Esercizi

①
$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"'] & \mathbb{D} \arrow[r, bend left, "G", ""{name=U, below}] \arrow[r, bend right, "G'"', ""{name=D, above}] & \mathbb{E} \arrow[from=U, to=D, phantom, "\Downarrow\psi"] \end{tikzcd} \quad \overset{?}{\rightsquigarrow} \quad \begin{tikzcd} \mathbb{C} \arrow[r, bend left, "GF", ""{name=U, below}] \arrow[r, bend right, "G'F"', ""{name=D, above}] & \mathbb{E} \arrow[from=U, to=D, phantom, "\Downarrow\psi F"] \end{tikzcd}$$

$$(\psi F)_X : GFX \longrightarrow G'FX \quad =_{def} \quad \psi_{FX} : GFX \longrightarrow G'FX$$

(ricordiamo : $\forall Y \in \mathbb{D}$ $\quad \psi_Y : GY \longrightarrow G'Y$)

Naturalità

$$\begin{array}{c} \mathbb{C} \\ \begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \end{array} \qquad \begin{array}{c} \mathbb{E} \\ \begin{tikzcd} GFX \arrow[r, "\psi_{FX}"] \arrow[d, "GFf"'] & G'FX \arrow[d, "G'Ff"] \\ GFY \arrow[r, "\psi_{FY}"'] & G'FY \end{tikzcd} \end{array}$$

✓

(2) $F : \mathbb{C} \longrightarrow \mathbb{D}$, $f : X \longrightarrow Y$ isomorfismo in $\mathbb{C}$

$\Longrightarrow$ $Ff : FX \longrightarrow FY$ isomorfismo.

<u>Dim</u> : $f : X \to Y$ iso $\Rightarrow$ $\exists g : Y \longrightarrow X$ t.c.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, bend right, "1_X"'] & Y \arrow[d, "g"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, bend right, "1_Y"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd} \quad \text{in } \mathbb{C}$$

$\Longrightarrow$ $Fg : FY \longrightarrow FX$ inversa di $Ff : FX \longrightarrow FY$

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[dr, bend right, "F1_X = 1_{FX}"'] & FY \arrow[d, "Fg"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} FY \arrow[r, "Fg"] \arrow[dr, bend right, "F1_Y = 1_{FY}"'] & X \arrow[d, "Ff"] \\ & FY \end{tikzcd}$$

③ $F : \mathbb{C} \longrightarrow \mathbb{D}$ equivalenza di categorie.

$\mathbb{C}$ ha un oggetto terminale $\Longrightarrow$ $\mathbb{D}$ ha un oggetto terminale.

④

$$\begin{tikzcd}
\underline{\mathrm{Grp}} \arrow[r, "U"] & \underline{\mathrm{Set}} \\
(G, \cdot, 1) \arrow[r, mapsto] \arrow[d, "f"] & G \arrow[d, "f"] \\
(H, \cdot, 1) \arrow[r, mapsto] & H
\end{tikzcd}$$

? pieno

? fedele

? essenzialmente suriettivo

? pieno ✗ $\quad \underline{\mathrm{Grp}}[(G,\cdot,1),(H,\cdot,1)] \longrightarrow \underline{\mathrm{Set}}[G,H]$

? fedele ✓

? ess suriettivo ✗

Esempio di equivalenza

$$\underline{\mathrm{Fin\,Set}}_{bij} = \begin{cases} \text{insiemi finiti} \\ \text{biezioni} \end{cases}$$

$$\mathbb{P} = \begin{cases} \text{numeri naturali} \\ \mathbb{P}[n,m] =_{def} \begin{cases} \emptyset & \text{se } n \neq m \\ \Sigma_n & \text{se } n = m \end{cases} \end{cases}$$

$$\begin{aligned} \mathbb{P} &\longrightarrow \underline{\mathrm{Fin\,Set}}_{bij} \\ n &\longmapsto \{1, \dots, n\} \end{aligned}$$

ess. suriettivo ✓

pienamente fedele

$\Longrightarrow$ $\underline{\mathrm{Fin\,Set}}_{bij}$ è "essenzialmente piccola".

