An adaptive unstructured solver for shallow granular flows

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Abstract: Free surface flows of granular material show features well known from shallow water flow or, more generally, from gas dynamics: traveling or steady state jumps in material depth, depth diminishing waves and formation of zero depth regions.

An adaptive Euler code using unstructured grids has been used to simulate supercritical shallow granular flows.

The case of supercritical flow past a wedge, in which oblique shocks have been observed in granular materials, was selected to test the computations.

The numerical method was verified against the analytical solution of the supercritical shallow water wedge flow case, and is compared to experimental thickness data for shallow granular flow. An excellent agreement with the analytical solution indicates the validity of the method, while a good correlation with the experiments shows correctness of the model.

Key words: Adaptive finite volume method, Unstructured grids, Riemann-Solver, Shallow granular flow, Depth-Averaged equations, Granular shocks

1. Introduction

Free surface shallow granular flows occur widely in nature and numerous industrial applications. When the depth of granular material flowing is small compared to its length some flow features are observed, which have a close analogy to the shallow water flow, and, more generally, to gas dynamics. These features include traveling or steady-state jumps in the depth (analogous to hydraulic jumps or shock waves), depth diminishing waves (expansion waves), and formation of zero depth regions (vacuum), Gray and Tai (1998). A depth-averaged approach Savage and Hutter (1989) similar to that employed in shallow water theory appears to be promising for a numerical model to simulate shallow granular flows.

An adaptive unstructured Euler code (Voinovich 1993) was therefore used in our present work. The above similarities in the flow patterns make the techniques employed in high-resolution shock capturing, in this solver, attractive for the simulation of shallow granular flow.

2. Governing equations

Gray et al. (1999) derived a continuum theory for the gravity driven free surface flow of granular material. Applying the depth integration procedure used in the shallow water theory to the mass and momentum conservation laws written for free surface flow of granular material the two-dimensional governing equations are obtained involving the flow depth h and two cartesian velocity components u and v as dependent variables.

The granular material slides down a slope of constant inclination angle ζ. Without loss of generality this inclination is assumed to lie in x-direction. Parameters of the granular material motion are the basal friction angle δ and the internal friction angle ϕ.

Dimensionless variables are defined by a reference height h_0 and a reference wave speed $c = \sqrt{gh_0}$, sub- and supercritical flow regimes are determined by the Froude number $\mathrm{Fr} = u/c$.

The set of conservation laws containing the continuity and momentum equations are:

$$
\begin{aligned}
\int_V \frac{\partial h}{\partial t}\,\mathrm{d}V &= -\int_S h\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S \\
\int_V \frac{\partial hu}{\partial t}\,\mathrm{d}V &= -\int_S hu\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S - \int_S \cos\zeta\,K_x\,\frac{h^2}{2}\,n_x\,\mathrm{d}S \\
&\quad - \int_V h\frac{u}{|\vec{u}|}\tan\delta\,\cos\zeta\,\mathrm{d}V + \int_V h\sin\zeta\,\mathrm{d}V \\
\int_V \frac{\partial hv}{\partial t}\,\mathrm{d}V &= -\int_S hv\,(\vec{u}\cdot\vec{n})\,\mathrm{d}S - \int_S \cos\zeta\,K_y\,\frac{h^2}{2}\,n_y\,\mathrm{d}S \\
&\quad - \int_V h\frac{v}{|\vec{u}|}\tan\delta\,\cos\zeta\,\mathrm{d}V
\end{aligned}
\tag{1}
$$

This system of equations is of hyperbolic type when $\phi = \delta$ and it is nonlinear in the stress and of parabolic type otherwise.

The system in integral form is well suited for a finite volume computational method. The left hand side of equations 1 stands for the time rate of change of a property in the volume, the first integral on the right hand side gives the flux across the volume boundary. The momentum equations contain additional integrals of the pressure, the basal friction acting against the flow direction and the driving gravity. The earth pressure coefficients K_x and K_y are defined as the ratio of the basal downslope and cross slope pressures to the normal basal pressure, see Gray et al. (1999), and are determined in a nonlinear fashion by the granular material constitutive law.

Despite the source terms (volume integrals) on the right hand side the system shows a strong analogy to the shallow water equations.

3. Numerical method

The main points of the resulting finite-volume method are as follows.

