Plane and oblique shocks in shallow granular flows

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Abstract: Although the Savage-Hutter theory for shallow granular flows has certain similarities with the shallow water equations of hydrodynamics, shock waves have not been observed until relatively recently. In this paper laboratory experiments are presented, in which plane travelling shock waves and stationary oblique shocks develop. The experiments also show the formation of expansion waves and granular vacua. A simple travelling wave solution to the Savage-Hutter equations is derived for flow on non-accelerative slopes and it is used to test a series of simple one-dimensional shock capturing numerical methods. Finally it is shown that plane shocks play an important role in the formation of stratification patterns in heaps and partially filled slowly rotating drums.

Key words: granular flow, avalanches, plane shocks, oblique shocks, pattern formation

1. Introduction

Rapid shallow granular free surface flows occur when a surface layer of static granular material becomes unstable or when granular material is released by some mechanism onto an inclined surface. They are abundant all around us from salt in a salt cellar and flow at the free surface of stockpiles, to landslides, rock-falls and snow slab avalanches. Although the length scales vary dramatically the dynamics of the avalanches are similar.

Savage & Hutter (1989, 1991) exploited the shallowness of these flows to derive a one-dimensional depth averaged theory for the flow of an incompressible Mohr-Coulomb granular material sliding down a rigid surface. Using a slope fitted curvilinear coordinate system Oxz, with the z axis normal to the slope and the x-axis parallel to it, the depth integrated mass and momentum balances reduce to

$$\partial_t h + \partial_x(hu) = 0, \qquad (1)$$

$$\partial_t(hu) + \partial_x(hu^2 + \cos\zeta K h^2/2) = hd, \qquad (2)$$

where h is the avalanche depth, u is the depth averaged downslope velocity and, ∂_t and ∂_x denote differentiation with respect to time and the downslope coordinate, respectively. The net driving force is the difference between the gravitational acceleration and the Coulomb rate-independent dry friction sliding law

$$d = \sin\zeta - (u/|u|)\tan\delta(cos\zeta + \kappa u^2), \qquad (3)$$

where $\zeta(x)$ is the slope inclination angle, $\kappa = -\partial_x\zeta$ is the slope curvature and δ is the basal friction angle. The system is similar to the shallow water equations of fluid dynamics, however, the earth pressure coefficient K, which determines the ratio of in-plane to normal pressure, is a non-linear function of $\partial_x u$ rather than a constant. The earth pressure coefficient approaches two limiting values, *active* and *passive*, dependent on whether the downslope motion is divergent or convergent. These are given by

$$K_{\mathrm{act/pas}} = 2\sec^2\phi\left(1 \mp \sqrt{1-\cos^2\phi\sec^2\delta}\right) - 1, \qquad (4)$$

where ϕ is the internal angle of friction in the Mohr-Coulomb constitutive relation. The transition between the two states can be achieved either by a jump (Savage & Hutter 1989) or a smooth transition (Tai & Gray 1998). When the internal and basal friction angles are equal, then $K_{\mathrm{act}} = K_{\mathrm{pas}}$, and the equations reduce to a non-strictly hyperbolic system with characteristic wavespeeds $\lambda = u \pm (\cos\zeta K h)^{1/2}$. In this situation the *granular Froude number*, $\mathrm{F_r} = |u|/(\cos\zeta K h)^{1/2}$, determines whether the flow is sub- or super- critical. A two-dimensional extension of the theory for modelling granular flow over complex three-dimensional topography has been derived by Gray et al (1999).

Despite the similarity with the shallow water equations shock waves have not been observed till recently. Gray & Hutter (1997, 1998) observed diffuse plane shocks on weakly accelerative slopes, and these play an important role in the process of stratification pattern formation. In addition, grain free regions are often generated during experimental flows. These regions are similar to vacua in gas dynamics and we now term such regions *granular vacua*.