Richiamo: date $\mathbb{C}, \mathbb{D}$ con $\mathbb{C}$ piccola

$$[\mathbb{C}, \mathbb{D}] = \left\{ \begin{array}{l} \text{funtori } \mathbb{C} \longrightarrow \mathbb{D} \\ \text{trasf. naturali} \end{array} \right.$$

<u>Prefasci</u> : data $\mathbb{C}$ piccola

$$\mathrm{Psh}(\mathbb{C}) = [\mathbb{C}^{op}, \underline{\mathrm{Set}}]$$

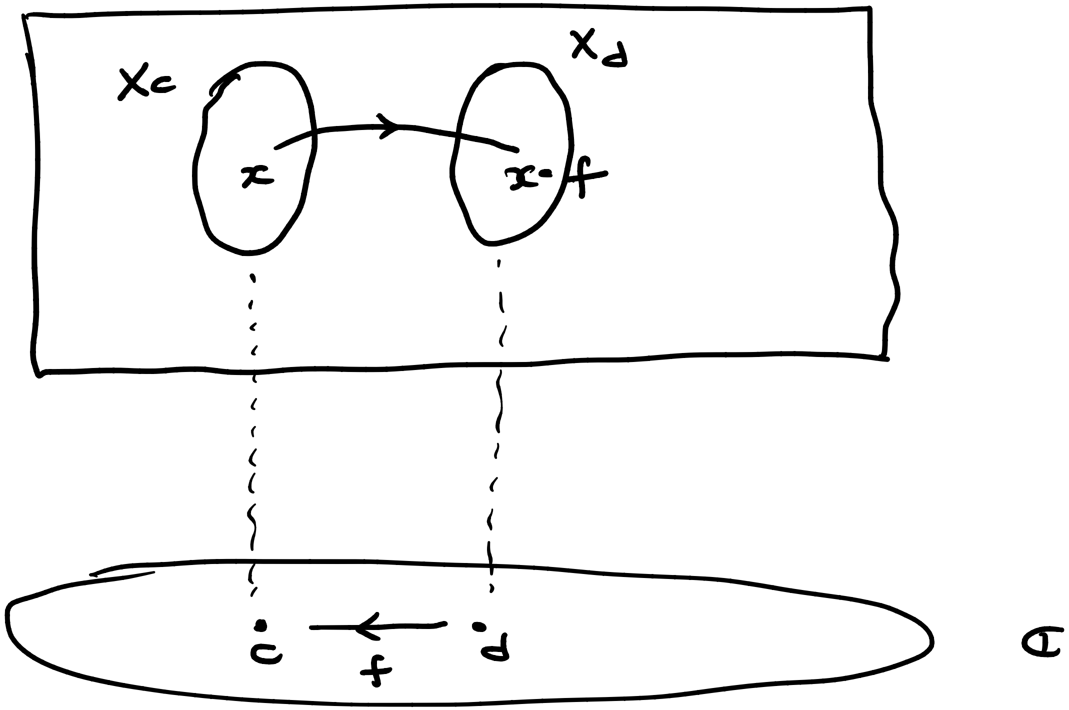
ove $\mathbb{C}^{op}$ e' l'opposta di $\mathbb{C}$

- $\mathrm{Ob}(\mathbb{C}^{op}) = \mathrm{Ob}(\mathbb{C})$
- $\mathbb{C}^{op}(X, Y) = \mathbb{C}(Y, X)$

$$\begin{array}{ccc} \mathbb{C} & \qquad & \mathbb{C}^{op} \\ X & & X \\ \uparrow f & & \downarrow f \\ Y & & Y \end{array}$$