The computational domain is triangulated using an unstructured grid generator, the one by Galyukov and Voinovich (1993) has been applied in this work. The control volumes constructed about the grid nodes (triangles' vertices) are bounded by median segments.

A linear distribution of the dependent variables is assumed within triangles and control volumes result in second-order spatial accuracy of the method except in vicinity of discontinuous solutions. The gravitational acceleration and the basal friction are treated as source terms and do not contribute to the fluxes. Flow variables on the cell faces are evaluated by an exact Riemann solver (Toro 1990) for the shallow water equations. The Riemann solver processes the values of dependent variables extrapolated from a pair of adjacent grid nodes to the common boundary of the respective control volumes. The output values of the flow depth and velocity and the earth pressure coefficients evaluated by the granular material constitutive law on the control volume boundary are used to compute fluxes through the control volume boundary.

To prevent spurious oscillations in the solution resulting from higher order approximation close to a singularity, a TVD limiter is applied when computing averaged gradients in control volumes. A predictor-corrector technique is used to enhance the accuracy of the scheme to second order in time.

A fast non-conservative central-difference scheme provides a preliminary evaluation of the dependent variables at a new time step, which are then used to compute only the fluxes, so that the compound scheme remains conservative.

The unstructured grids provide a tool for handling computational domains of complex geometry. Moreover, they allow local modifications in the grid structure. The classical reversible refinement/derefinement technique is incorporated in our numerical model allowing a tremendously sharp resolution at the depth jumps and other singularities in the solution. The grid is refined dynamically at each time step by subdivision of a grid triangle either into 2 or 4 smaller ones, giving rise to one or three new grid nodes at the midpoints of triangle's sides. The new triangles produced by cutting one into four can be subdivided further at a next time step, thus assuring the desired resolution. The refinement procedure is reversible; the smaller triangles can confluence restoring their common source triangle. The grid adaptation is controlled by a sensor involving first and second derivatives of the flow depth along the triangles sides. The effect of the local grid adaptation is a sharp resolution of the depth jumps on a grid with limited amount of total grid nodes which essentially contributes to the high efficiency of the method. Figure 4 shows a dynamically refined grid.

4. Results and discussion

4.1. Wedge flow problem

The wedge flow problem, well known from gas dynamics and shallow water theory, consists of a supercritical flow over a wedge of a given angle forming an inclined jump in height and velocity of the flow.

4.1.1. Shallow water model

For the shallow water case the analytical solution of the supercritical wedge flow problem is well known and plotted in figure 8. The shallow water model is achieved by neglecting the basal friction term ($\delta = 0$) and the driving term ($\zeta = 0$) and using $K_x = K_y = 1$ in the pressure term of equations 1. The inlet flow conditions and values of the corresponding analytical solution are given in table 1.

Wedge angle	φ	3°
Inclination angle	ζ	0°
Inlet height	h_1	1.0
Inlet Froude number	Fr_1	1.9
Jump angle	θ	35°
Height behind jump	h_2	1.12
Froude number behind jump	Fr_2	1.84

Table 1. Inlet flow conditions and analytical solution: supercritical wedge flow, shallow water flow.

The geometry (see figure 1) was chosen such that the inclined jump meets the upper right corner of the computational domain, hence the computed jump angle can be compared directly (see figure 2).

The initial grid contained 1007 nodes, ten thousand time steps were performed and after adaptive refinement the number of nodes reached 9260. The computed jump angle of 35° fits the analytic one exactly. A comparison between the analytical results for the flow height h and the Froude number Fr with the computational results shows that there is very good agreement.

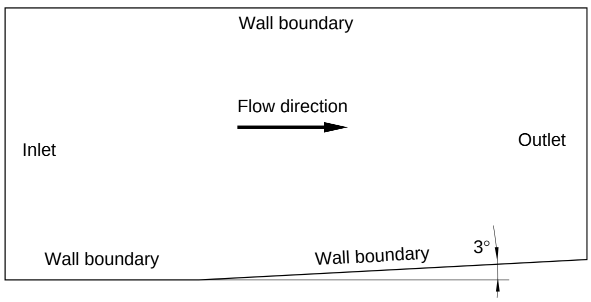

Figure 1. Computational domain of the wedge flow problem for the shallow water case.