2. Plane shocks in debris flows

Shock waves are also observed in water saturated granular flows. These debris flows are initiated by slowly saturating the granular material and, once saturated, rapidly increasing the flow rate, for a short time, until the whole body starts to move. Once mobile the debris

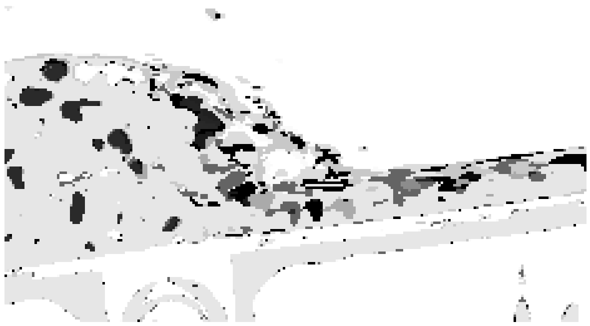

Figure 1. A CCD image of a travelling shock wave in a debris flow. A supercritical saturated mixture enters from the right and is brought to rest as it passes through the shock wave, which propagates upslope from left to right.

flow accelerates downslope and is brought rapidly to rest when the inclination angle of the slope decreases below the angle of basal friction. If the flow is supercritical a shock wave can form, which travels upslope. Figure 1 shows a CCD image of such a travelling shock wave. The material moves from right to left. The shock position is marked by a rapid change in the debris flow thickness, which propagates in the opposite direction to the flow.

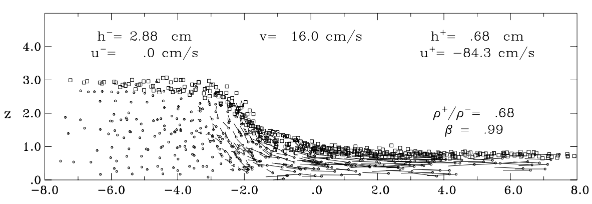

Figure 2. The debris flow thickness and velocity vectors determined from 16 CCD images are translated and superimposed upon one another to build up a detailed image of the travelling wave. The free surface of the granular material is indicated by a square at each data point. The initial position of each particle is indicated by a circle and its position $1/100^{\text{th}}$ of a second later is indicated by a line segment. All distances are measured in cm and velocities in cm/s.

Figure 2 shows data obtained from 16 CCD images that have been translated and superimposed upon one another under the assumption that the shock travels at a constant speed $v = 16$ cm/s upslope. The incoming debris is 0.68 cm thick and has a velocity of 84.3 cm/s. After the material passes through the shock it is over 4 times thicker and $u = 0$. Discontinuous solutions to the Savage-Hutter model exist, and provide a simple explanation of the jumps. Consider the case when $\phi = \delta$ and $d = 0$ in the governing equations 1–4. A simple solution of the resulting equations is that the avalanche velocity and thickness are constant, i.e. $u = u_0$ and $h = h_0$. Different constant states may be joined together by the mass and momentum jump conditions

$$[\![h(u-v)]\!] = 0, \quad (5)$$

$$[\![hu(u-v)]\!] = -[\![\cos\zeta K h^2/2]\!], \quad (6)$$

where $[\![\,]\!]$ are the jump brackets and v is the normal speed of the non-material singular surface. With these conditions it is a simple exercise to construct a travelling wave solution with $u = 0$ behind the shock.

3. Shock capturing numerical methods

A moving grid (Lagrangian) finite difference scheme was developed to solve a reduced non-conservative form of the equations by Savage & Hutter (1989). This method has proved to be a particularly effective numerical integration scheme for smooth solutions (Hutter & Koch 1991, Hutter & Greve 1993, Greve et al 1994, Koch et al 1994, Wieland et al 1999). However, when there are large gradients, or non-smooth solutions, numerical oscillations develop which degrade the quality of the solution and cause numerical instabilities. The top panel of figure 3 shows that the Lagrangian method is not able to reproduce the travelling shock wave solution.

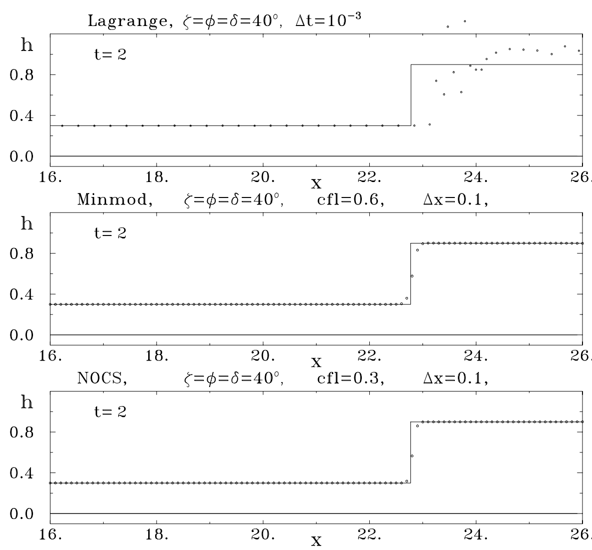

Figure 3. Comparison of analytic plane shock wave solution (solid line) for the thickness, h, with the computed thickness distribution (circles at each grid node) as a function of space x at $t = 2$ non-dimensional units. The granular material flows from left to right and increases rapidly in thickness at the shock. The shock wave propagates upslope (from right to left) and the material on the downslope side is at rest. The results are illustrated for the original Lagrangian method (top) the modified TVDLF method (middle) and the NOC scheme (bottom).