Idee:

$$X : \mathbb{C}^{op} \longrightarrow \underline{\mathbf{Set}}$$



$$x \cdot 1_c = x$$

$$(x \cdot f) \cdot g = x \cdot (fg)$$

Esempio Dato $(X, \mathcal{O}(X))$ spazio topologico

$$\begin{tikzcd}
\mathcal{O}(X) & \mathcal{O}(X)^{op} \arrow[r] & \mathbf{Set} & \\
U \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] & U \arrow[r, mapsto] & C(U, \mathbb{R}) & f_{|U} \arrow[l, phantom, "\ni"] \\
V & V \arrow[u, dotted] \arrow[r, mapsto] & C(V, \mathbb{R}) \arrow[u] & f \arrow[l, phantom, "\ni"] \arrow[u, mapsto]
\end{tikzcd}$$

Esempio / Esercizio

Sia G un gruppo, lo si consideri come una categoria

$\underline{G}$.

Si descriva in maniera esplicita

$$[\underline{G}, \underline{Set}] \quad \begin{cases} \text{funtori} & \underline{G} \xrightarrow{X} \underline{Set} \\ \text{trasformazioni naturali} & \begin{tikzcd} \underline{G} \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\Downarrow f"] & \underline{Set} \end{tikzcd} \end{cases}$$

Nota : cambio di programma

24/8

- 11:15 – 12:30 A. Cantini
- 14:30 – 16:15 M. Viale

25/9

- Nessun cambiamento

26/9

- 9:00 – 10:45 N. Gambino
- 11:15 – ... N. Gambino

Logica costruttiva / λ-calcolo tipato

Fissiamo un insieme di formule atomiche $\mathcal{L}$

L'insieme delle formule $Frm_{\mathcal{L}}$ è definito induttivamente dalle seguenti clausole:

- Se $A \in \mathcal{L}$ allora $A \in Frm_{\mathcal{L}}$
- $\top \in Frm_{\mathcal{L}}$
- Se $A, B \in Frm_{\mathcal{L}}$ allora $A \wedge B$, $A \Rightarrow B$ sono in $Frm_{\mathcal{L}}$

Deduzione naturale : permette di costruire alberi di derivazione per sequenti della forma

$$A_1, \ldots, A_n \vdash A$$

ove $A_1, \ldots, A_n, A \in \mathrm{Form}_{\mathcal{L}}$.

Regole :

$$\frac{}{\Gamma, A \vdash A}\ \mathrm{id}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge\text{-intro} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge\text{-elim}_1 \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ \wedge\text{-elim}_2$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ \Rightarrow\text{-intro} \qquad \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ \Rightarrow\text{elim}$$

Regole ammissibili :

$$\frac{\Gamma \vdash A \qquad \Gamma, A \vdash B}{\Gamma \vdash B}\ \text{cut} \qquad\qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ \text{weak}$$

Esercizio

① $A \Rightarrow B,\ B \Rightarrow C \vdash A \Rightarrow C$

② $A \Rightarrow B_1,\ A \Rightarrow B_2 \vdash A \Rightarrow B_1 \wedge B_2$

③ $A \Rightarrow (B \Rightarrow C),\ B \vdash A \Rightarrow C$

$$\dfrac{\dfrac{\dfrac{}{A\Rightarrow B,\ B\Rightarrow C,\ A \vdash B\Rightarrow C}\,\mathrm{Id} \qquad \dfrac{\dfrac{}{\ldots \vdash A\Rightarrow B}\,\mathrm{Id} \qquad \dfrac{}{\ldots \vdash A}\,\mathrm{Id}}{A\Rightarrow B,\ B\Rightarrow C,\ A\vdash B}\Rightarrow_E}{A\Rightarrow B,\ B\Rightarrow C,\ A\vdash C}\Rightarrow_E}{A\Rightarrow B,\ B\Rightarrow C\vdash A\Rightarrow C}\Rightarrow_I$$

Uguaglianza / riduzione di prove

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi\\ \Gamma, A \vdash B\end{array}}{\Gamma \vdash A \Rightarrow B} \qquad \Gamma, A \vdash A}{\Gamma, A \vdash B} \quad \rightsquigarrow \quad \begin{array}{c}\vdots\,\pi\\ \Gamma, A \vdash B\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash A \wedge B\end{array}}{\Gamma \vdash A} \qquad \dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash A \wedge B\end{array}}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B} \quad \rightsquigarrow \quad \dfrac{\vdots}{\Gamma \vdash A \wedge B}$$

