Superdecomposable pure-injective modules

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Abstract. Existence of superdecomposable pure-injective modules reflects complexity in the category of finite-dimensional representations. We describe the relation in terms of pointed modules. We present methods for producing superdecomposable pure-injectives and give some details of recent work of Harland doing this in the context of tubular algebras.

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1. Introduction

Every finite-dimensional module is pure-injective. A **superdecomposable** module is one with no indecomposable direct summand. So it might seem that, in the context of modules over finite-dimensional algebras, the two adjectives in the title have little connection, at least as far as the finite-dimensional representation theory is concerned. In fact, the existence of superdecomposable pure-injective modules is a strong reflection of some aspects of the structure of morphisms in the category of finite-dimensional modules. In particular, the evidence to date is consistent with existence of superdecomposable pure-injectives being equivalent to having non-domestic representation type. In this expository paper, we explain this, describe some methods, due to Ziegler and Puninski, for proving existence of these modules and outline the proof of a recent result of Harland that, if R is a tubular algebra and r is a positive irrational, then there is a superdecomposable pure-injective module of slope r.

Pointed modules and morphisms between them, as well as (the essentially equivalent) pp formulas and associated methods from the model theory of modules, figure heavily in Harland's proof. Puninski's proofs also use these methods, which we will explain, though we won't use much model theory *per se*. When we consider pointed modules, we are interested not just in morphisms between modules but also in what these do to specified individual elements. This fits well with model theory, which has the notion of the *type* of an element in a structure; types keep track of what changes, as regards that element, when the structure changes. In model theory, the usual changes to a structure are to (elementarily) embed it in a larger one and/or to restrict automorphisms to those which fix some specified substructure. In the model theory of modules, however, the allowed changes are unrestricted: any homomorphism is fine. This still fits, in the module context, with the model theory because every homomorphism preserves pp formulas and there is a result (pp-elimination of quantifiers) which says that the model theory in modules is determined by pp formulas. We will explain what pp formulas, and the corresponding pp-types, are but we don't need to go into model theory as such; all we need is this convenient terminology and technology for handling pointed modules plus some useful theorems that have been proved in this context.

Our default is that modules are right R-modules. The category of these is denoted Mod-R; mod-R denotes the category of finitely presented modules. We write (M, N) rather than $\mathrm{Hom}(M, N)$. We will use the notation pp for the lattice of pp formulas, equivalently pointed finitely presented modules, possibly decorated with a subscript to denote the ring and/or a superscript to denote the number of elements we're pointing to.

2. Pointed modules

A **pointed module** is a pair (M, a) consisting of a module and an element in that module. We preorder pointed modules by setting $(M, a) \geq (N, b)$ if there is a morphism from M to N taking a to b. Since a pointed module is equivalently a morphism with domain R (that which takes $1 \in R$ to the element of the pair), this is just the ordering on such morphisms given by $f \geq f'$ if f' **factors initially through** f (that is, $f' = gf$ for some g). Write $(M, a) \sim (N, b)$ if both $(M, a) \geq (N, b)$ and $(N, b) \geq (M, a)$. If (A, a) is a pointed module with A finitely presented then we refer to this as a **pointed finitely presented module**. The partial order we obtain on pointed finitely presented modules is naturally isomorphic to the lattice of pp formulas in one free variable, which is one of the basic structures in the model theory of modules. We explain this so that we can move easily between pointed modules and pp formulas.

Before doing that, notice that we can generalise to n-pointed modules - pairs $(M, \overline{a})$ consisting of a module and a specified n-tuple of its elements, thus replacing the top point $(R, 1)$ in the preordering by $(R^n, \overline{e})$ where $\overline{e} = (e_1, \ldots, e_n)$ is a basis for R^n. Even more generally, if A is a finitely presented module then we have the notion of A-pointed modules, equivalently morphisms with domain A. If we choose a generating tuple, $\overline{a} = (a_1, \ldots, a_n)$, of elements of A then the poset of A-pointed finitely presented modules is thereby identified with the interval between $(A, \overline{a})$ and $(0, \overline{0})$ in the lattice of n-pointed finitely presented modules. The most general notion is that of an $(A, \overline{a})$-pointed module, where the tuple $\overline{a}$ does not necessarily generate A; again, this is just a representative of a point in the interval $[(A, \overline{a}), (0, \overline{0})]$. In all these cases it is easy to check that the resulting partial order on (appropriately-)pointed finitely presented modules is a modular lattice, with join given by direct sum and meet given by pushout; note that if A and B are finitely presented then so is C in the pushout diagram below.

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, "c"] & A \arrow[d, "\beta"] \\ B \arrow[r, "\alpha"'] & C \end{tikzcd}$$

We will write $(A, a) \wedge (B, b)$ for (any pointed module in the equivalence class of) (C, c) and $(A, a) + (B, b)$ for $(A \oplus B, a + b)$.

We will use pointed modules where the module is not necessarily finitely presented but, when it comes to considering a lattice structure, we restrict to pointed finitely presented modules. There will be a few occasions when we need to consider pp-types and not just pp formulas; then we use pointed pure-injective modules. This restriction is because the structure that we actually need, and that can be exactly expressed in terms of pp formulas (or finitely presented functors from mod-R to $\mathbf{Ab}$) is perfectly reflected in existence of morphisms between finitely presented modules, or between pure-injective modules, but not between arbitrary modules where there are obstructions to existence of the morphisms that would faithfully reflect the pp formula/finitely presented functor structure.

2.1. Pointed finitely presented modules and pp formulas. Now we explain what pp formulas are and how they arise from pointed finitely presented modules. Fix a pointed finitely presented module (A, a). Choose a finite generating tuple $\overline{c} = (c_1, \ldots, c_k)$ for A. Since A is finitely presented there are finitely many linear equations $\sum_1^k x_i r_{ij} = 0$ $(j = 1, \ldots, m)$ which are satisfied by $\overline{c}$ and generate all other linear equations satisfied by $\overline{c}$. Combine these into the single matrix equation $\overline{c}H = 0$ where H is the matrix with ij-entry r_{ij}. Since $\overline{c}$ generates A, the element a can be expressed as an R-linear combination of the c_i, say $a = \sum_i c_i s_i$. So the following assertion is true in A: there are elements $x_1, \ldots, x_n$ such that $\overline{x}H = 0$ and $a = \sum_i x_i s_i$. That assertion is expressible by a formula in the standard formal language which one sets up when dealing with the model theory of R-modules, namely the formula $\exists x_1, \ldots, x_n\, (\overline{x}H = 0 \wedge a = \sum_i x_i s_i)$, where the symbol $\wedge$ is read as "and". One might write the notation $\phi(a)$ for this formula, showing the element a because it is a parameter coming from a particular module (whereas the symbols for multiplication by individual elements of the ring R are fixed, being built into the formal language). The corresponding formula without parameters would replace a by a variable, say v, to give the formula $\phi(v)$ which is $\exists x_1, \ldots, x_n\, (\overline{x}H = 0 \wedge v = \sum_i x_i s_i)$, and then we would regard $\phi(a)$ as the result of substituting the free variable v in $\phi(v)$ by a particular element of a particular module. This is a fairly representative pp formula[1]. Because the free variable v in $\phi(v)$ is just a place-holder and the choice of variable name is irrelevant[2] we may write just ϕ. Any formula ϕ constructed from (A, a) as above (note that there were choices made) is said to **generate** the pp-type of a in A; we will discuss pp-types later and explain this terminology.

[1]The abbreviation pp (for positive primitive) is used for formulas such as $\phi(v)$ which are existentially quantified conjunctions (repeated "and") of linear equations; using matrix equations is just a convenient way of arranging a conjunction of equations.

[2]as long as we avoid using one already in use

What we have done here for (1-)pointed modules works as well for n-pointed modules; it will give us pp formulas with n free variables. The extension of the notion of pp formula to allow for more than one free (=unquantified =substitutable-by-parameters variable) should be clear and the similarly-constructed pp formula is said to **generate the pp-type** of $\overline{a}$ in A; the notation typically used is $\mathrm{pp}^A(\overline{a}) = \langle\phi\rangle$.

If ϕ is a pp formula with one free variable, such as that above, if M is any module and if $m \in M$, then either m satisfies the condition expressed by ϕ or it doesn't; if it does, then the notation used in model theory is $M \models \phi(m)$ but we will rather write $m \in \phi(M)$ where we denote by $\phi(M)$ the solution set of ϕ in M - the set of elements of M which satisfy the condition expressed by ϕ. Note that whether or not m satisfies ϕ depends on the containing module M (at least, if there are existentially quantified variables in ϕ). Of course, if ϕ has n free variables then the solution set $\phi(M)$ will be a subset of M^n. The assigment of $\phi(M)$ to M actually defines a functor from Mod-R to $\mathbf{Ab}$, the point being the following (which his easily proved, or see, e.g. [37, 1.1.7, §1.1.1]).

