

# Cubic flux models for size segregation in dense granular free surface flows

Parmesh Gajjar, Nico Gray

School of Mathematics and  
Manchester Centre for Nonlinear Dynamics,  
University of Manchester, UK

SIZE segregation is an almost ubiquitous feature of dense granular free surface flows. A bi-disperse mixture consisting of large and small sized particles of the same density will quickly separate into two distinct normally graded layers, with the large particles lying above the small particles. *Kinetic sieving* and *squeeze expulsion* combine to give a net effect with small particles percolating downwards, and large particles rising upwards.

A number of authors have developed continuum models for this segregation process, most notably Savage & Lun (1988) and Gray & Thornton (2005). These existing models all share a hyperbolic structure with a quadratic  $\phi(1-\phi)$  segregation flux, where  $\phi$  is the small particle concentration. However, although Wiederseiner *et al.* (2011) found a reasonable match between their experimental flume results and the quadratic segregation flux model of Gray & Thornton (2005), they also suggested that the flux-function may take a more complicated form. In this poster we present analysis of a **cubic segregation flux**, which demonstrates higher-order effects not captured by the quadratic flux models.

Intuitively, **segregation rates would be higher when there is a single small particle in a matrix of large particles, than when there is a single large particle in a region of smalls**, as shown in figure 1. A similar phenomena was observed by Golick & Daniels (2009) in their annular shear cell experiments. This motivates the use of an asymmetric cubic flux function that is skewed towards  $\phi = 0$ ,

$$F(\phi) = A\phi(1-\phi)(1-\gamma\phi) \quad 0 \leq \phi \leq 1, \quad (1)$$

with parameter  $\gamma$  controlling the amount of skewness, and  $A$  a normalisation constant to give the same maximum amplitude as the quadratic flux model.

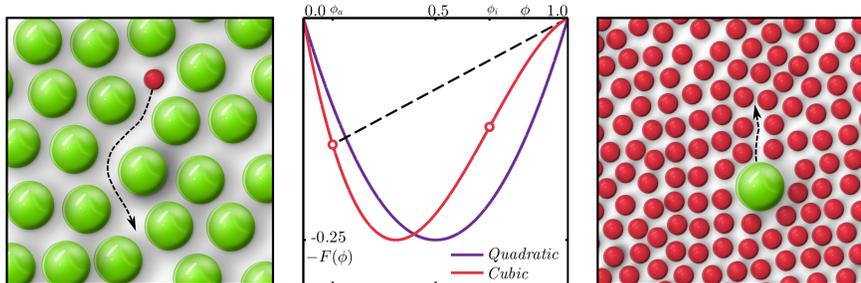


Figure 1: The segregation rate is faster when there is a single small particle in a region of large particles (*left*) than when there is a single large particle in a region of small particles (*right*). This phenomena cannot be captured by the symmetric quadratic flux, but can be captured by the simple asymmetric cubic flux function (*centre*) shown here with  $\gamma = 0.9$  and inflexion point  $\phi_i$ . The intersection with the tangent at  $\phi = 1$  is  $(\phi_a, F(\phi_a))$ .

Using mixture theory and a lithostatic pressure decomposition between the two phases, the non-dimensional governing segregation equation can be derived to be

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) + \frac{\partial}{\partial z} (-S_r F(\phi)) = \frac{\partial}{\partial z} \left( D_r \frac{\partial \phi}{\partial z} \right), \quad (2)$$

where  $\mathbf{u}$  is the velocity field, and  $S_r$  and  $D_r$  are non-dimensional segregation and diffusive-remixing numbers respectively.