Two one-dimensional methods have been applied to numerically capture plane shock waves. The first is a modified Total Variational Diminishing Lax Friedrichs (TVDLF) method (Yee 1989, Tóth 1996) and the second is a Non-Oscillating Central (NOC) Scheme (Nessyahu & Tadmor 1990, Jiang & Tadmor 1997). Both of these methods use Eulerian grids and do not rely on a known solution to the Riemann problem. A comparison between the exact plane shock wave solution on non-accelerative slopes and the TVDLF and NOC schemes is illustrated in figure 3, and demonstrates that these high resolution methods are very effective at capturing the travelling wave.

4. Oblique shocks and granular vacua

Steady oblique shocks can also develop in shallow granular avalanches. Figure 4 shows the measured avalanche velocity and thickness in a laboratory experiment on an inclined plane. These fields are determined from particle imaging velocimetry techniques and stereo-photogrammetric, respectively.

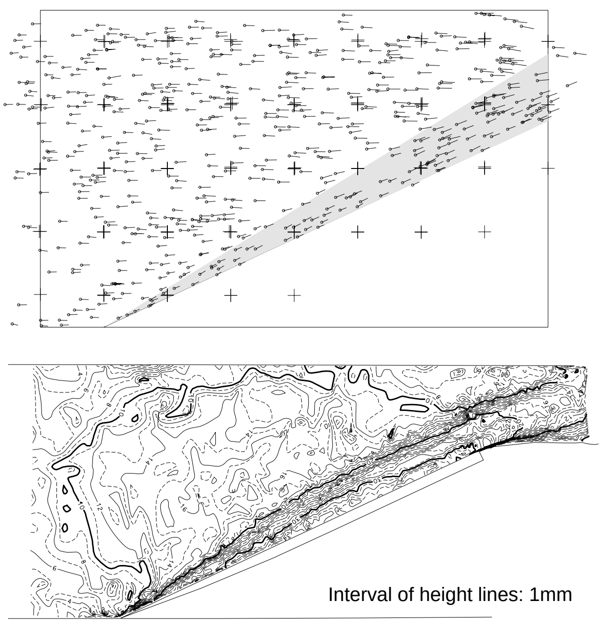

Figure 4. Experimental thickness contours (bottom) and free surface velocity vectors (top) for steady supercritical flow past a wedge. The granular material moves from left to right and an oblique shock is shed from the tip of the wedge. Behind the shock (in the grey region of the right image) the avalanche is thicker and the velocity vectors have decreased in magnitude and rotated so that they lie parallel to the wedge.

Prior to the shock the granular material flows downslope from left to right at approximately constant speed and with approximately constant thickness. An oblique shock is generated from the tip of the wedge. After the shock the avalanche is about twice as thick as the incoming flow, the velocity magnitude decreases and the direction of flow lies parallel to the wedge. This is very similar to oblique shocks generated in shallow hydrodynamic flows. Irmer et al (this issue) have used a two-dimensional extension of the Savage-Hutter theory (Gray et al 1999) to simulate the formation of an oblique shock at the wedge. Their results are in very good agreement with the experiments and show that the source terms, and the earth pressure coefficients modify the flow from that of the classical shallow water case.

Figure 5. An oblique shock is also shed on either side of an avalanche defence structure (grey), which can be clearly seen from the streamlines. The avalanche flows from the top to the bottom of the image. On the lee side of the defence a granular vacuum is formed (white) which protects the structure below (black) from contact with the avalanche

In the experiments the wedge is of finite length and an expansion wave forms immediately after it. This can be seen in the top right-hand corner of the thickness contour plot in figure 4. The avalanche rapidly spreads in the fan and a granular vacuum forms in the lee of the wedge. Granular vacua often form in the lee of solid objects, and these grain free regions close only slowly in the wake as the pressure in the expansion waves is not strong. Such structures can be used as snow slab avalanche defences. Figure 5 shows laboratory experiments for a pyramid, that is designed to divert an avalanche around a building placed in its grain free wake. The use of wedges as avalanche defences is a significant step towards better avalanche protection.

