor product of operad bimodules
- ▶ Differentiation

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