## Homogeneous Inflow

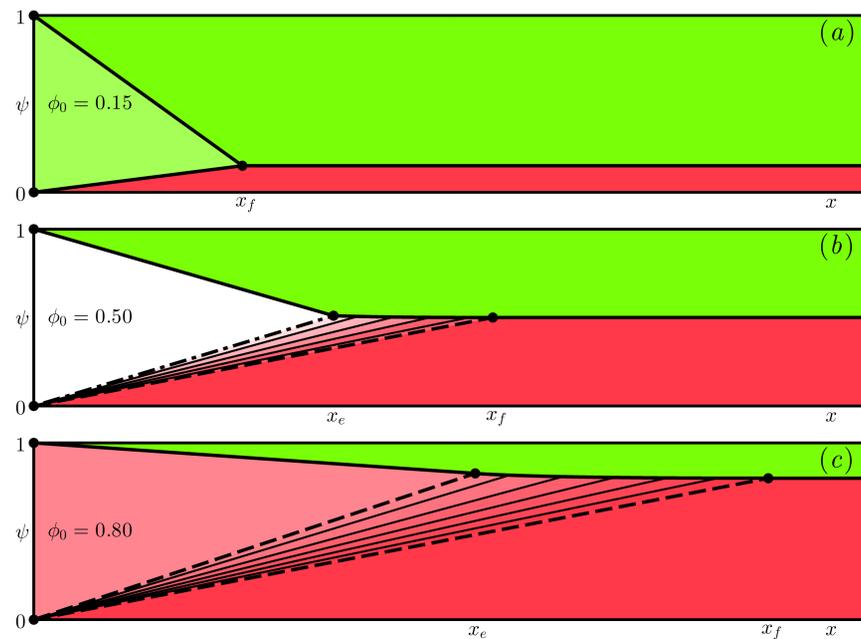


Figure 2: Analytic solutions in the case of homogeneous inflow. For low initial concentrations  $\phi_0$ , the solution consists of three shocks between the initial homogeneous region and the regions of small and large particles (*a*). As the initial concentration is increased, a one sided discontinuity develops between the homogeneous region and layer of small particles (*b*). For highly concentrated inflows, there is a smooth transition from the homogeneous region to the region of small particles (*c*). It can be seen that **as the initial concentration  $\phi_0$  is increased, the final distance  $x_f$  for the flow to fully segregate also increases**, a feature which is not captured by the quadratic flux models.

Consider the steady segregation equation (2) in a velocity field of the form  $\mathbf{u} = (u(z), 0, 0)$  and the absence of diffusion ( $D_r = 0$ ). Using a velocity averaged vertical co-ordinate  $\psi = \int_0^z u(z') dz'$ , equation (2) reduces to

$$\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial \psi} (-S_r F(\phi)) = 0. \quad (3)$$

This may be solved using the method of characteristics; characteristics are straight lines, and the cubic-flux function (1) gives rise to shocks, rarefaction fans and one-sided discontinuities.

Analytic solutions for homogeneous inflow and inversely graded inflow are shown in figures 2 and 3. There are several unique features compared to the analytical solutions for the quadratic flux model of Gray & Thornton (2005) and Thornton, Gray & Hogg (2006):

- The **non-convexity** of the cubic flux function leads to rarefaction fans in the analytical solution. This corresponds to regions where a few large particles remain 'stuck' in the layer of small particles and **take longer to rise to their pure phase layer**. In the case of normally graded inflow, under certain conditions, this leads to a second rarefaction fan with a beautiful 'catherine wheel' structure.
- For homogeneous inflow, the distance  $x_f$  until the flow is fully separated is now dependent on the inflow concentration  $\phi_0$ , with **the distance increasing as the inflow concentration increases**.
- In the case of inversely graded inflow, the cubic flux function causes **asymmetry** between the distance  $x_t$  for the first large particle to reach the surface and distance  $x_b$  for the first small particle to reach the base,

$$\frac{x_s}{x_b} = 4\gamma \frac{1-\psi_r}{\psi_r}, \quad (4)$$

where  $\psi_r$  is the separation height of the initial inversely graded layers.