Lemma 2.1. *If M is any module and $\phi = \phi(v_1, \ldots, v_n)$ is a pp formula then the solution set, $\phi(M)$, of ϕ in M is a subgroup of M^n - a* **pp-definable subgroup** *of M.*[3] *If, moreover, $f : M \to N$ is a morphism and if $\overline{a} \in \phi(M)$ then $f\overline{a} \in \phi(N)$. That is, $M \mapsto \phi(M)$ defines a subfunctor of the n-th power, $(R^n, -)$, of the forgetful functor.*

If ϕ is a pp formula defined from the pointed module (A, a) as above then this functor picks out the **trace of** (A, a) **on** a module M, that is, the set of images of a under morphisms from A to M. Let us write $\mathrm{tr}(A, a)(M)$ for this set which, by the next lemma, equals $\phi(M)$.

Lemma 2.2 ((see [37, 1.2.7])). *If $(A, \overline{a})$ is a(n n-)pointed finitely presented module and if ϕ is a pp formula constructed from $(A, \overline{a})$ as above then, for any module M and $\overline{m} \in M^n$, $\overline{m} \in \phi(M)$ iff there is a morphism $f : A \to M$ with $f\overline{a} = \overline{m}$.*

This also shows that the ordering on pointed finitely presented modules is equivalent to the ordering on pp formulas, the latter being defined by $\phi \geq \psi$ if $\phi(M) \geq \psi(M)$ for every $M \in$ mod-R. (This implies that $\phi(M) \geq \psi(M)$ for every $M \in$ mod-R because every module is a direct limit of finitely presented modules, and functors of the form $M \mapsto \phi(M)$ commute with direct limits[4].) We regard pp formulas with the same solution set in every (finitely presented) module as equivalent and don't distinguish between them, nor between them and their equivalence classes. We write pp^n (or pp^n_R) for the lattice of pp formulas in n free variables (for right R-modules) and usually drop the superscript when $n = 1$. By the discussion above (also see [37, 10.2.19]) we have the following.

[3] or, more carefully if $n > 1$, a subgroup of M^n pp-definable in M

[4] This relation between ϕ and ψ is one that can be checked fairly effectively, just in terms of the systems of equations they involve, in particular, without having to test it at every finitely presented module, see [37, 1.1.13].

Proposition 2.3. *The lattice of n-pointed finitely presented modules is naturally isomorphic to the lattice* pp^n *of pp formulas with n free variables. This lattice is modular.*

More generally, if $(A, \overline{a})$ is a pointed finitely presented module and if ϕ generates the pp-type of $\overline{a}$ in A then the lattice of $(A, \overline{a})$-pointed modules is naturally isomorphic to the interval $[\phi, 0]$ in pp^n (where 0 denotes the class of the pp formula $v_1 = 0 \wedge \ldots \wedge v_n = 0$).

The terminology used in the model theory of modules when making use of this bijection is as follows. As we have seen already, starting with a pointed finitely presented module $(A, \overline{a})$ we can construct a pp formula ϕ which generates the pp-type of $\overline{a}$ in A: $\mathrm{pp}^A(\overline{a}) = \langle \phi \rangle$. In the other direction, starting with a pp formula ϕ we can construct a **free realisation** of ϕ, that is a pair $(A_\phi, \overline{a}_\phi)$ with A_ϕ finitely presented and ϕ generating the pp-type of $\overline{a_\phi}$ in A_ϕ (see, e.g., [37, 1.2.14]).

We mention that the functor associated to an n-pointed finitely presented module $(A, \overline{a})$ (equivalently to the associated pp formula), that is, the subfunctor of $(R_R^n, -)$ defined by taking $B \in \text{mod-}R$ to $\{g\overline{a} : g \in (A, B)\}$, is finitely generated, indeed, finitely presented, as an object of the functor category $(\text{mod-}R, \mathbf{Ab})$. For we have the exact sequence $R \xrightarrow{a} A \to A/aR \to 0$ in mod-R and this induces an exact sequence of functors $0 \to (A/aR, -) \to (A, -) \to (R, -)$, where the image of the last morphism is exactly this functor. Since $(A/aR, -)$ and $(A, -)$ are finitely generated projectives in $(\text{mod-}R, \mathbf{Ab})$ this proves the claim. Indeed (see, e.g., [37, 10.2.2]) every finitely generated subfunctor of $(R^n, -)$ has this form.

2.2. Relativising to a definable subcategory. Fix any module M. We define a quotient of the lattice of pointed finitely presented modules relative to M. This will be naturally isomorphic to the lattice of pp-definable subgroups of M, for which the usual notation is $\mathrm{pp}(M)$, which notation we may as well use for the lattice thought of in terms of pointed finitely presented modules. That is, for pointed finitely presented modules (A, a) and (B, b) we set $(A, a) \geq_M (B, b)$ if for every morphism $f : B \to M$ there is a morphism $g : A \to M$ with $ga = fb$. Clearly this is equivalent to $\mathrm{tr}(A, a)(M) \geq \mathrm{tr}(B, b)(M)$. Then we set $(A, a) \sim_M (B, b)$ if both $(A, a) \geq_M (B, b)$ and $(B, b) \geq_M (A, a)$ hold; we can then take $\mathrm{pp}(M)$ to be the resulting quotient lattice. Of course there are corresponding definitions for n-pointed modules obtained by putting a bar over a and b and a superscript n to "pp".

Lemma 2.4. *Let (A, a) and (B, b) be pointed finitely presented modules with pushout diagram as shown and let M be any module. Then $(A, a) \geq_M (B, b)$ iff every $f : B \to M$ factors through $\alpha : B \to C$.*

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[d, "b"'] & A \arrow[d, "\beta"] \\ B \arrow[r, "\alpha"'] & C \end{tikzcd}$$

Proof. ($\Rightarrow$) Take $f : B \to M$. Then we have the commutative diagram shown where g exists by assumption and so, by the pushout property, there is h as shown. In particular f factorises through α as required.

$$\begin{tikzcd}
R \arrow[r, "a"] \arrow[d, "b"'] & A \arrow[d, "\beta"] \arrow[ddr, dotted, bend left, "g"] & \\
B \arrow[r, "\alpha"'] \arrow[drr, bend right, "f"'] & C \arrow[dr, dashed, "h"] & \\
& & M
\end{tikzcd}$$

($\Leftarrow$) Given $f : B \to M$ there is, by assumption, $h : C \to M$ with $f = h\alpha$, so then $fb = h\alpha b = h\beta a$ so set $g = h\beta : A \to M$. $\square$

Corollary 2.5. *Let (A, a) and (B, b) be pointed finitely presented modules with pushout diagram as above. Then $(A, a) \sim_M (B, b)$ iff every morphism from A to M factors through β and every morphism from B to M factors through α.*

We comment here that we can assume when necessary that in a pointed finitely presented module $(C, \overline{c})$ the module C is **pp-essential over $\overline{c}$**, meaning that there is no proper direct summand of C containing every entry of $\overline{c}$ (if there is such a proper direct summand then it lies in the same equivalence class as $(C, \overline{c})$ in the sense of the ordering in 2.3 so we may use it to replace $(C, \overline{c})$). The reason for the terminology (which is from [31]) is that it is equivalent to the image of the functor $(\overline{c} \otimes -) : (R_R^n \otimes -) \to (C \otimes -)$ being essential in the functor $(C \otimes -)$ (which is an injective object of $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$, [19, 5.5]), so we are actually looking here at injective hulls - which exist if R is an artin algebra - in the subcategory of finitely presented functors on finitely presented *left* modules (equivalently, see, e.g., [37, §10.2.3], projective covers in the dual category of finitely presented functors on finitely presented right modules).

A **definable subcategory** of Mod-R is one closed under direct limits, products and pure submodules[5]. Relativising the pp-lattice to modules is equivalent to relativising to definable subcategories because if two modules M, M' generate the same definable subcategory then the induced orderings $\leq_M$ and $\leq_{M'}$ on pointed finitely presented modules are identical (e.g. by [37, 3.4.11]), so $\mathrm{pp}(M)$ and $\mathrm{pp}(M')$ are naturally isomorphic (or identical, depending on how we view these). Every definable subcategory $\mathcal{D}$ is generated as such by some module (the direct sum, or direct product, of one copy of each indecomposable pure-injective in $\mathcal{D}$ will do) so we extend the notation and terminology to definable subcategories writing, for example, $\mathrm{pp}(\mathcal{D})$. If M' is in the definable subcategory generated by M, in particular if M' is a pure submodule of M, and $(A, a) \geq_M (B, b)$ then $(A, a) \geq_{M'} (B, b)$ (see [37, 3.4.11]).

2.3. Elementary duality. If M is any module, $S \to \mathrm{End}(M)$ is a homomorphism of rings and ${}_SE$ is an injective cogenerator for left S-modules then we set $M^* = \mathrm{Hom}_S({}_SE, {}_SM)$ to be the corresponding dual of M.

[5] The definable subcategories that we consider will arise naturally in particular contexts; for the general theory of definable categories one may look at [37] or [39].

