On the commuting tensor product of symmetric multicategories and their bimodules

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Plan

1. Context and motivation

2. Commuting tensor product

3. Bimodules

Reference

N. Gambino, R. Garner, C. Vasilakopoulou, A unified treatment of commuting tensor products of categories, operads, sym. multicategories and their bimodules.

ArXiv, 2025.

1. Context and motivation

	$\mathbb{R}\mathrm{ing}$	$\mathbb{P}\mathrm{rof}_V$	$\mathbb{S}\mathrm{MultProf}_V$
Objects	rings	V- categories	sym. V-multicategories
Vertical Maps	ring hom.	V- functors	sym. V-multifunctors
Commuting tensor	$R \otimes S$	$A \otimes B$	$A \otimes B$ (Boardman-Vogt)
Horizontal maps	bimodules	V-profunctors	V-sym. multiprofunctors
Commuting tensor of h. maps	$R \otimes S \xrightarrow{M \otimes M'} R' \otimes S'$	$A \otimes B \xrightarrow{F \otimes F'} A' \otimes B'$	? ✓ for operad bimodules [Dwyer-Hess]

Goal : To provide a unified account of these examples, in particular extending [Dwyer-Hess].

Motivation : Interest in symmetric multicategories

- $a_1, \ldots, a_n \xrightarrow{f} a \quad \rightsquigarrow \quad a_1, \ldots, a_n \vdash a$

- SMultProf and some of its subcategories have rich structure : cartesian closed [FGHW '08, GJ '18], fixpoints [Galal '23], differential [FGH '24], cf. quantitative semantics [Olimpieri '21].

2. Commuting tensor product

Fact: All the examples are of the form $\mathbb{B}\mathrm{im}(\mathbb{C})$ for $\mathbb{C}$ a double category.

Recall: a double category $\mathbb{C}$ has

- objects: $A, B, C, \ldots$
- vertical maps:

$$\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd}$$

- horizontal maps: $A \overset{x}{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\;} A'$
- squares:

$$\begin{tikzcd} A \arrow[r, "x", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\phi"] & A' \arrow[d, "f'"] \\ B \arrow[r, "y"', "|" marking] & B' \end{tikzcd}$$

Example : the double category $\mathbb{M}\mathrm{at}$ of matrices

- **Objects**: sets
- **Vertical maps**: functions
- **Horizontal maps**: matrices $A \overset{x}{\nrightarrow} A'$, i.e.

$$\begin{aligned} A' \times A &\xrightarrow{\ x\ } \mathbf{Set} \\ (a', a) &\longmapsto x(a', a) \end{aligned}$$

- **Squares** :

$$\begin{tikzcd} A \arrow[r, "x", "|"' marking] \arrow[d, "f"'] \arrow[dr, phantom, "\phi"] & A' \arrow[d, "f'"] \\ B \arrow[r, "y"', "|" marking] & B' \end{tikzcd} \quad \rightsquigarrow \quad \begin{gathered} \text{families of functions} \\ \phi_{a', a} : x(a', a) \longrightarrow y(f'a', fa) \end{gathered}$$

For a double category $\mathbb{C}$, define $\mathbb{B}\mathrm{im}(\mathbb{C})$ as follows:

double category of bimodules

- **Objects**: monads (A, a, μ, η) in $\mathbb{C}$, where

$$A \overset{a}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} A \qquad \begin{tikzcd} A \arrow[r, "1", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & A \arrow[d, equal] \\ A \arrow[r, "a"', "|" marking] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "a", "|" marking] \arrow[d, equal] & A \arrow[r, "a", "|" marking] \arrow[d, phantom, "\mu"] & A \arrow[d, equal] \\ A \arrow[rr, "a"', "|" marking] & & A \end{tikzcd}$$

- **vertical maps**: monad morphism $(f, \phi) : (A, a) \longrightarrow (B, b)$

where $f : A \longrightarrow B$ and

vertical map

$$\begin{tikzcd} A \arrow[r, "a", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\phi"] & A \arrow[d, "f"] \\ B \arrow[r, "b"', "|" marking] & B \end{tikzcd}$$

- **horizontal maps** : monad bimodules

$$(p, \rho, \lambda) : (A, a) \longrightarrow\!\!\!\!\!\!\!\!\!\mapsto (B, b)$$

where $A \overset{p}{\longrightarrow\!\!\!\!\!\!\!\mapsto} B$

$$\begin{tikzcd} A \arrow[r, "a", "|" marking] \arrow[d, equal] & A \arrow[r, "p", "|" marking] \arrow[d, phantom, "\rho"] & B \arrow[d, equal] \\ A \arrow[rr, "p"', "|" marking] & & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "p", "|" marking] \arrow[d, equal] & B \arrow[r, "b", "|" marking] \arrow[d, phantom, "\lambda"] & B \arrow[d, equal] \\ A \arrow[rr, "p"', "|" marking] & & B \end{tikzcd}$$

$\wr$ $\qquad\qquad$ $\wr$

$pa \Rightarrow p$ $\qquad\qquad$ $bp \Rightarrow p$

- **squares** : bimodule morphisms.