λ- calcolo tipato

Fissiamo un insieme $\mathcal{L}$, i cui elementi saranno chiamati tipi atomici

Definiamo induttivamente l'insieme $Ty_{\mathcal{L}}$ con le seguenti clausole

- $A \in \mathcal{L}$, allora $A \in Ty_{\mathcal{L}}$
- $1 \in Ty_{\mathcal{L}}$
- Se $A, B \in Ty_{\mathcal{L}}$ allora $A \times B$, $A \Rightarrow B \in Ty_{\mathcal{L}}$

sequenti hanno la forma

$$x_1 : A_1 , \ldots , x_n : A_n \vdash b : B$$

$$x_1 : A_1 , \ldots , x_n : A_n \vdash \underline{b_1 = b_2} : B$$

<u>Regole di deduzione</u>

$$\frac{}{\Gamma, x : A \vdash x : A}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathrm{pair}(a,b) : A \times B} \qquad \frac{\Gamma \vdash c : A \times B}{\Gamma \vdash \pi_1(c) : A} \qquad \frac{\Gamma \vdash c : A \times B}{\Gamma \vdash \pi_2(c) : B}$$

$$\frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash (\lambda x : A)\, b : A \Rightarrow B} \qquad \frac{\Gamma \vdash t : A \Rightarrow B \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathrm{app}(t, a) : B}$$

Note:

$$A \xrightarrow{f \,=\, (\lambda x : A) b} B$$

$$x \longmapsto b$$

in cui la x può comparire libera

$$\mathbb{R} \xrightarrow{(\lambda x : \mathbb{R}) x^2} \mathbb{R}$$

$$x \longmapsto x^2$$

Regole ammissibili

① $$\frac{\Gamma \vdash b : B}{\Gamma, x : A \vdash b : B}$$

② $$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A \vdash b : B}{\Gamma \vdash b[a/x] : B}$$

Isomorfismo di Curry-Howard

Logica	Teoria dei tipi
$A \wedge B$	$A \times B$
$A \Rightarrow B$	$A \Rightarrow B$
$\top$	1
$A_1, \ldots, A_n \vdash B$	$x_1 : A_1, \ldots, x_n : A_n \vdash b : B$
$\Gamma \vdash A$	$\Gamma \vdash a : A$

Equazioni tra termini

β-regole

$$\dfrac{\dfrac{\Gamma, x : A \vdash b : B}{\Gamma \vdash (\lambda x : A) b : A \Rightarrow B} \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathrm{app}\big((\lambda x : A) b,\ a\big) = b[a/x] : B}$$

$$\dfrac{\dfrac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathrm{pair}(a, b) : A \times B}}{\Gamma \vdash \pi_1\big(\mathrm{pair}(a, b)\big) = a : A} \qquad \dfrac{\dfrac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathrm{pair}(a, b) : A \times B}}{\Gamma \vdash \pi_2\big(\mathrm{pair}(a, b)\big) = b : B}$$

η- regole

$$\frac{\Gamma \vdash t : A \Rightarrow B}{\Gamma \vdash t = (\lambda x : A)\, app(t, x) : A \Rightarrow B}$$

$$\frac{\Gamma \vdash c : A \times B}{\Gamma \vdash c = pair(\pi_1 c, \pi_2 c) : A \times B}$$

Esercizio

① $u : A \Rightarrow B, \quad v : B \Rightarrow C \vdash ? : A \Rightarrow C$

② $u_1 : A \Rightarrow B_1, \quad u_2 : A \Rightarrow B_2 \vdash ? : A \Rightarrow B_1 \wedge B_2$

③ $u : A \Rightarrow (B \Rightarrow C), \quad y : B \vdash ? : A \Rightarrow C$