On the 2-category of symmetric 2-rigs

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Context and main contributions

Symmetric 2-rigs

- Categorification of commutative rings
- Used in algebraic geometry, category theory, enumerative combinatorics, ...

Main contributions

- New results on symmetric 2-rigs ($\Rightarrow$ Coexponentiability)
- Study of various sub-2-categories ($\Rightarrow$ Operadic 2-rigs)

References

- [GGV 25] = www.arxiv.org/abs/2511.144402
- [AFG 26a] = www.arxiv.org/abs/2607.12683
- [AFG 26b] = www.arxiv.org/abs/2607.12705

Plan

1. Symmetric 2-rigs
2. Coexponentiability
3. Operadic 2-rigs
4. Differentiation

Commutative rings

$$\begin{tikzcd}
 & \mathbf{CRing} \arrow[dl] \arrow[dr] & \\
\mathbf{Ab} \arrow[dr] & & \mathbf{CMon} \arrow[dl] \\
 & \mathbf{Set} &
\end{tikzcd}$$

Recall:

- Distributive law
- $(\mathbf{Ab}, \otimes, \mathbb{Z})$ symmetric monoidal closed
- $\mathbf{CRing} = \mathbf{CMon}(\mathbf{Ab}, \otimes, \mathbb{Z})$ has binary coproducts, given by $R \otimes S$

Symmetric 2-rigs

Definition. A **symmetric 2-rig** is a category R such that

- R cocomplete
- R is symmetric monoidal
- the tensor product preserves colimits in each variable.

$$\begin{tikzcd} & \mathbf{Rig} \arrow[dl] \arrow[dr] & \\ \mathbf{Coc} \arrow[dr] & & \mathbf{SMCat} \arrow[dl] \\ & \mathbf{Cat} & \end{tikzcd}$$

Note: size issues, presentability.

Examples of symmetric 2-rigs

- **Free 2-rigs:** $\mathrm{Psh}(\mathrm{S}(A))$, for a category A
  - $A = 0$ gives $\mathbf{Set}$ = the initial 2-rig
  - $A = 1$ gives $[\mathbf{P}, \mathbf{Set}]$ = symmetric sequences (cf. single-colour operads)
- **Convolution 2-rigs:** $\mathrm{Psh}(M)$, for a symmetric monoidal category M
- **Operadic 2-rigs:** $\mathrm{Psh}(\mathrm{Env}(P))$, for an operad P

We obtain sub-2-categories of **Rig**:

$$\mathbf{Rig}^{free} \hookrightarrow \mathbf{Rig}^{opd} \hookrightarrow \mathbf{Rig}^{conv}$$

Question: Can we describe these 2-categories intrinsically?

Duality and analytic functors

For a 2-rig R, define its category of points

$$\mathrm{pt}(R) = \mathbf{Rig}[R, \mathbf{Set}]$$

Examples

- points of free 2-rig on $A \simeq [A^{\mathrm{op}}, \mathbf{Set}]$
- points of convolution 2-rig on $M \simeq \mathbf{SMCat}[M^{\mathrm{op}}, \mathbf{Set}]$
- points of operadic 2-rig on $P \simeq P\text{-Alg}$

We obtain

$$\mathrm{pt}\colon \mathbf{Rig}^{\mathrm{op}} \to \mathbf{SiftCat}$$

Analytic functors [Joyal 86], [FGHW 08], [GJ 18] $\subseteq$ functors $\mathrm{pt}(f)\colon \mathrm{pt}(S) \to \mathrm{pt}(R)$

Coexponentiability for 2-rigs [AFG 26a]

$(\mathbf{Coc}, \otimes, \mathrm{Set})$ symmetric monoidal closed.

Proposition. A cocomplete category is dualizable in **Coc** if and only if it is a deformation retract of a presheaf category.

Rig has binary coproducts, given by $R \otimes S$.

Theorem. A 2-rig is coexponentiable in **Rig** if and only if its underlying cocomplete category is dualizable in **Coc**.