Theorem 2.6 (e.g. [37, 3.4.17]). *If M and N generate the same definable subcategory, $\mathcal{D}$, of* Mod-R *then M^* and N^* generate the same definable subcategory of* R-Mod, *denoted $D(\mathcal{D})$ and termed the (***elementary***)* **dual** *of $\mathcal{D}$. Furthermore, M and M^{**} generate the same definable subcategory of* Mod-R*; that is, $D^2(\mathcal{D}) = \mathcal{D}$.*
In particular $D(\text{Mod-}R) = R\text{-Mod}$.

If $\phi(v_1, \dots, v_n)$ is a pp formula, say it is $\exists y_1, \dots, y_m \ (\overline{v}, \overline{y}) \begin{pmatrix} A \\ B \end{pmatrix} = 0$, where A, B are matrices (matching $\overline{v}$, resp. $\overline{y}$) with entries in R, then the **elementary dual** [32] of ϕ is the pp formula for left modules which is $\exists \overline{z} \begin{pmatrix} I_n & A \\ 0 & B \end{pmatrix} \begin{pmatrix} \overline{v} \\ \overline{z} \end{pmatrix} = 0$ where I_n is the $n \times n$ identity matrix and 0 denotes a zero matrix of the correct size (and where $\overline{v}$ and $\overline{z}$ now are column vectors). In a moment we will rephrase this in terms of pointed modules but first we state the relation between $\text{pp}(\mathcal{D})$ and $\text{pp}(D(\mathcal{D}))$.

Theorem 2.7 (e.g. [37, 3.4.18]). *If $\mathcal{D}$ is a definable subcategory of* Mod-R *then elementary duality of pp formulas induces an anti-isomorphism of lattices* $\text{pp}(D(\mathcal{D})) \simeq \big(\text{pp}(\mathcal{D})\big)^{\text{op}}$.

Duality can be described in terms of pointed modules, as follows. If (A, a) is a pointed finitely presented module, then choose a morphism $R^m \to R^n$, given by left multiplication by the matrix H, with cokernel A but also with H chosen so that the image of the element $(1, 0, \dots, 0) \in R^n$ is a. Now use the transpose map, right multiplication by H, to define a morphism ${}_RR^n \to {}_RR^m$. Let $\pi : {}_RR^n \to R$ be projection to the first coordinate. Then form the pushout shown.

$$\begin{tikzcd} R^n \arrow[r, "-\times H"] \arrow[d, "\pi"'] & R^m \arrow[d] \\ R \arrow[r, "l"] & L \end{tikzcd}$$

If (A, a) is a free realisation of the pp formula ϕ then (L, l) will be a free realisation of its dual $D\phi$.

3. Pure-injective modules

Over artin algebras the simplest definition of pure-injectivity is that a module N is **pure-injective** if it is a direct summand of a direct product of finite length modules. As the name suggests, the general definition is that a module is pure-injective if it is injective over pure monomorphisms. A **pure monomorphism** is one that begins a **pure-exact sequence**, that is, a short exact sequence $0 \to A \xrightarrow{f} B \to C \to 0$ (of right modules) which, on tensoring with any (finitely presented) left A-module, gives an exact sequence of abelian groups. There is a conceptually quite distinct notion, algebraic compactness, which coincides with pure-injectivity (for modules, e.g. [66, Thm. 2], and for many other types of structure). A module N is **algebraically compact** if every collection of cosets of pp-definable subgroups of

N with the **finite intersection property** (that is, the intersection of any finitely many of these cosets is non-empty) has non-empty intersection.

Since every pp-definable subgroup of a module M is (by 2.1) closed under the action of $\mathrm{End}(M)$, any module which is of finite length over its endomorphism ring (**finite endolength** for short) is algebraically compact. So the **generic** modules over artin algebras - those indecomposables which are of finite endolength but not of finite length - are, together with the finite-length modules, examples of pure-injectives. Any injective module is pure-injective. Any functor between module categories which commutes with direct sums and direct products preserves pure-injectivity ([23, 7.35]), so the Prüfer modules ([56]) over tame hereditary (and other) algebras are pure-injective, being the images of injective modules under such functors. Similarly, rather dually, adic modules, being the images of suitably compactified modules, are pure-injective (see, e.g., [7]).

Hom-duals of modules are pure-injective: if M is any right R-module and $M^* = \mathrm{Hom}_S({}_SM, {}_SE)$ is as in Section 2.3, then M^* is a pure-injective left R-module (see, e.g., [37, §4.3.4]). Moreover, the natural map from M to M^{**} is a pure embedding, so every module purely embeds in a pure-injective module. Indeed there is a unique minimal-to-isomorphism-fixing-M such embedding, $M \to H(M)$; we say that $H(M)$ is the **pure-injective hull** (or **pure-injective envelope**) of M. Definable subcategories are closed under taking pure-injective hulls (see [37, 3.4.8]).

The easiest way to get the existence of pure-injective hulls (various proofs are referenced at [37, 4.4.2]), and the general structure theorem, 3.2, for pure-injectives is to make use of the full embedding $M \mapsto (M \otimes_R -)$ of Mod-R into the functor category (R-mod, $\mathbf{Ab}$) ([18], see [23, B16], [37, §12.1.1]).

Theorem 3.1. *The right module N is pure-injective iff the functor $N \otimes -$ is injective.*

Because the functor $M \mapsto (M \otimes -)$ is full it follows, from the fact that this is true for injectives, that an indecomposable pure-injective has local endomorphism ring. The general decomposition theorem for injective objects in Grothendieck categories pulls back to the following theorem for pure-injective modules (the first proof seems to be that in [11, 7.21]).

Theorem 3.2. *Every pure-injective module has a decomposition $H\big(\bigoplus_\lambda N_\lambda\big) \oplus N'$ where each N_λ is indecomposable pure-injective and N' is superdecomposable.*

It is the existence of superdecomposable pure-injectives in the category Mod-R and, more generally, in definable subcategories of Mod-R, that we are interested in here. For pure-injectivity in general, there are already surveys such as [22], [38] and, accounts with full proofs such as [23], [37]. In Section 5.1 we indicate what is known about existence of superdecomposable pure-injectives.

3.1. Pointed pure-injectives and pp-types. A key concept in the model theory of modules is that of a pp-type (a modification of the notion of *type* which is pervasive in model theory) but, just as pp formulas are equivalent to pointed finitely presented modules, pp-types are equivalent to pointed pure-injective modules. The actual definition of the **pp-type**, $\mathrm{pp}^M(m)$, **of** an element m **in** a module M is that it is the set of pp formulas ϕ such that $m \in \phi(M)$ (similarly for pp-types of n-tuples) but below we will reformulate this algebraically. We say that a pp formula ϕ **generates** a pp-type p if $p = \{\psi : \psi \geq \phi\}$. We order pp-types so as to agree with the ordering on pp formulas, setting $p \geq q$ iff $p \subseteq q$.[6] The connection with pointed modules is as follows (see [37, 4.3.9]).

Theorem 3.3 (e.g. [37, 4.3.9]). *Suppose that (M, m) and (N, n) are pointed modules and that N is pure-injective. Then $\mathrm{pp}^M(m) \geq \mathrm{pp}^N(n)$ iff there is a morphism $(M, m) \to (N, n)$.*

Pure-injectivity of N is needed for this.

Theorem 3.4. *The poset of pointed pure-injective modules is naturally isomorphic to the poset (indeed, modular lattice) of pp-types which, in turn, is naturally isomorphic to the lattice of subfunctors of the forgetful functor $({}_RR, -) \simeq (R \otimes -)$ in $(R\text{-mod}, \mathbf{Ab})$.*

The first statement is immediate from the previous result. For the other equivalence see, for instance, [37, 12.2.1]; it uses elementary duality (Section 2.3) to give that pp-types - that is filters in the lattice of finitely generated subfunctors of the forgetful functor $(R_R, -) \in (\text{mod-}R, \mathbf{Ab})$ - are in natural bijection with arbitrary subfunctors of the forgetful functor $({}_RR, -) \in (R\text{-mod}, \mathbf{Ab})$. We won't really use this result but perhaps it makes the link between the model-theoretic and algebraic concepts clearer.

Here is the algebraically reformulated definition of the pp-type of an element m in a module M: it is the collection of all pointed finitely presented modules (A, a) such that there is a morphism $(A, a) \to (M, m)$. We write $\mathrm{pp}^M(m)$ for this also (and whether you want to think of this as a set of pp formulas or a set of pointed finitely presented modules, or both, is up to you). This (rather this collection up to equivalence) is not just a set: it's actually a filter[7] in the lattice pp_R, in the sense that if $(A', a') \geq (A, a)$ and $(A, a) \in \mathrm{pp}^M(m)$ then $(A', a') \in \mathrm{pp}^M(a)$ (trivial from the definitions) and if (A, a) and (B, b) are in $\mathrm{pp}^M(m)$ then so is their pushout/meet $(A, a) \wedge (B, b)$ (immediate). And conversely, given any set of pointed finitely presented modules closed under (finite) pushout we can take the direct limit of this directed system to get a module (M, m) with pp-type the closure of this set under precomposition. We state this (the proof is from the fact that a module C is finitely presented iff the functor $(C, -)$ commutes with direct limits).

