Aspects of 2-categorical logic

Nicola Gambino Giacomo Tendas

The University of Manchester

Fiore Fest

University of Cambridge

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"Isoregular theories, accessible 2-categories, and free constructions"
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Context, motivation, and goals

Accessible 1-categories

- Categories of models of (infinitary) first-order theories

See Makkai-Paré [1989]

Accessible 2-categories

- Lack and Rosický [2012]
- Lack and Tendas [2022, 2023]

Open problem: Show that 2-categories of interest are accessible with some limits

- Makkai [1997]
- Bourke [2021]: pseudo-algebra morphisms are key

Main contribution: A new method, many applications.

Plan

1. Review of 1-categorical logic
2. Isoregular theories
3. Main results and applications

Regular theories

$$\top, \qquad s = t, \qquad A \wedge B, \qquad (\exists x\colon X)A(x).$$

Theorem. Let T be a regular theory.

(i) $\mathbf{Syn}(T)$ is a regular category

(ii) $\mathbf{Mod}(T) \cong \mathbf{Reg}[\mathbf{Syn}(T), \mathbf{Set}]$

(iii) $\mathbf{Mod}(T)$ is accessible with finite products.

Examples.

- Theory of Abelian groups
- Theory of divisible Abelian groups: $(\exists y)ny = x$
- Theory of categories

Finite limit theories

$$\top, \qquad s = t, \qquad A \wedge B, \qquad (\exists! x \colon X) A(x).$$

Theorem. Let T be a finite limit theory.

(i) $\mathbf{Syn}(T)$ is a finite limit category
(ii) $\mathbf{Mod}(T) \cong \mathbf{Lex}[\mathbf{Syn}(T), \mathbf{Set}]$
(iii) $\mathbf{Mod}(T)$ is finitely accessible with limits, i.e. locally finitely presentable.

Examples.

- ✓ Theory of Abelian groups
- ✗ Theory of divisible Abelian groups: $(\exists y) ny = x$
- ✓ Theory of categories

Isoregular 2-categories

Definition. A category $\mathcal{C}$ is **regular** if it has finite limits and

1. every kernel pair has a coequaliser
2. regular epimorphisms are stable under pullback.

Definition. A 2-category $\mathcal{K}$ is **isoregular** if it has finite 2-categorical limits and

1. every fully faithful kernel pair has a coequaliser
2. fully faithful regular epimorphisms are stable under pullback.

A kernel pair

$$\Delta_f \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} A \xrightarrow{f} B$$

is said to be **fully faithful** if π_1 (or π_2) is fully faithful.

Characterisation

Proposition. Let $f\colon A \to B$ be a morphism in $\mathcal{K}$. The following are equivalent:

(i) the morphism f is an FFK-morphism, i.e. it has a fully faithful kernel pair

(ii) for every $X \in \mathcal{K}$, $\mathcal{K}(X, f)\colon \mathcal{K}(X, A) \to \mathcal{K}(X, B)$ has a fully faithful kernel pair

(iii) the diagram
$$\begin{array}{ccc} A & = & A \\ \| & & \downarrow f \\ A & \xrightarrow[f]{} & B \end{array}$$
is a semistrict pullback

(iv) the morphism f is faithful and full on identities

(v) the unique $r\colon A \to \Delta_f$ such that $\pi_1 \circ r = 1_A$ and $\pi_2 \circ r = 1_A$ is an equivalence

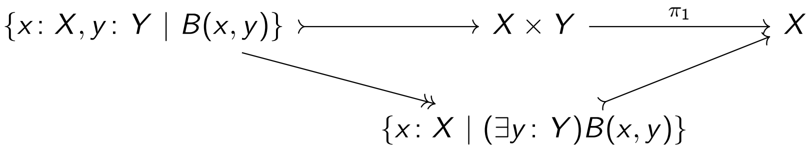
(vi) the morphism $f\colon A \to B$ is a subcontractible object of the slice 2-category $\mathcal{K}_{/B}$

Example. $\{\ (d, c, \lambda\colon \Delta c \Rightarrow d) \mid d \in C^I,\ c \in C,\ (c, \lambda) \text{ limit of } d\ \} \xrightarrow{\pi_1} C^I.$

Factorisation

Proposition. Let $\mathcal{K}$ be an isoregular 2-category. Every $\textsc{ffk}$-morphism factors as a fully faithful regular epimorphism followed by a monomorphism.

