Spectra of categories of modules

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$\mathrm{Mod}\text{-}R$

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Mod-R

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Or find some organising structures which at least give some insight into the category and how it hangs together?

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Taking the $[A] = \{E : (A, E) = 0\}$ where A is a finitely presented R-module as basic open sets gives the same topology.

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The points of $\mathrm{Zar}(\mathcal{C})$ are the (isomorphism classes of) indecomposable injective objects of $\mathcal{C}$. Let $\mathrm{inj}(\mathcal{C})$ denote the set of these.

And the basic open sets are the $[A] = \{E \in \mathrm{inj}(\mathcal{C}) : (A, E) = 0\}$ as A ranges over finitely presented objects of $\mathcal{C}$.

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This functor category is abelian, locally coherent, Grothendieck. So we may consider the space $\mathrm{Zar}(\mathrm{mod}\text{-}R, \mathrm{Ab}) = \mathrm{Zar}(\mathrm{Fun}\text{-}R)$.

Theorem

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Thus the Gabriel-Zariski topology on $\mathrm{inj}(\mathrm{Fun}\text{-}(R^{\mathrm{op}}))$ induces a topology on pinj_R, which we denote by Zar_R and refer to as the **rep-Zariski spectrum** of R.

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This gives a much richer structure than what we would get by looking just at injective R-modules. A good pair of examples is $R = k[T]$ (the affine line) and $R = k\widetilde{A_1}$ (more or less the projective line).

The structure sheaf

Given a ring R we define a presheaf (defined on a basis) over Zar_R by $[F] = \{N : FN = 0\} \mapsto \mathrm{End}((R, -)/\langle F \rangle)$, where $\langle F \rangle$ denotes the result of "localisation at F".

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This presheaf-on-a-basis, Def_R, is separated (this follows from the interpretation of $\mathrm{End}((R,-)/\langle F\rangle)$ as a certain ring of functions on a certain module) so embeds in its sheafification, which we denote by LDef_R and refer to as the **sheaf of locally definable scalars** (for R-modules).

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And there's no reason to stop there: we may replace $\mathrm{Mod}\text{-}R$ by any functor category $\mathrm{Mod}\text{-}\mathcal{R} = (\mathcal{R}^{\mathrm{op}}, \mathbf{Ab})$ or even by any definable subcategory of such a category (that is, a subcategory closed under products, direct limits and pure subobjects).

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R (commutative) $\mapsto \mathrm{Ab}(R^{(\mathrm{op})}) = \mathcal{A}$ (small abelian category)
$\mapsto \mathrm{Ex}(\mathcal{A}, \mathbf{Ab}) = \mathcal{D}$ (definable category, in fact $\mathrm{Mod}\text{-}R$)
$\mapsto \big(\mathrm{Zar}(\mathcal{D}), \mathbf{LDef}(\mathcal{D})\big)$ $(= \big(\mathrm{Zar}_R, \mathbf{LDef}_R\big))$: a rather large sheaf of categories, but it contains, as a subsheaf, over the subspace of injective points of Zar_R, the thread consisting of localisations of $(R, -)$ and that is exactly $\big(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}\big)$, the usual structure sheaf of R.