Remark We assume $\mathbb{C}$ has stable local reflexive coequalisers. (!)
to define composition in $\mathbb{B}\mathrm{im}(\mathbb{C})$.

Example $\mathbb{B}\mathrm{im}(\mathbb{M}\mathrm{at}) = \mathbb{P}\mathrm{rof}$

- a monad in $\mathbb{M}\mathrm{at}$ is a category, e.g.

$$A_0 \overset{A_1}{\longrightarrow\!\!\!\!\!\!\!+\;\;} A_0 \iff A_1 : A_0 \times A_0 \longrightarrow \underline{\mathrm{Set}} \qquad A_1(a', a)$$

$$A_1 \circ A_1 \overset{\mu}{\Longrightarrow} A_1 \iff \sum_{b \in C_0} A_1(b,c) \times A_1(a,b) \xrightarrow{\mu_{a,c}} A_1(a,c)$$

- a bimodule in Mat is a profunctor

$$\mathbb{A} \overset{P}{\longrightarrow\!\!\!\!\!\!\!+\;\;} \mathbb{B} \iff \mathbb{B}^{op} \times \mathbb{A} \xrightarrow{P} \mathrm{Set} \iff B_0 \times A_0 \xrightarrow{P} \mathrm{Set}$$

$$\wr$$

$$A_0 \overset{P}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B_0$$

plus 'actions'

$$P(b,a) \times A_1(a,a') \xrightarrow{\rho} P(b,a')$$

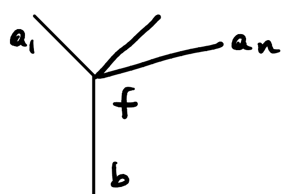
$$B_1(b',b) \times P(b,a) \xrightarrow{\lambda} P(b',a)$$

Example $\mathrm{Bim}(\mathrm{Sym}) = \mathrm{SMultProf}$, where Sym

- objects : sets
- horizontal maps : symmetric sequences
- vertical maps : functions
- squares : ...

$\Rightarrow$ a monad in Sym is a symmetric multicategory

$\Rightarrow$ a bimodule in Sym is a symmetric multiprofunctor



[FGHW '08, GJ '18]

Question Under what conditions on $\mathbb{C}$ do we get a 'commuting tensor product' on $\mathbb{B}\mathrm{im}(\mathbb{C})$?

We start from the vertical category of $\mathbb{B}\mathrm{im}(\mathbb{C})$, which is the category of monads and monad morphisms.

We build on R. Garner & I. Lopez-Franco, Commutativity JPAA, 2016.

Example In $\underline{\mathbf{Cat}}$

$$\begin{tikzcd} A, B \arrow[r] \arrow[dr, bend right, "\forall F"'] & A \times B \arrow[d, dashed, "\exists !"] \\ & C \end{tikzcd}$$

commuting functor in two variables

① $\begin{cases} \forall a & F(a,-) : B \longrightarrow C \\ \forall b & F(-,b) : A \longrightarrow C \end{cases}$ s.t. $F(a,-)(b) = F(-,b)(a) \underset{def}{=} F(a,b)$

② $\forall$ $\begin{array}{c} a \\ f \downarrow \\ a' \end{array}$ $\begin{array}{c} b \\ \downarrow g \\ b' \end{array}$

$$\begin{tikzcd} F(a,b) \arrow[r, "F(f,1)"] \arrow[d, "F(1,g)"'] & F(a',b) \arrow[d, "F(1,g)"] \\ F(a,b') \arrow[r, "F(f,1)"'] & F(a',b') \end{tikzcd}$$

commuting condition

Note There is also a 'funny' tensor product, which classifies functors in two variables, not necessarily commuting

$$\begin{tikzcd}
A, B \arrow[r] \arrow[dr, bend right, "\forall F"'] & A \square B \arrow[d, dashed, "\exists !"] \\
 & C
\end{tikzcd}$$

satisfying (1), not nec (2)

Goal: We want to do this in our general setting.

Definition Let $(\mathbb{C}, \boxtimes, I)$ be a symmetric normal oplax monoidal double category.

- A moned multimorphism "functor in two vars"

$$(f, \phi, \psi) : (A, a), (B, b) \longrightarrow (C, c)$$

consists of a vertical map $f : A \boxtimes B \longrightarrow C$ and

$$\begin{tikzcd} A \boxtimes B \arrow[r, "a \boxtimes 1"] \arrow[d, "f"'] \arrow[dr, phantom, "\varphi"] & A \boxtimes B \arrow[d, "f"] \\ C \arrow[r, "c"'] & C \end{tikzcd} \qquad \begin{tikzcd} A \boxtimes B \arrow[r, "1 \boxtimes b"] \arrow[d, "f"'] \arrow[dr, phantom, "\psi"] & A \boxtimes B \arrow[d, "f"] \\ C \arrow[r, "c"'] & C \end{tikzcd}$$

s.t. (f, φ) and (f, ψ) are moned morphisms.