Proposition 3.5 ([37, 3.2.5]). *Suppose that $\mathcal{A}$ is a set of pointed finitely presented modules closed under pushout (or even just closed under pushout up to equivalence). Let (M, m) be the direct limit of this system of morphisms with domain R. Then, if (C, c) is a pointed finitely presented module, there is a morphism $(C, c) \to (M, m)$*

[6] The convention varies, over time as well as between people, for example this agrees with [37] but not [31]!

[7] or, in more category-theoretically, a sieve closed under pushout

iff there are $(A_1, a_1), \dots, (A_n, a_n) \in \mathcal{A}$ *and a morphism* $(C, c) \to (A_1, a_1) \wedge \dots \wedge (A_n, a_n)$.

In particular, given any set in pp_R*, the filter it generates (by taking finite meets and closing upwards) is the pp-type of some element in some module (which may be taken to be pure-injective).*

For the last statement just replace M by its pure-injective hull.

Corollary 3.6. *If* p *is a pp-type then there is a (pure-injective) module* N *and an element* $n \in N$ *which* **realises** p*, that is, such that* $\mathrm{pp}^N(n) = p$.

Note that if (A, a) is a pointed finitely presented module then the pp-type $\mathrm{pp}^A(a)$ of a in A consists of all the pointed finitely presented modules which are greater than or equal to (A, a) (just by definition). Let us state this.

Remark 3.7. If (A, a) is a pointed finitely presented module then $\mathrm{pp}^A(a) = \{(B, b) : B \in \text{mod-}R \text{ and } (B, b) \geq (A, a)\}$.

A pp-type which consists of all the pointed modules above a single[8] pointed finitely presented module (A, a) is said to be **finitely generated** (by (A, a)).

Of course everything said here for 1-pointed modules works for n-pointed modules.

3.2. Hulls of pp-types and Ziegler's criterion. Arguments using the model theory of modules often work to produce a pp-type with some particular properties and then finish with an appeal to taking the hull of that type so as to get a pure-injective module whose (desired) properties are algebraic reflections of those properties of the pp-type. We describe what is happening here in terms of pointed modules.

Suppose that p is a pp-type. By 3.6 there is a pointed pure-injective (N, n) which realises p, that is, such that if (A, a) is a pointed finitely presented module then there is a morphism from (A, a) to (N, n) iff $(A, a) \in p$. There is a minimal direct summand[9] of N which contains n and this is unique-to-isomorphism over n. We can see this by considering the image of $(n \otimes -) : (R \otimes -) \to (N \otimes -)$; by 3.1 the injective hull of this image has the form $(N' \otimes -)$ for some direct summand N' of N. We refer to this minimal pure-injective, N', over n as the **hull of** n **in** N, writing $H(n)$ for (any copy of) this. Since the pp-type of n is p we also write $H(p)$ for this and call it the **hull** of p. That this is well-defined follows from the next result (and the fact that if $m \in M'$ and M' is pure in M, in particular if M' is a direct summand of M, then $\mathrm{pp}^{M'}(m) = \mathrm{pp}^M(m)$).

Proposition 3.8 (see [37, 4.3.42]). *Suppose that* (N, n) *and* (M, m) *are pointed pure-injective modules and that* $\mathrm{pp}^N(n) = \mathrm{pp}^M(m)$*. Then (by 3.3) there is a morphism* $(N, n) \to (M, m)$*; for any such morphism the image in* M *of any copy of the hull of* n *in* N *is a direct summand of* M.

[8] equivalently, by taking their pushout, finitely many pointed finitely presented modules

[9] That is, there is no direct summand of N which contains n and is properly contained in this.

Say that a pp-type is **irreducible**[10] if its hull $H(p)$ is indecomposable. That is, the pp-type of an element m in a module M is irreducible iff the smallest direct summand of the pure-injective hull of M containing m is indecomposable. This can be characterised purely in terms of the lattice properties of p, as follows.

Theorem 3.9 (Ziegler's criterion, [67, 4.4]). *Let $p = \mathrm{pp}^M(m)$ be a pp-type. Then p is irreducible iff whenever (B, b) and (B', b') are pointed finitely presented modules not in p there is a pointed finitely presented module (A, a) in p such that there is no morphism $(C, c) \oplus (C', c') \to (M, m)$ where (C, c) and (C', c') are the pushouts shown.*

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, "c"] & A \arrow[d] \\ B \arrow[r] & C \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "a"] \arrow[d, "b'"'] \arrow[dr, "c'"] & A \arrow[d] \\ B' \arrow[r] & C' \end{tikzcd}$$

That is, p is irreducible iff given $(B, b), (B', b') \notin p$ there is $(A, a) \in p$ such that $((A, a) \wedge (B, b)) + ((A, a) \wedge (B', b')) \notin p$. This criterion, which is an expression of the fact that the endomorphism ring of an indecomposable pure-injective is local, is extremely useful both in proving general results and in computations. We will see, in Section 5, that there is a derived criterion for detecting when the hull $H(p)$ of a pp-type contains *no* indecomposable summand, that is, is superdecomposable.

4. Dimensions on lattices

There is a general process which assigns ordinal-valued dimensions to modular lattices modulo a specified class $\mathcal{L}$ of modular lattices which is closed under sublattices and quotient lattices. Two choices of $\mathcal{L}$ will be of interest to us. (Our **convention** will be that the the requirement of having a top and a bottom element is part of our definition of modular lattice.) The first takes $\mathcal{L}$ to consist of the trivial, one-point, lattice together with the simple, two-point, lattice. The second takes $\mathcal{L}$ to be the class of all chains (totally ordered sets). The basic process is to take a lattice L, then collapse all intervals in L which are in $\mathcal{L}$. More precisely, we define a congruence on L by setting $a \sim b$ if there is a chain $a + b = a_0 \geq a_1 \geq \cdots \geq a_n = a \wedge b$ with each interval $[a_i, a_{i+1}]$ in $\mathcal{L}$. We then form the quotient lattice $L/\sim$ which consists of the $\sim$-equivalence classes. The quotient $L/\sim$ is again a modular lattice so we may apply the same process to $L/\sim$. We continue this inductively, transfinitely (at limits we take the quotient of the original lattice L by the union of the pullbacks to L of the congruences on its successive quotients). Two things can happen. At some ordinal (necessarily of the form $\alpha + 1$ for some ordinal α) the top and bottom elements are identified and we obtain the trivial lattice, so we stop and set the **$\mathcal{L}$-dimension** of L to be α. Or we reach a quotient of L which has no nontrivial interval in $\mathcal{L}$, that is, in which the congruence $\sim$ is the identity relation, so the process stops but not at the trivial lattice. In that case we set the $\mathcal{L}$-dimension of L to be ∞ and say that the $\mathcal{L}$-dimension of L is **undefined**

[10] the terminology "indecomposable" also is used

(or that L **does not have $\mathcal{L}$-dimension**). Let us look at the two special cases mentioned (the general treatment can be found in [31, §10.2] or [37, §7.1]).

4.1. m-dimension/Krull-Gabriel dimension. In the case where $\mathcal{L}$ consists of the trivial lattice and the two-point lattice, the congruence $\sim$ is that which identifies a and b iff the interval $[a+b, a \wedge b]$ has finite length; the resulting dimension is termed **m-dimension**. It was introduced (though not in this way) in [67, p. 191] and is an ambidextrous version of the dimension of Gabriel and Rentschler ([55]). These dimensions - m-dimension and Krull dimension in the sense of noncommutative ring theory (see [15]) - coexist: one is defined iff the other is, but m-dimension grows more slowly and also has the useful property that its value for a lattice is equal to its value for the opposite lattice.

Proposition 4.1 (see, e.g., [37, 7.2.3]). *The following are equivalent for any modular lattice L:*
(i) the m-dimension of L is defined;
(ii) the Krull dimension of L is defined;
(iii) L contains no subset (with more than one point) which is densely ordered.

Garavaglia had already proved that if the Krull dimension of the lattice of pp formulas is defined then there are no superdecomposable pure-injectives so we have the following (for the lattice of pp formulas/pointed finitely presented modules relativised to a definable subcategory see 2.2).

Theorem 4.2 ([12, Thm. 1], see [37, 7.3.6]). *Suppose that $\mathcal{D}$ is a definable subcategory of* Mod-R*. If the lattice of pp formulas for $\mathcal{D}$ (in one free variable) has m-dimension then there is no superdecomposable pure-injective in $\mathcal{D}$.*

It follows by elementary duality (Section 2.3) that the m-dimension for right modules equals that for left modules, so we may further conclude that there is no superdecomposable pure-injective left module in the elementary dual definable category (see, e.g., [37, §3.4.2]) of $\mathcal{D}$ which, in the case $\mathcal{D} = \text{Mod-}R$, is R-Mod.