Idea: we can form



if the fibers $\{y\colon Y \mid B(x,y)\}$, for $x\colon X$, are subcontractible (cf. HoTT).

Example: $\{d \in C^I \mid \exists\,(c,\lambda) \text{ limit of } d \text{ in } C\} \rightarrowtail C^I$.

Isoregular theories

Enriched categorical logic for $\mathcal{V} = \mathbf{Cat}$. [Rosický–Tendas 2025 & 2026]

Arities = finitely presented categories.

Definition. A **finitary language** $\mathbb{L}$ is given by:

- basic sorts: $S, T, \ldots$
- function symbols: $f \colon S_1^{\mathbb{C}_1}, \ldots, S_n^{\mathbb{C}_n} \to S^{\mathbb{C}}$
- relation symbols: $R \rightarrowtail S_1^{\mathbb{C}_1}, \ldots, S_n^{\mathbb{C}_n}$

Example. The language $\mathbb{L}_{\mathbf{GFib}}$ for Grothendieck fibrations:

- two basic sorts: E and B
- one function symbols: $p \colon E \to B$
- one relation symbol: $\mathrm{Cart} \rightarrowtail E^{[1]}$

Isoregular theories

Forms of judgement: $s\colon X$, $A\colon \text{prop}$, $A_1, \ldots, A_n \vdash A$

Propositions: $\top$, $s = t$, $A \wedge B$, $(\exists x\colon X)A(x)$, $A^{[1]}$

Deduction rules: for example

$$\frac{(x\colon X, y\colon Y) \quad B(x,y)\colon \text{prop} \qquad \mathcal{J}_{\text{faithful}}(B) \qquad \mathcal{J}_{\text{full-id}}(B)}{(x\colon X) \quad (\exists y\colon Y)B(x,y)\colon \text{prop},}$$

$$\frac{A_1, \ldots, A_n \vdash A}{A_1^{[1]}, \ldots, A_n^{[1]} \vdash A^{[1]}}$$

Example: The theory $\mathbb{T}_{\mathbf{GFib}}$ has

$$(y\colon E, u\colon B^{[1]}) \quad \text{cod}(u) = p(y) \vdash (\exists v\colon E^{[1]})\ \text{Cart}(v) \wedge \text{cod}(v) = y \wedge p^{[1]}(v) = u$$

Main results

Theorem. Let T be an isoregular theory.

(i) $\mathbf{Syn}(T)$ is an isoregular 2-category

(ii) $\mathbf{Mod}(T) \cong \mathbf{IsoReg}[\mathbf{Syn}(T), \mathbf{Cat}]$

(iii) $\mathbf{Mod}(T)$ is accessible with flexible limits.

Proof.

(i) Use deduction rules for isoregular theories.

(ii) Direct calculation.

(iii) 2-dimensional category theory. □

Applications: general category theory

Theorem.

(i) The 2-categories **Lex**, **Rex**, **RLex** are accessible with flexible limits.

(ii) The forgetful 2-functors

$$\begin{array}{ccc} \mathbf{RLex} & \longrightarrow & \mathbf{Lex} \\ \downarrow & & \downarrow \\ \mathbf{Rex} & \longrightarrow & \mathbf{Cat} \end{array}$$

have left biadjoints (they are accessible and preserve flexible limits).

Proof of (i). Write $\mathbb{T}_{\mathbf{Lex}}, \ldots$.

- Finite case of Joyal's bicompletion

Applications: categorical logic

Theorem.

(i) The 2-categories **GFib**, **Cmp**, **Clan** are accessible with flexible limits.

(ii) The forgetful 2-functors

$$\mathbf{Cmp} \to \mathbf{GFib}, \qquad \mathbf{Clan} \to \mathbf{Cat}$$

have left biadjoints (they are accessible and preserve flexible limits).

Proof of (i). Write $\mathbb{T}_{\mathbf{GFib}}, \ldots$

Note. Many other applications:

- multicategories (**Mult**, **RepMult**, . . .).
- categorical algebra (**Reg**, **Ex**, **Mal**, **SmAb**, . . .)

Summary

- Isoregular 2-categories & isoregular theories
- 2-categories of models are accessible with flexible limits
- Applications to category theory, categorical logic, categorical algebra

N. Gambino and G. Tendas
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