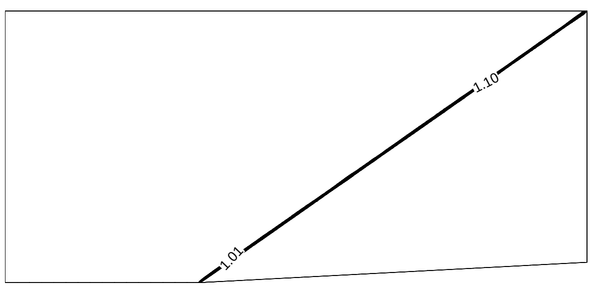

Figure 2. Computed height h of the supercritical wedge flow problem for the shallow water case.

Basal friction angle	δ	31.0°
Internal friction angle	ϕ	38.0°
Wedge angle	φ	24.9°
Inclination angle	ζ	32.5°
Inlet height	h_1	1.0
Inlet Froude number	Fr_1	7.0

Table 2. Properties and inlet conditions of the granular supercritical wedge flow case.

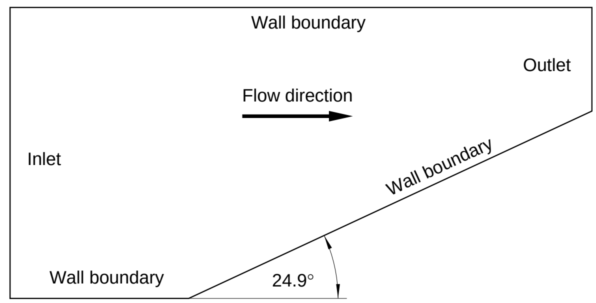

Figure 3. Geometry of the wedge flow problem for granular material: computational domain and experimental setup.

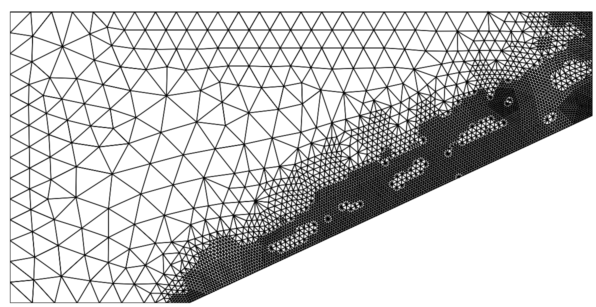

Figure 4. Adaptively refined grid after 10 000 time steps, containing 5150 nodes.

4.1.2. Shallow granular flow

The wedge flow problem was transfered to the shallow granular flow case. Unfortunately the analytical solution for the shallow granular supercritical wedge flow problem is unknown, but experimental results are available.

The flow region has an inclination angle ζ of 32.5°, the computational domain is shown in figure 3 and the properties of the granular material, the geometrical properties and the inlet conditions are shown in table 2. The flow direction is from left to right.

Figure 4 shows the adaptive refined grid containing 5150 nodes after 10 000 computed time steps. Computed height h and Froude number for the granular supercritical wedge flow is shown in figures 5 and 6.

The computation captured the curvature of the granular shock very well indeed. Granular shocks are less sharp than the jumps in the shallow water case and the well known shocks of gas dynamics.

For comparison an experimental result is included, see figure 7. Assuming an inlet height $h_0 = 7$ mm, the relative height h/h_0 in the granular shock region is about 4.6 compared to $h = 5.8$ in our computational result. This quantitative difference is due to the internal friction angle φ, the basal friction angle δ and the inlet Froude number, which are not known exactly.

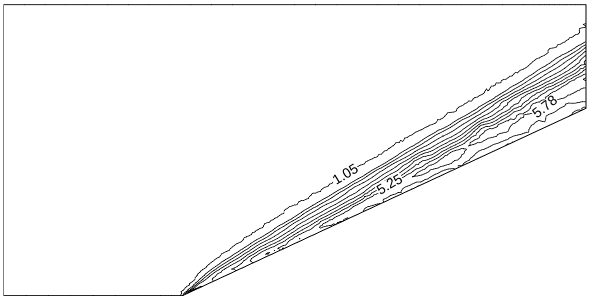

Figure 5. Computed solution of the supercritical wedge flow problem: height h.