Prop + Thm $\Rightarrow$ Simple criterion for coexponentiability (cf. examples)

Cf. [Niefield–Wood 17]

Free 2-rigs

$\mathbf{Rig}^{\textit{free}}$ = full sub-2-category of **Rig** spanned by free 2-rigs.

$$\begin{aligned}
\mathbf{Rig}^{\textit{free}}\big[\mathsf{Psh}(\mathsf{S}(A)), \mathsf{Psh}(\mathsf{S}(B))\big] &\simeq \mathbf{SMCat}\big[\mathsf{S}(A), \mathsf{Psh}(\mathsf{S}(B))\big] \\
&\simeq \mathbf{Cat}\big[A, \mathsf{Psh}(\mathsf{S}(B))\big] \\
&\simeq \mathbf{Prof}\big[A, \mathsf{S}(B)\big] \\
&= \mathbf{CatSym}\big[A, B\big]
\end{aligned}$$

Proposition. $\mathbf{Rig}^{\textit{free}} \simeq \mathbf{CatSym}$ = bicategory of categories and symmetric sequences.

$\rightsquigarrow$ $\mathbf{CatSym}^{\mathrm{op}}$ is cartesian closed [FGHW 08].

Note: Monad (A, P) in $\mathbf{CatSym}$ = A-coloured operad P.

Operadic 2-rigs [AFG 26b]

$\mathbf{Rig}^{opd}$ = full sub-2-category of $\mathbf{Rig}$ spanned by operadic 2-rigs.

Theorem. $\mathbf{Rig}^{opd} \simeq \mathbf{OpdBim}$ = bicategory of operads and bimodules.

Key ideas of proof.

- $\mathbf{OpdBim}$ is Eilenberg–Moore–Kleisli completion of $\mathbf{CatSym}$
- $\mathbf{Rig}^{opd}$ is Eilenberg–Moore–Kleisli completion of $\mathbf{Rig}^{free}$
- $\mathrm{Psh}(\mathrm{Env}(P))$ is Eilenberg–Moore & Kleisli object of P. $\square$

$\rightsquigarrow$ $\mathbf{OpdBim}^{\mathrm{op}}$ is cartesian closed [GJ 18]

Boardman–Vogt tensor product of operad bimodules [GGV 24], [GGV 25]

CatSym and **OpdBim** embed in *double* categories $\mathbb{C}$**atSym** and $\mathbb{O}$**pdBim**.

Theorem.

1. $\mathbb{C}$**atSym** admits a normal oplax monoidal structure
   $\rightsquigarrow$ arithmetic (Dirichlet) tensor product of symmetric sequences
2. $\mathbb{O}$**pdBim** admits an oplax monoidal structure
   $\rightsquigarrow$ Boardman–Vogt tensor product of operad bimodules.

Note: extends [Maia & Mendez 08], [Dywer & Hess 14], [Lopez-Franco & Garner 16]

Differentiation

The embeddings

$$\mathbf{CatSym} \hookrightarrow \mathbf{Rig} \qquad \mathbf{OpdBim} \hookrightarrow \mathbf{Rig}$$

allow us to exploit analogy with commutative algebra: Kähler differentials $\Omega_R, \ldots$

For $\mathbf{Rig}^{\mathit{free}} \simeq \mathbf{CatSym}$, one recovers the theory of differentiation of [FGH 24]:

$$\Omega_{\mathsf{Psh}(\mathsf{S}(A))} = \mathsf{Psh}(A \otimes \mathsf{S}(A))$$

The differential $\mathsf{Psh}(\mathsf{S}(A)) \to \mathsf{Psh}(A \otimes \mathsf{S}(A))$ induced by the profunctor

$$\mathsf{d}_A \colon \mathsf{S}(A) \nrightarrow A \otimes \mathsf{S}(A) \qquad \mathsf{d}_A((a, \alpha), \beta) =_{\mathrm{def}} \mathsf{S}(A)[a \oplus \alpha, \beta]$$

Cf. [Blute & Lucyshyn-Wright & O'Neill 16]

Summary

- Characterisation of coexponentiable 2-rigs
- Sub-2-category of **Rig** spanned by operadic 2-rigs
- Boardman–Vogt tensor product of operad bimodules
- Differentiation

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