## Inversely Graded Inflow

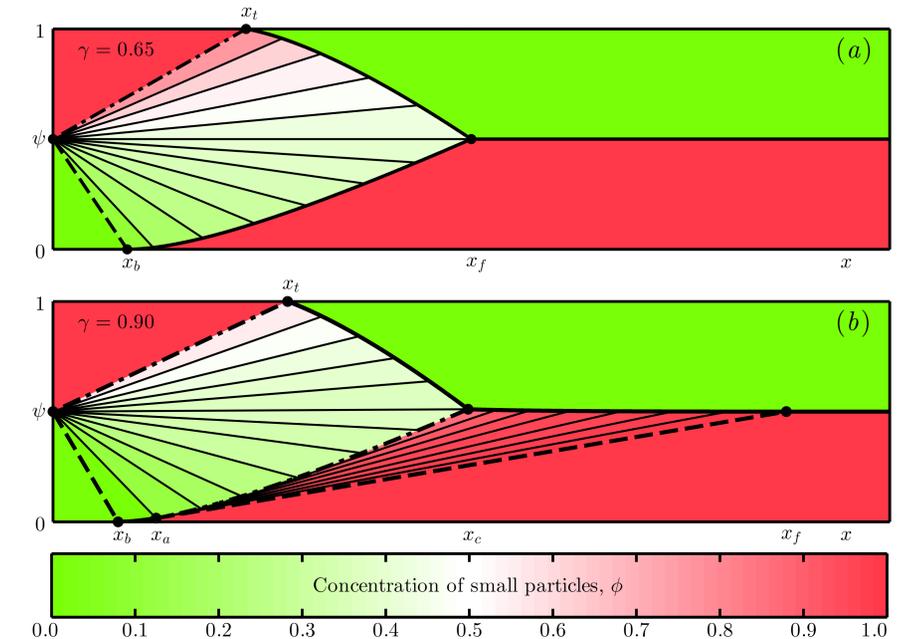


Figure 3: There are two distinct analytic solutions in the case of inversely graded inflow with an initial interface height of  $\psi_r = 0.5$ . The first type of solution resembles the solution for the quadratic, but with the layer of small particles initially at the top separated from the slowly varying concentration field by a one-sided discontinuity (*a*). For larger  $\gamma$ , a second rarefaction fan is formed in a 'catherine wheel' like structure (*b*). **The distance  $x_t$  for the first large particle to reach the surface is larger than the distance  $x_b$  for the first small particle to reach the base**. This is a new feature of the cubic-flux model.

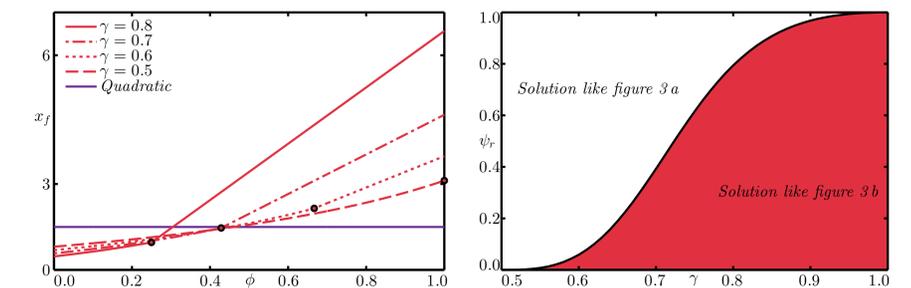


Figure 4: The distance  $x_f$  until the final segregated state increases with initial homogeneous concentration  $\phi_0$  (*left*). The  $(\gamma, \psi_r)$  parameter space for the two different types of solution for inversely graded inflow (*right*).

- **A three-phase mixture theory for particle size segregation in shallow granular free-surface flows.** Thornton A.R., Gray J.M.N.T. & Hogg A.J. (2006) J. Fluid Mech. **550**, 1–25.
- **A theory for particle size segregation in shallow granular free-surface flows.** Gray J.M.N.T. & Thornton A.R. (2005) Proc. Roy. Soc. A **461**, 1447–1473.
- **Experimental investigation into segregating granular flows down chutes.** Wiederseiner S., Andreini N., Epely-Chauvin G., Moser G., Monnereau M., Gray J.M.N.T. & Ancey C. (2011) Phys. Fluids **23**, 013301.
- **Mixing and segregation rates in sheared granular materials.** Golick, L.A. and Daniels, K.E. (2009) Phys. Rev. E **80**, 042301.
- **Particle size segregation in inclined chute flow of dry cohesionless granular solids.** Savage S.B. & Lun C.K.K. (1988) J. Fluid Mech. **189**, 311–335.