Corollary 4.3. *If M is an R-module and if the lattice of pp-definable subgroups of M has m-dimension then the definable subcategory of* Mod-R *generated by M contains no superdecomposable pure-injective.*

These results also may be phrased in terms of Krull-Gabriel dimension. This dimension, introduced in [13], see also [23, p. 197ff.], is defined on the functor category associated to $\mathcal{D}$, which is just the localisation of the functor category $(\text{mod-}R, \mathbf{Ab})$ for Mod-R at the functors which are 0 on every member of $\mathcal{D}$. This is defined like Gabriel dimension (see [16]), which inductively (and transfinitely if necessary) localises at the torsion class generated by the the simple objects, except that at each stage it uses only the *finitely presented* simple objects in the current category to generate the torsion class[11]

[11] In fact, for locally coherent categories such as these, Gabriel dimension is defined iff Krull-Gabriel dimension is defined, [5, 5.1] (see [37, 13.2.9]).

We saw in 2.3 that the lattice of pp formulas is isomorphic to the lattice of pointed modules so, of course, the results above all can be said in terms of these. That was done in [35], using the notion of a **factorisable system** of morphisms in mod-R, that is, a collection of pointed finitely presented modules $(A_q, a_q)_{q\in\mathbb{Q}}$ indexed by the rationals (or the rationals in $[0, 1]$, as one prefers) such that $(A_q, a_q) > (A_p, a_p)$ iff $q < p$ (the (optional) reversal of ordering is to fit with the direction of morphisms).

Corollary 4.4. *If there is no factorisable system of pointed modules in* mod-R *then there is no superdecomposable pure-injective R-module.*

For more details, see that paper or [37, §7.2.2].

The implications in the corollaries above are definitely not reversible: if R is a nearly simple uniserial domain then its m-dimension is undefined but there is no superdecomposable pure-injective R-module ([46, 4.1, 6.3]). The above dimensions being defined is reflected to some extent in the category of pure-injectives in the sense that, over a countable ring, the m-dimension is defined iff there are only countably many indecomposable pure-injectives; whereas if it is undefined then there are continuum many ([67, 8.1, 8.4]). Over rings of arbitrary cardinality, if m-dimension is defined then so is the Cantor-Bendixson rank of the Ziegler spectrum; conjecturally this is an equivalence, and it has been proved under various additional assumptions (see [37, 5.3.60]).

4.2. Width/breadth. If we take $\mathcal{L}$ to be the class of chains then the corresponding $\mathcal{L}$-dimension is termed **breadth**. This terminology was introduced in [31, §10.2] (where defining such dimensions by successive collapsing was introduced) in order to distinguish it from *width* which was defined in [67, p. 183] (more in the style of the usual definition of noncommutative Krull dimension). These coexist in the sense that if one of breadth or width is defined then so is the other ([31, 10.7]). In practice the term width is most used (but it has no connection with width in the sense of the size of a maximal antichain!). We write $\mathrm{w}(L) = \infty$ if the width of L is undefined. Of course, if the width of L is undefined then so is the m-dimension of L.

For width/breadth there is an analogue to 4.1. Here we will say that a lattice L is **wide**[12] if, given any pair $a > b$ in L there are incomparable $c, d \in L$ between a and b. In such a lattice the $\mathcal{L}$-congruence is just equality, so it does not have width; indeed this is the obstruction to having width.

Proposition 4.5 (see [37, 7.3.1]). *The following are equivalent for a modular lattice L:*
(i) the width (equivalently breadth) of L is defined;
(ii) L has no wide (sub)quotient;
(iii) L contains no wide subposet[13]

[12] Sometimes the term wide is used for any lattice which has width undefined.

[13] The definition of a poset, as opposed to lattice, being wide is slightly more complicated but we don't need it here and I have added this equivalent only for completeness.

Theorem 4.6 ([67, 7.1]). *Let $\mathcal{D}$ be a definable subcategory of* Mod-R*; if there is a superdecomposable pure-injective in $\mathcal{D}$ then the width of the lattice* $\mathrm{pp}(\mathcal{D})$ *is undefined. If this lattice is countable, in particular if R is countable, then the converse is true.*

It is unknown whether the converse of 4.6 is true in general, and that is currently a significant obstruction to proving existence of superdecomposable pure-injectives: one might be able to prove that the width of the lattice of pp formulas/pointed finitely presented modules is undefined but, to go beyond that to existence of superdecomposable pure-injectives, either one has to assume countability or find a direct construction (as Puninski has done, as discussed in Section 5.2). On the other hand, if one's main concern is with finite-dimensional modules then showing that width is undefined is already showing that morphisms in the category of finite-dimensional modules contain a complex, continually branching, structure. For this dimension one could write down a definition analogous to that of factorisable system of pointed modules, just replacing reference to pp formulas by reference to pointed modules. A useable sufficient condition is described just before 5.6.

As with m-dimension, by elementary duality, the width for right modules equals that for left modules.

Corollary 4.7. *If the lattice of pp formulas/pointed finitely presented modules for right R-modules has width $= \infty$ then the same is true of the lattice for left modules.*

In particular, if R is a countable ring then there is a superdecomposable pure-injective right R-module iff there is a superdecomposable pure-injective left R-module.

We give some idea of the proof of 4.6.

First we consider the direction that needs no countability hypothesis. Recall that $\mathrm{pp}(N)$ denotes the lattice of pointed finitely presented modules modulo N (equivalently, as the notation indicates, the lattice of pp-definable subgroups of N).

Theorem 4.8 ([67, §7]). *Suppose that N is a pure-injective module and that $(A, a) \to (B, b)$ is a pair of pointed finitely presented modules which is open on N, that is* $\mathrm{tr}(A, a)(N) \gneqq \mathrm{tr}(B, b)(N)$. *Suppose that the interval $[(A, a), (B, b)]$ is totally ordered by the relation $\geq_N$. Then there is an indecomposable summand N_0 of N (on which the pair $(A, a) \geq (B, b)$ is open).*

Proof. Our assumption is that if $(A, a) \geq (C, c) \geq (B, b)$ and $(A, a) \geq (D, d) \geq (B, b)$, with C, D finitely presented then either $\mathrm{tr}(C, c)(N) \geq \mathrm{tr}(D, d)(N)$ or *vice versa*. Since the pair $(A, a) \geq (B, b)$ is open on N we can choose and fix some $m \in \mathrm{tr}(A, a)(N) \setminus \mathrm{tr}(B, b)(N)$. We use this element to define a cut on the interval with upper part $U = \{(C, c) : (A, a) \geq (C, c) \geq (B, b) \text{ and } m \in \mathrm{tr}(C, c)(N)\}$ and lower part $L = \{(C, c) : (A, a) \geq (C, c) \geq (B, b) \text{ and } m \notin \mathrm{tr}(C, c)(N)\}$. We then use Zorn's lemma to extend U maximally to a pp-type p which contains no pointed module in L. It is then a standard use of Ziegler's criterion, 3.9, to deduce that p is irreducible and so $H(p)$ is an indecomposable pure-injective. One does need to do more work, for which see the references, to show that a copy of $H(p)$ is a direct summand of N but what we have said here shows how we use the fact that some interval is a chain. $\square$

Corollary 4.9 ([67, 7.8]). *If N is a superdecomposable pure-injective module then the lattice of pp-definable subgroups of N is wide.*

Proof. Otherwise there would be a non-trivial interval in that lattice which is a chain but then, by the result above, N would have an indecomposable direct summand, contradicting superdecomposability. $\square$

It follows (by the comments in the last paragraph of Section 2.2) that if $\mathcal{D}$ is any definable category containing N then the width of $\mathrm{pp}(\mathcal{D})$ is undefined.

The current proof of the other direction of 4.6 requires countability and is quite technical. The original proof is [67, 7.8(2)] and is in terms of pp formulas. The proof was slightly reformulated in [31, 10.13] and then that version was cast into the form of an argument in the functor category in [37, §12.6] using the translation which replaces pure-injective modules by injective functors (cf. 3.1), pp formulas by finitely presented functors etc. but it becomes no less technical in the functor category, so I just describe the shape of that proof now.

Suppose then that R is countable. Then there are just countably many finitely presented modules (up to isomorphism), each of which has just countably many elements. Therefore there are only countably many pointed finitely presented modules, so they can be listed using the natural numbers as index set:
$(A_0, a_0), (A_1, a_a), \ldots, (A_n, a_n), \ldots$.
We will construct a pp-type p by working through that list, deciding at each stage whether to put (A_n, a_n) into p or whether to exclude it (permanently). The order is entirely immaterial; all we need is that each pointed finitely presented module occurs at some stage and that, at that stage, we have already had to consider only finitely many other pointed modules. In the limit we will have dealt with every pointed finitely presented module. We need to ensure that (1) the resulting set p of "included" pointed finitely presented modules is a pp-type and (2) no pointed finitely presented module is large for p (see Section 5 for the definition of "large"). For instance if, when we come to consider (A_n, a_n), it is the case that $(A_n, a_n) \geq (A_i, a_i)$ for some $i < n$ which has been put into p, then (A_n, a_n) also must go into p.