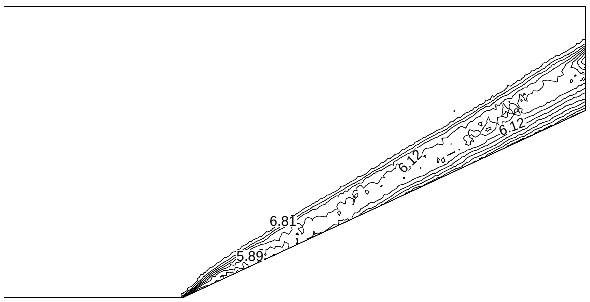

Figure 6. Computed solution of the supercritical wedge flow problem: Froude number Fr.

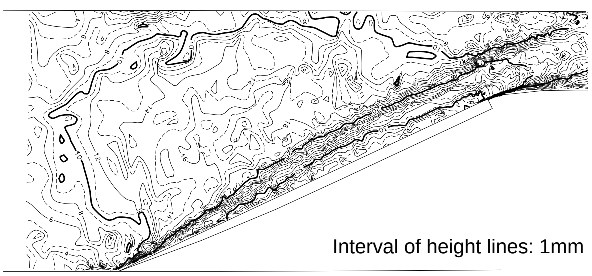

Figure 7. Measured height (in mm) of granular material flowing down a chute.

5. Conclusions

An adaptive Euler code using unstructured grids was used to compute shallow granular flows. Oblique granular shocks and oblique hydraulic jumps in the supercritical granular and shallow water flow past a wedge were computed. The numerical method, wich uses techniques known from high-resolution shock capturing, was successfully applied in simulating shallow granular flows.

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References

Galyukov, A., Voinovich, P. (1993) Two-dimensional unstructured triangular grid generator. Advanced Technology Center, St. Petersburg, Russia (unpublished)

Gray J. M. N. T., Tai Y.-C. (1998) Particle Size Segregation, Granular Shocks and Stratification Patterns. NATO ASI Series, H.J. Herrmann et al. (eds.), Physics of Dry Granular Media, pp. 697–702. Kluwer Academic.

Gray, J. M. N. T., Wieland, M., Hutter, K. (in press) Free surface flow of cohesionless granular avalanches over complex basal topography. Proc. Roy. Soc. Series A, London

Savage, S. G., Hutter, K. (1989) The motion of a finite mass of granular material down a rough incline. J. Fluid Mech., 199.

Toro, E. F. (1990) Riemann problems and the WAF-Method for the Two-Dimensional Shallow Water equations. Cranfield College of Aeronautics Report No. 9005

Voinovich, P. (1993) Two-dimensional locally adaptive unstructured unsteady Euler code. Advanced Technology Center, St. Petersburg, Russia (unpublished)

Appendix

A Analytical solution of the shallow water wedge flow problem

The analytical solution for the computed case $\varphi = 3°$ and $\theta = 35°$ is marked by the thick line in figure 8. Values of the jump in height (h_2/h_1) and Froude number ($\mathrm{Fr}_2/\mathrm{Fr}_1$) for this solution are given in table 1.

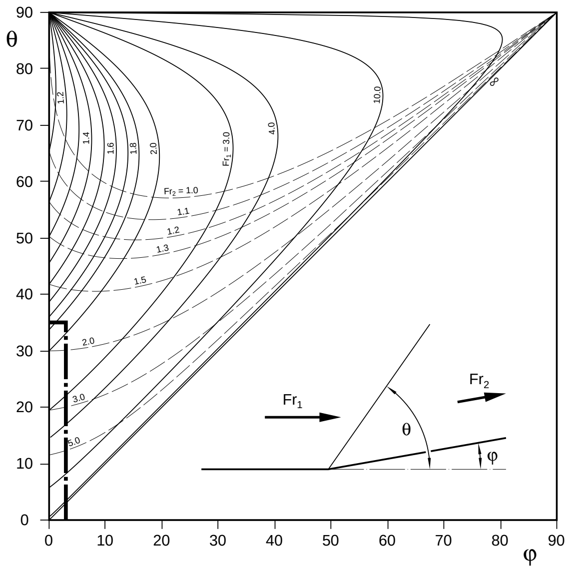

Figure 8. Analytical solution of the supercritical wedge flow problem for shallow water jumps.