5. Stratification patterns in heaps

When a granular material with grains of two differing sizes is poured into the vertical gap between thin plates from a point source a triangular heap is built up. Although the material is continuously deposited at the top of the pile, it does not flow immediately down the faces because of the difference between the static and dynamic internal friction angles. Once the static friction angle is exceeded the avalanche flows down the face of the pile and *kinetic sieving* takes place (Savage & Lun 1988, Savage 1993), which sorts the granular material by grain size. Kinetic sieving is the name given to the process whereby gaps open up between the grains, as they are sheared, and the smaller particles are more likely to fall into open space beneath them, than the large particles, because they are more likely to fit into the available space. An *inverse-grading* of the particles rapidly develops with the larger particles overlying the smaller ones. In a bi-disperse granular mixture this creates a two-layer stripe close to the free surface. When the front of the avalanche reaches the run-out zone a plane shock wave is generated, similar to those in the debris flow experiments, which propagates upslope and *freezes* the stripe into the granular deposit as shown in figure 6. Successive releases create a large scale stratification pattern.

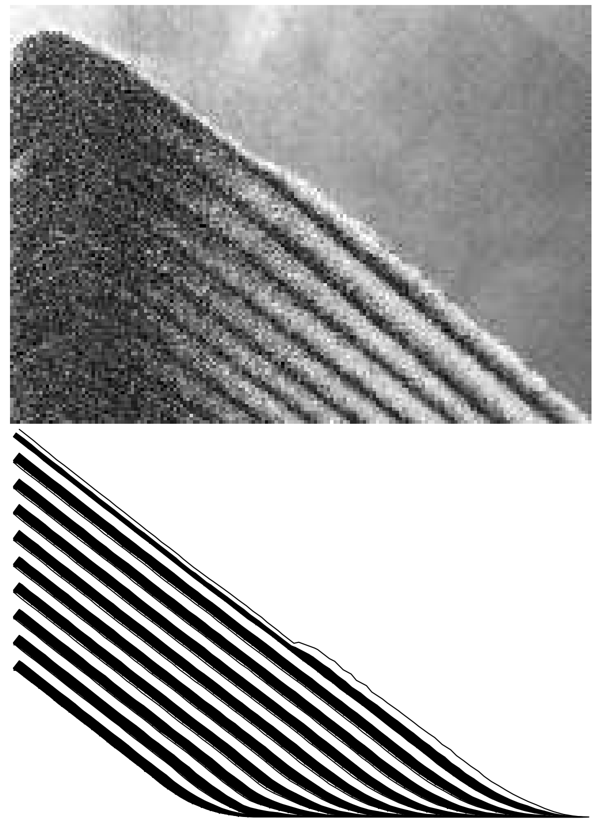

Figure 6. Stratification pattern formation in a bi-disperse grain size population of large (white) particles and small (grey) particles. The uppermost pair of dark and white layers prior to the upslope propagating shock are avalanching downslope. Behind the shock, which can be seen as a rapid increase in the avalanche thickness in the experiments (top) and the simulation (bottom), the material is at rest. All the material below the top two layers is also at rest.

A simple mixture theory has been developed to model stratification pattern formation. The volume fraction of small particles per unit mixture volume, $0 \le \nu \le 1$, is defined and it is assumed that the remaining space $1 - \nu$ is occupied by large particles. Assuming that the volume fraction of small particles has no influence on the dynamics of the bulk mixture the Savage-Hutter theory can be used to compute the avalanche thickness and the downslope velocity. The normal velocity can then be inferred from the incompressibility relation together with the basal boundary condition. The experiments suggest that the time-scale for kinetic sieving to take place is much faster than the time-scale for the avalanche motion. It is therefore assumed that the grains are initially presorted when they are input into the flow and a tracer equation is used to follow the local particle size distribution. The results of numerical simulations are shown in figure 6. The avalanche propagates downslope transporting the initial particle size distribution with it. When it reaches the run-out plane the gravity forcing ceases and a granular shock develops, which propagates upslope with approximately constant speed. Successive layers are built up by defining the old free surface of the avalanche to be the new basal topography for the next avalanche to run down. In this way large scale stratification patterns are built up.