- A monad multimorphism is commuting if

$$
\begin{tikzcd}
A \boxtimes B \arrow[rr, "a \boxtimes b"] \arrow[d, equal] \arrow[drr, phantom, "\sigma"] & & A \boxtimes B \arrow[d, equal] \\
A \boxtimes B \arrow[r, "a \boxtimes 1"'] \arrow[d, "f"'] \arrow[dr, phantom, "\varphi"] & A \boxtimes B \arrow[r, "1 \boxtimes b"'] \arrow[d, "f"] \arrow[dr, phantom, "\psi"] & A \boxtimes B \arrow[d, "f"] \\
C \arrow[r, "c"'] \arrow[d, equal] \arrow[drr, phantom, "\mu"] & C \arrow[r, "c"'] & C \arrow[d, equal] \\
C \arrow[rr, "c"'] & & C
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \boxtimes B \arrow[rr, "a \boxtimes b"] \arrow[d, equal] \arrow[drr, phantom, "\tau"] & & A \boxtimes B \arrow[d, equal] \\
A \boxtimes B \arrow[r, "1 \boxtimes b"'] \arrow[d, "f"'] \arrow[dr, phantom, "\psi"] & A \boxtimes B \arrow[r, "a \boxtimes 1"'] \arrow[d, "f"] \arrow[dr, phantom, "\varphi"] & A \boxtimes B \arrow[d, "f"] \\
C \arrow[r, "c"'] \arrow[d, equal] \arrow[drr, phantom, "\mu"] & C \arrow[r, "c"'] & C \arrow[d, equal] \\
C \arrow[rr, "c"'] & & C
\end{tikzcd}
$$

Theorem

1. Monads and monoid multimorphisms form a representable closed symmetric multicategory

$$\frac{(A,a),\ (B,b) \longrightarrow (C,c)}{(A,a) \mathbin{\square} (B,b) \longrightarrow (C,c)}$$

non-commuting tensor product

2. Monads and commuting monoid multimorphisms form a representable closed symmetric multicategory

$$\frac{(A,a),\ (B,b) \longrightarrow (C,c) \quad \text{commuting}}{(A,a) \otimes (B,b) \longrightarrow (C,c)}$$

commuting tensor product

Example

1. For $\mathbb{C} = \mathrm{Mat}_{\vee}$, we re-obtain the commuting tensor product of $\mathrm{Cat}_{\vee}$

2. For $\mathbb{C} = \mathrm{Sym}$, we re-obtain the Boardman-Vogt tensor product of symmetric multicategories, cf Elmendorff and Mandell '09.

3. Bimodules

Question How can we extend this to bimodules, i.e.

$$(A_1, a_1) \xrightarrow{P_1} (B_1, b_1) \quad , \quad (A_2, a_2) \xrightarrow{P_2} (B_2, b_2)$$

$$\overset{?}{\rightsquigarrow}$$

? how to define this ?

$$(A_1, a_1) \otimes (A_2, a_2) \xrightarrow{P_1 \otimes P_2} (B_1, b_1) \otimes (B_2, b_2)$$

commuting tensor

commuting tensor

Two approaches :

① Follow Dwyer & Hess :

- define commuting tensor of bimodules on free ones using $\boxtimes$
- extend to general bimodules via reflexive coequalisers

② Again, use multicategorical approach

in the paper

- define a notion of multimorphism of bimodules

$$P_1, \ldots, P_n \Rightarrow Q$$

- prove these can be represented. $P_1 \otimes P_2$

Theorem $(\mathbb{B}\mathrm{im}(\mathbb{C}), \otimes, I)$ is a symmetric oplax monoidal double category.

Remark : bimodule composition vs commuting tensor

$$(A_1, a_1) \xrightarrow{p_1} (B_1, b_1) \xrightarrow{q_1} (C_1, c_1) \quad , \quad (A_2, a_2) \xrightarrow{p_2} (B_2, b_2) \xrightarrow{q_2} (C_2, c_2)$$

$$\begin{tikzcd}
(A_1, a_1) \otimes (A_2, a_2) \arrow[rr, "(q_1 \circ p_1) \otimes (q_2 \circ p_2)"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \xi"] & & (C_1, c_1) \otimes (C_2, c_2) \arrow[d, equal] \\
(A_1, a_1) \otimes (A_2, a_2) \arrow[r, "p_1 \otimes p_2"'] & (B_1, b_1) \otimes (B_2, b_2) \arrow[r, "q_1 \otimes q_2"'] & (C_1, c_1) \otimes (C_2, c_2)
\end{tikzcd}$$

Applications

Theorem The double categories

① $\mathbf{SMultProf}$ of symmetric multicategories, symmetric multifunctors, and symmetric multiprofunctors

② $\mathbf{OpdBim}$ of operads, operad morphisms, and operad bimodules

admit a symmetric oplax monoidal structure, given on objects by the Boardman-Vogt tensor product.

Remarks

- Part ② is also new.
- Method applies also to cartesian multicategories.
- $\underline{\underline{Q}}$: counterexample to normality of $\mathbb{B}\mathrm{im}(\mathbb{C})$

Reference :

N. Gambino, R. Garner, C. Vasilakopoulou, A unified treatment of commuting tensor products of categories, operads, sym. multicategories and their bimodules.

ArXiv, 2025.