That shape of argument is used elsewhere in model theory (of modules and in general). In this case we also develop, through the stages, a set of non-trivial intervals in the lattice of pointed finitely presented modules, each of which relativised to N, has width ∞ (by assumption we can start with the interval $[(R, 1), (0, 0)]$). At the stage n, when we consider (A_n, a_n), we take each of our current finitely many intervals: let $[(C_k, c_k), (C'_k, c'_k)]$ be one of them; if the width of $[(A_n, a_n) + (C_k, c_k), (A_n, a_n) + (C'_k, c'_k)]$, relativised to N, still is undefined then this will go into our set of intervals to be used at stage $n + 1$ and (A_n, a_n) will be put into p. Otherwise we exclude (A_n, a_n) from p and we choose two incomparable intervals within $[(C_k, c_k), (C'_k, c'_k)]$, each still with undefined width, and put each of these into our set of intervals to be used at stage $n + 1$ (this is part of making sure that the excluded (A_n, a_n) is not large for p). Then one shows that the conditions

(1) and (2) above are indeed satisfied. (The obstacle to extending this argument to the uncountable case, by using transfinite induction, is in trying to carry the set of intervals through any limit stage.)

5. Superdecomposable pure-injectives

Sometimes superdecomposable (pure-)injective modules can be produced directly: injective hulls of modules without uniform submodules are superdecomposable and these can be moved, using functors which commute with direct products and direct sums, hence preserve pure-injectivity, and are full, hence preserve superdecomposability, to produce superdecomposable pure-injectives in other categories. This method is used in [23, p. 211ff.] and shows, for example, that strictly wild algebras have superdecomposable pure-injectives.

Beyond that, our main method of obtaining superdecomposable pure-injectives is to produce a pp-type p, the hull of which has no indecomposable direct summand. As in 3.9 this can be detected just from the lattice structure of pointed finitely presented modules. Say that a pointed finitely presented module (B_0, b_0) is **large for**[14] p if it is not in p and the condition of 3.9 holds above (B_0, b_0); that is, if for every $(B, b), (B', b') \notin p$ with $(B, b), (B', b') \geq (B_0, b_0)$ there is $(A, a) \in p$ such that $((A, a) \wedge (B, b)) + ((A, a) \wedge (B', b')) \notin p$.

Theorem 5.1 ([67, 7.6]). *A pp-type has superdecomposable hull iff it has no large pointed finitely presented module.*

As one would expect, if there is a pointed finitely presented module (B_0, b_0) which is large for p then one can say something about the relationship between this and a corresponding indecomposable direct summand of $H(p)$ (see e.g. [37, 4.3.79]) but we don't need that detail here.

If the lattice pp has width ∞ and is countable then, as discussed in the previous section, there is an argument of Ziegler which produces a pp-type with no large formula. Of course, one has to have a method for showing that the lattice pp has width ∞. It is also desirable to avoid having to use the countability hypothesis. For example, Puninski characterised those commutative valuation rings which have superdecomposable pure-injective modules; for domains this is as follows.

Theorem 5.2 ([41, 4.2], also see [44, 12.12]). *Let R be a commutative valuation domain. Then there is a superdecomposable pure-injective R-module iff the Krull dimension of R is ∞ iff the value group of R contains a densely ordered subset.*

In this case, a densely ordered chain of (right) ideals gives rise to a similarly-ordered chain of divisibility pp formulas and, in the opposite direction, a densely ordered chain of annihilation pp formulas. It turns out that if two such chains freely generate a modular (indeed, distributive) lattice within the lattice pp_R then that

[14] The usual terminology is "large in" p but that doesn't sit very well with the fact that the pointed module is definitely not in p; the usual terminology makes more sense in the original context where a distinction is made between types and pp-types.

lattice with width $= \infty$. This underlying lattice theory has been developed into a method which we describe in Section 5.2. This method has proved to be effective in a number of contexts, both in showing that the lattice of pointed finitely presented modules does not have width and, with a stronger hypothesis, in directly producing pp-types without large pointed modules, hence superdecomposable pure-injectives, in the absence of any countability hypothesis.

5.1. Existence of superdecomposable pure-injectives. Here we summarise much of what is known about existence of superdecomposable pure-injectives over various kinds of ring. When the results apply to more general definable subcategories we state them in that way but one may always take the definable subcategory to be Mod-R itself.

• If we ask about the existence of superdecomposable, not necessarily pure-injective, modules then these exist even among abelian groups (see [14, p. 606]). This reflects the fact that the general infinitely-generated representation theory of most rings is wild (see e.g. [60]).

• Rings of finite representation type, and the possibly more general pure-semisimple rings, have no superdecomposable pure-injectives, although every module is pure-injective (for a proof and references, see, e.g. [37, §4.5.1]).

• Tame hereditary artin algebras have no superdecomposable pure-injectives (this follows from [13] and 4.2 see also [36], [59]).

• It is conjectured that all domestic string algebras have Krull-Gabriel dimension (for which see Section 4.1) and hence (4.3) have no superdecomposable pure-injectives.

• There are results in [43], [42] for differential polynomial rings over fields of characteristic 0, somewhat analogous to those for hereditary finite-dimensional algebras, in particular implying that there are no superdecomposable pure-injectives for "tame" such rings.

• Non-domestic string algebras have width undefined hence, if countable, have superdecomposable pure-injectives ([49, 3.2]). The countability hypothesis is conjecturally unnecessary and has been bypassed in some cases ([51]).

• Tubular algebras have width undefined hence, if countable, superdecomposable pure-injectives, indeed, for each positive irrational there is a superdecomposable pure-injective with that slope (see Section 6).

• Tame strongly simply connected algebras of non-polynomial growth, over a field of characteristic $\neq 2$ have width undefined, hence superdecomposable pure-injectives if countable ([25]).

• The free associative algebra $K\langle X, Y\rangle$ in two generators has superdecomposable injective hull (since R_R has no uniform submodule).

• Strictly wild algebras have superdecomposable pure-injectives (the image of the injective hull of $K\langle X, Y\rangle$ will be pure-injective and, by fullness of the representation embedding, superdecomposable).

• PI Dedekind domains, in particular $\mathbb{Z}$ and $K[X]$, more generally, PI hereditary noetherian prime rings, have no superdecomposable pure-injectives (the commutative case goes back to Kaplansky [24], the general PI case is, noting 4.3, [36, 1.6,

3.3]).
• The polynomial ring $K[X, Y]$ has superdecomposable pure-injectives, as does the power series ring $K[[X, Y]]$ and any regular local ring of dimension ≥ 2 ([23, p. 218]).
• For many generalised Weyl algebras (in the sense of [3]), including the first Weyl algebra A_1 and its localisation B_1, the quantum Weyl algebra A_q ($q \neq 0, 1$) and the universal enveloping algebra Usl_2, there is a wide poset of pointed finitely presented modules, hence if the field is countable, a superdecomposable pure-injective ([40, 7.5, §6], using a construction from [26]).
• If G is a finite non-trivial group then there is a superdecomposable $\mathbb{Z}G$-module ([52]).
• If R is the pullback (in the sense of [30]) of two commutative Dedekind domains, neither of which is a field, then the width of pp_R is undefined so, if the Dedekind domains are countable, there is a superdecomposable pure-injective R-module ([52, 5.8]).
• If R is von Neumann regular then there is a superdecomposable pure-injective R-module iff R is not semiartinian ([65, §1]); in the case that R is commutative this is iff the boolean algebra of idempotents of R is superatomic ([31, 16.26]).
• Over serial rings, the model theory of modules and the structure of pure-injectives (as well as pure-projectives, that is, direct summands of direct sums of finitely presented modules) has been extensively studied by Puninski in a series of papers, see especially [45], [46] and the book [44]. If R is a serial ring Puninski gives a criterion, purely in terms of the structure of the lattices of right and left ideals of R, equivalent to existence of a superdecomposable pure-injective ([50, 5.2]). The criterion is right/left symmetric, so it follows that there is a superdecomposable pure-injective right module iff there is such a left module [50, 6.1]. It is also the case, [50, 6.2], that if the lattice of two-sided ideals of a serial ring R has Krull dimension (in the Gabriel-Rentschler sense) then there is no superdecomposable pure-injective R-module. Furthermore, [50, 6.4], in the case of serial rings the countability restriction in 4.6 can be dropped: width being undefined is equivalent to existence of a superdecomposable pure-injective.
• Those group rings KG where K is a field and G is a finite group such that the category of KG-modules does not have width are almost completely characterised in [53], the only unresolved case being that where G is the quaternion group and K has characteristic 2 and does not contain a primitive cube root of 1 (the conjecture is that the width is undefined in this case).