6. Stratification and segregation patterns in thin rotating drums

In rotating drums the granular flow is divided into two distinct regions that are separated by a non-material singular surface, at which there are discontinuities (jumps) in the velocity and density fields. Close to the inclined free surface there is a thin fluid-like avalanche, and below it there is a solid layer that rotates about the axis of revolution. Strong mass transfer takes place between the fluid and solid regions. The material that is transported by the avalanche to the base of the inclined slope, is then absorbed by the solid body and rotated back up to the top of the slope, where it replenishes the avalanche.

A series of particle size segregation experiments have also been performed in thin rotating drums. To emphasise the patterns the disk was laid horizontally and gently shaken so that all the small particles fell to the bottom. Once returned to the vertical one side of the disk is filled with large (white) particles and the other side is filled with small (grey) ones. When the disk is rotated at a constant rate (110 seconds per revolution) intermittent avalanches are formed at the free surface. The intermittency again stems from the difference between static and dynamic internal friction angles. The central circular core of material remains completely undisturbed by slow rotation of the drum. Each avalanche release sorts the material, forming a stripe, which is frozen into the deposit by the shock wave and subsequently rotated and buried in the undisturbed material below the free surface. Subsequent releases create a sequence of stripes tangent to the central core, which create a *Catherine wheel* effect as shown in figure 7.

At faster rotation rates ($<$ 20 seconds per revolution) the intermittency of the avalanche release, the shock waves and the stripes disappear and a steady flow regime dominates. The material is continuously released on the upper side and continuously deposited on the lower side of the concave free surface and is transported between the two positions by a quasi-steady avalanche in which kinetic sieving takes place. Since the smaller particles are concentrated at the bottom of the avalanche they are the first to get deposited on the lower half of the free surface and a new pattern develops in which the central core is undisturbed, and there is a continuous distribution of grain sizes outside the central core, starting with a high concentration of small particles near the core and ending with a high concentration of large particles near the outer wall.

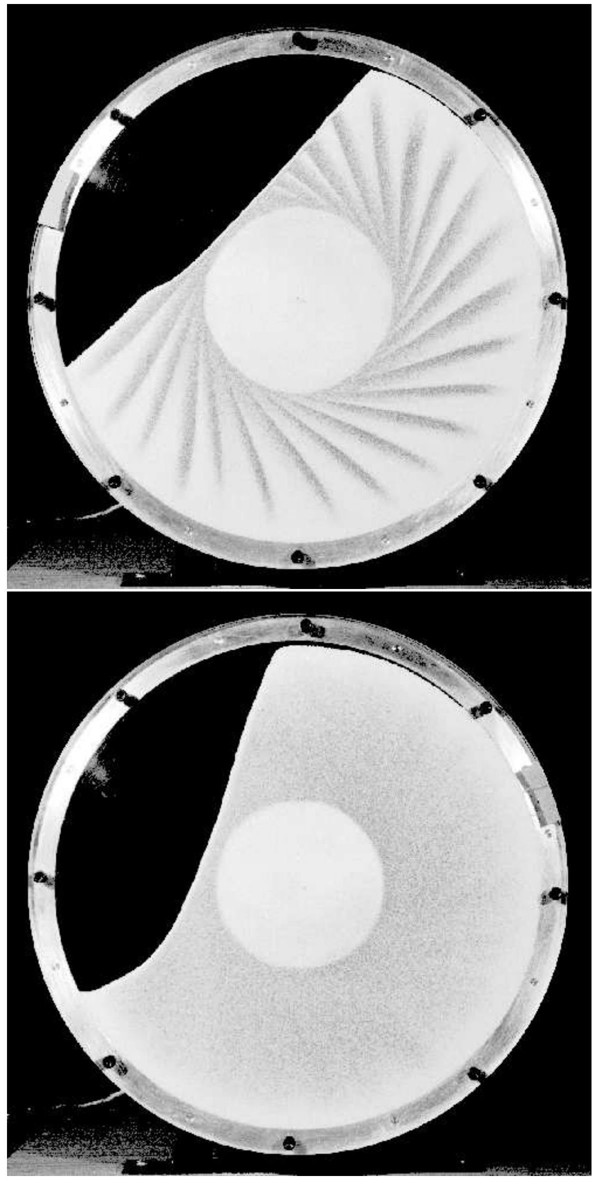

Figure 7. Pattern formation due to particle size segregation in a thin rotating disk at low (left) and fast (right) revolution rates. The large particles are white and the small particles are grey. The drum is rotated slowly counter clockwise and contains a bi-disperse mixture of grain sizes.

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