5.2. Lattices freely generated by chains. Suppose that L_1 and L_2 are chains with top and bottom. Denote by $L_1 \otimes L_2$ the modular lattice freely generated by L_1 and L_2 subject to identifying their respective tops and bottoms. This is a distributive lattice (see [17, Thm. 13]) with a canonical form (indeed, two canonical forms) for its elements: each element of $L_1 \otimes L_2$ has a unique representation of the form $(a_n \wedge b_1) + (a_{n-1} \wedge b_2) + \cdots + (a_1 \wedge b_n)$ with $a_1 < \cdots < a_n$ in L_1 and $b_1 < \cdots < b_n$ in L_2; by distributivity this is equal to $a_n \wedge (b_1 + a_{n-1}) \wedge \cdots \wedge (b_{n-1} + a_1) \wedge b_n$ and that is the second canonical form.

Lemma 5.3 ([52, p. 61]). *If $a_1 < a_2$ in the chain L_1 and $b_1 < b_2$ in the chain L_2 then, in $L_1 \otimes L_2$, the elements $a_1 + b_1$ and $a_2 \wedge b_2$ are not comparable.*

Proof. The meet of these two elements is $(a_1 + b_1) \wedge a_2 \wedge b_2 = b_2 \wedge (a_1 + b_1) \wedge a_2$ which is a canonical form of the second type and different from the (second-type) canonical form $b_2 \wedge a_2$, hence these elements are not equal and so $a_2 \wedge b_2 \not\leq (a_1 + b_1) \wedge a_2 \wedge b_2$. Also, by distributivity, $(a_1 + b_1) \wedge a_2 \wedge b_2 = (a_1 \wedge b_2) + (b_1 \wedge a_2)$ which is in first canonical form and different from $a_1 + b_1$, which is also in first canonical form. So we deduce that $a_1 + b_1 \not\leq (a_1 + b_1) \wedge a_2 \wedge b_2$ and the claim is established. □

It is useful to have a description of the ordering in $L_1 \otimes L_2$, which is as follows (see [51, p. 708]). To check whether two elements of this lattice satisfy $l \leq l'$, write the first as a sum of meets (for example, using the first canonical form) $l = \sum_i (a_i \wedge b_i)$, and write the second as a meet of sums $l' = \bigwedge_j (a'_j + b'_j)$ (for example, using the second canonical form). Then $l \leq l'$ iff for each i and j we have $a_i \wedge b_i \leq a'_j + b'_j$ and that, by the lemma above, is the case iff $a_i \leq a'_j$ or $b_i \leq b'_j$. With this clear description of $L_1 \otimes L_2$, Puninski proves the following [49, 3.1, 3.2] (he also gives a visual version of the proof, using a graphical representation for the elements of $L_1 \otimes L_2$).

Proposition 5.4. *Suppose that L_1 and L_2 are chains, each containing a densely ordered subset. Then $L_1 \otimes L_2$ has width ∞.*

In fact, the result proved in [49] is more general, being an estimate of the m-dimension (rather, a slightly modified version of that) of $L_1 \otimes L_2$ in terms of the m-dimensions of L_1 and L_2.

To use this in practice, one must be able to come up with two candidate chains of pointed finitely presented modules/pp formulas for L_1 and L_2 and then show that the lattice they generate within pp_R is freely generated by them. Densely ordered chains, for many types of algebra, are not hard to find (see [35]). For instance in [52] the modules are string modules over a nondomestic string algebra so one may fix a string, extend it to the left to get a densely ordered chain L_1, alternatively extend it to the right to get L_2. But how do we show that these chains really are independent? The following lemma is the key.

Lemma 5.5 ([52, 5.4]). *If L_1 and L_2 are chains then a factor L' of the lattice $L_1 \otimes L_2$ is a proper factor iff there are $a_1 < a_2 \in L_1$ and $b_1 < b_2 \in L_2$ such that the images of the elements $a_1 + b_1$ and $a_2 \wedge b_2$ in L' are comparable.*

Proof. One direction is 5.3. For the other, if L' is a proper factor then there are $l' < l$ in $L_1 \otimes l_2$ which become equal in L'. Write l' as a meet of sums $l' = \bigwedge_j (a'_j + b'_j)$ and l as a sum of meets $l = \sum_i (a_i \wedge b_i)$. Mapping to L' by the projection π (which preserves meets and sums), our assumption is that the images satisfy $\pi(l) \leq \pi(l')$, that is, $\sum_i (\pi(a_i) \wedge \pi(b_i)) \leq \bigwedge_j (\pi(a'_j) + \pi(b'_j))$, from which we deduce that $\pi(a_i \wedge b_i) \leq \pi(a'_j + b'_j)$ for all i, j.

If we have $a_i > a'_j$ and $b_i > b'_j$ for some i, j then we have the desired configuration in L'. Otherwise, for every pair i, j we have either $a_i \leq a'_j$ or $b_i \leq b'_j$ and hence $a_i \wedge b_i \leq a'_j + b'_j$ for all i, j, and hence $l \leq l'$, so l and l' already were equal, contradiction. □

In practice so far, in particular in [53] and [25], the following special case of the notion of a factorisable/densely ordered system of morphisms/pointed modules has been used. The extra requirement is that the modules A_q in the factorisable system should be indecomposable (and that no a_p should be 0). To keep the terminology reasonably consistent and brief, let us refer to what we define below as a **dense chain of indecomposable pointed modules** (the "finitely presented" being understood). The key definition, from [53, 5.4], is that of an **independent pair** of such dense chains, that is, two dense chains $(A_p, a_p)_{p\in\mathbb{Q}}$, $(B_s, b_s)_{s\in\mathbb{Q}}$[15] of indecomposable pointed modules such that:

- there is no pointed map between any (A_p, a_p) and any (B_s, b_s);
- for all p and s, in the pushout diagram

$$\begin{tikzcd} R \arrow[r, "a_p"] \arrow[d, "b_s"'] \arrow[dr, "c_{ps}"] & A_p \arrow[d] \\ B_s \arrow[r] & C_{ps} \end{tikzcd}$$

giving the pointed module (C_{ps}, c_{ps}) which is the meet of (A_p, a_p) and (B_s, b_s), the module C_{ps} is indecomposable;

- for fixed p and s, for all $q \in \mathbb{Q}^+$, $q \neq p$ we have $(C_{qs}, c_{qs}) \not\simeq (C_{ps}, c_{ps})$ and for all $t \in \mathbb{Q}^-$, $t \neq s$, we have $(C_{pt}, c_{pt}) \not\simeq (C_{ps}, c_{ps})$.

Theorem 5.6 ([53, 5.4]). *Let R be any ring. Suppose that there is an independent pair of dense chains of indecomposable pointed modules. Then the sublattice of pp_R generated by these is freely generated by them, hence pp_R has width ∞.*

Proof. To show free generation we check the condition of 5.5 (and then the second statement will follow from 5.4).

So suppose that there are $p < q$ and $s < t$ such that $(A_p, a_p) \wedge (B_s, b_s) \leq (A_q, a_q) + (B_t, b_t)$ (recall, re checking the criterion of 5.5, that our ordering of indices is opposite to that in the lattice of pointed finitely presented modules). Using distributivity we have $(A_p, a_p) \wedge (B_s, b_s) = (A_p, a_p) \wedge (B_s, b_s) \wedge \big((A_q, a_q) + (B_t, b_t)\big) = \big((A_p, a_p) \wedge (B_s, b_s) \wedge (A_q, a_q)\big) + \big((A_p, a_p) \wedge (B_s, b_s) \wedge (B_t, b_t)\big) = \big((B_s, b_s) \wedge (A_q, a_q)\big) + \big((A_p, a_p) \wedge (B_t, b_t)\big)$. This last is represented by the direct sum of the pushouts (C_{qs}, c_{qs}) and (C_{pt}, c_{pt}), so there is a morphism from that direct sum to (C_{ps}, c_{ps}) with c_{ps} =(say) $d_{qs}+d_{pt}$ (in reasonably obvious notation). Since the endomorphism ring of C_{ps} is, by assumption, local, and we have maps from (C_{ps}, c_{ps}) to each of those direct summands, this implies that we have, say, $c_{ps} = d_{qs}$ and so, in fact, (C_{ps}, c_{ps}) is equivalent to (C_{qs}, c_{qs}) in the ordering on pointed modules.

[15] In the source, one set is ordered by the positive rationals and the other inversely ordered by the negative rationals but this, and the orderings I have used here, are just matters of taste or of suitability to a particular situation.

$$\begin{tikzcd}
R \arrow[r, "a_p"] \arrow[d, "b_s"'] & A_p \arrow[r] \arrow[d] & A_q \arrow[d] \\
B_s \arrow[r] \arrow[d] & C_{ps} \arrow[r] \arrow[d] & C_{qs} \\
B_t \arrow[r] & C_{pt} &
\end{tikzcd}$$

That contradicts the incomparability assumption (the last of the bulleted conditions), as required. □

The methods above get us lattices with width ∞, but not superdecomposable pure-injectives directly. There is another version which was introduced by Puninski in [49] which bypasses thc countability restriction (cf. 4.6) but assumes that we know an interval essentially completely.

Theorem 5.7 ([49, 2.3]). *Suppose that M is an R-module such that there is an interval in the lattice* pp(M) *which is freely generated by two chains L_1, L_2 each of which contains a dense subchain. Then there is a superdecomposable pure-injective module in the definable subcategory generated by M.*

Since the hypothesis is self-dual, the dual definable category to that generated by M also contains a superdecomposable pure-injective (left) module.

The proof constructs a pp-type with no large pointed finitely presented module, making use of the control given by having a canonical form for all points of that interval in pp(M).

6. Superdecomposables over tubular algebras

In this long section I will outline Harland's proof of existence of superdecomposable pure-injectives over tubular algebras. This is given in the first part of his thesis [20] (the second part is about pure-injective modules over string algebras), see also [21]. Some parts of the argument are written making extensive use of the terminology and techniques around pp formulas so what I have done here is to outline the whole proof and give details of those particular parts but phrased in terms of pointed modules. The major terminological replacement is as follows. In the pp-versions of arguments, one takes a pp formula ϕ and choses a pointed module (A_ϕ, a_ϕ) of which it is a free realisation (Section 2.1) but one continues to use the notation $\phi(M)$ for the solution set of ϕ in a module M. In order to bias the emphasis and terminology of the proofs towards pointed modules, we can start instead with the pointed module (A, a) and, rather than go on to introduce the corresponding pp formula ϕ, as described in Section 2.1, refer to the trace of (A, a) in an arbitrary module M. This, recall, is $\mathrm{tr}(A, a)(M) = \{m \in M : m = fa \text{ for some } f : A \to M\}$; as discussed in Section 2.1, this is exactly the pp-definable subgroup $\phi(M)$. This is really just a translation - I have not made any significant changes to the actual proofs, indeed I follow [21] very closely - but perhaps it will make these arguments more accessible.

Throughout this section R is a tubular algebra; these algebras are defined in [57, Chpt. 5], also see [64, XIX.3.19]. For simplicity we assume that R is basic, connected and that the field K is algebraically closed. Let $S_1, \dots, S_n$ denote the simple modules, so dimension vectors of finite-dimensional modules live in $\mathbb{Z}^n$. By ind-R we denote the set of (isomorphism types of) indecomposable finite-dimensional R-modules.

We will use the standard bilinear form $\langle -, - \rangle$ for R-modules. Tubular algebras have global dimension 2 ([57, 3.1.5]) so the usual formula (see, e.g., [1, III.3.13]) becomes $\langle \underline{\dim}(M), \underline{\dim}(N) \rangle = \dim(M,N) - \dim \mathrm{Ext}^1(M,N) + \dim \mathrm{Ext}^2(M,N)$. Indeed, when we use it, either M will have projective dimension 1 or N will have injective dimension 1, so only the Hom and Ext^1 terms will be non-zero. The corresponding quadratic form is denoted $\chi_R(x) = \langle x, x \rangle$, the **radical** of χ_R is the subgroup $\mathrm{rad}(\chi_R) = \{x : \chi_R(x) = 0\}$ of $\mathbb{Z}^n$ and its elements are the **radical** vectors; the **roots** of χ_R are those $x \in \mathbb{Z}^n$ such that $\chi_R(x) = 1$.

There is ([57, §5.1]) a pair h_0, h_∞ of canonical linearly independent radical vectors. These generate a subgroup of $\mathrm{rad}(\chi_R)$ of finite index. Define the **slope** of $A \in$ ind-R to be the ratio $\iota(\underline{\dim}(A)) = -\dfrac{\langle h_0, \underline{\dim}(A)\rangle}{\langle h_\infty, \underline{\dim}(A)\rangle}$; then, if A is in neither $\mathcal{P}_0$ nor $\mathcal{Q}_\infty$, we have $\iota(\underline{\dim}(A)) = q$ iff $A \in \mathcal{T}_q$ (notation as just below).

The basic structure theorem for mod-R fibres ind-R over the values taken by slope.

Theorem 6.1 ([57, §5.2], ([57, 3.1.5]). *Let R be a tubular algebra; then* ind-R = $\mathcal{P}_0 \cup \bigcup\{\mathcal{T}_q : q \in \mathbb{Q}_0^\infty\} \cup \mathcal{Q}_\infty$ *(disjoint union) where each $\mathcal{T}_q$ is a tubular family separating $\mathcal{P}_q = \mathcal{P}_0 \cup \bigcup\{\mathcal{T}_{q'} : q' < q\}$ from $\mathcal{Q}_q = \bigcup\{\mathcal{T}_{q'} : q < q'\} \cup \mathcal{Q}_\infty$.*

Every module in $\bigcup\{\mathcal{T}_q : q \in \mathbb{Q}^+\}$ has both injective and projective dimension 1.

Note that the finite-dimensional modules A of slope q, meaning the modules in $\mathrm{add}(\mathcal{T}_q)$, are characterised by the conditions $(A, \mathcal{P}_q) = 0 = (\mathcal{Q}_q, A)$. More generally, we say that the **slope** of *any* module $M \in$ Mod-R is $r \in \mathbb{R}^+$ if $(M, \mathcal{P}_r) = 0 = (\mathcal{Q}_r, M)$ where $\mathcal{P}_r$ and $\mathcal{Q}_r$ are defined as in 6.1 but with r in place of the rational value q. By the Auslander-Reiten formula (see, e.g., [1, IV.2.15]) and the fact that $\mathcal{P}_r$ and $\mathcal{Q}_r$, being unions of Auslander-Reiten components, are closed under $\tau^{\pm 1}$, this is equivalent to $\mathrm{Ext}^1(\mathcal{P}_r, M) = 0 = (\mathcal{Q}_r, M)$.

Now, if B is finitely presented and C is FP_2 (i.e., has a projective presentation the first three terms of which are finitely generated) then each of the conditions $(B, M) = 0$ and $\mathrm{Ext}^1(C, M) = 0$ on M is expressible in the form $\phi(M) = \psi(M)$ for certain pp formulas ϕ, ψ ([33, p. 211-12], see [37, 10.2.35]). Therefore (see, e.g., [37, §3.4.1]) the modules which satisfy any collection of such conditions form a definable subcategory of Mod-R. In particular we will denote by $\mathcal{D}_r$ the definable category consisting of all modules of slope $r \in \mathbb{R}^+$. Note that if r is irrational then $\mathcal{D}_r$ contains no finite-dimensional module apart from 0.

By 6.1 every finite-dimensional module is a direct sum of modules with slope, but what about the infinitely generated modules? For indecomposables the answer is surprisingly strong (and implies, by [67, 6.9], that every module over a tubular algebra is elementarily equivalent to a direct sum of modules with slope).

Theorem 6.2 ([54, 13.1]). *Every indecomposable module over a tubular algebra has a slope.*

So what lives at irrationals r? There's something there in $\mathcal{D}_r$, apart from 0, because the observations above, combined with an application of the Compactness Theorem from model theory. The argument is, in brief, as follows.

If all the conditions $(B,M)=0$ with B of slope $<r$ and $\mathrm{Ext}^1(C,M)=0$ with C of slope $>r$ implied $M=0$ then, by the Compactness Theorem, some finite number of them would imply $M=0$ but these finitely many conditions would involve only finitely many modules B, C, so then we just appeal to 6.1 to find a non-zero finite-dimensional module satisfying these finitely many Hom and Ext conditions - giving a contradiction.

So now we know that the modules of positive irrational slope r form a nonzero definable subcategory of Mod-R and it follows that there are indecomposable pure-injectives of slope r. This much, but essentially nothing else about $\mathcal{D}_r$, had been known for a long time.

Harland showed that the width of each definable category $\mathcal{D}_r$ for r positive irrational is ∞ hence, if K is countable, there is a superdecomposable pure-injective in each $\mathcal{D}_r$. This is achieved by showing that the lattice of pp-formulas for $\mathcal{D}_r$ is wide (in fact, in the strong sense of the term as used in Section 4.2).

Before outlining the proof we mention that if q is a positive rational then the definable category $\mathcal{D}_q$ generated by the finite-dimensional indecomposables of slope q is well-known (see, e.g., [20, §3.5]), with structure essentially like that seen in the tame hereditary case ([36], [59], also see [27]). Namely, the infinite-dimensional indecomposable pure-injectives of slope q are of adic and Prüfer types, parametrised by the quasisimple modules of slope q, plus a generic module. There are, morever, no more generic R-modules apart from these ([29]), in particular none of irrational slope.

Fix a module M and recall, from Section 2.2, the relative ordering $(A,a)\geq_M (B,b)$ on pointed finitely presented modules. Note that, if every indecomposable summand of A and B has slope strictly less than that of every indecomposable summand of C in the pushout shown, then, if M is any module with slope strictly between those of summands of A, B and those of summands of C, it must be that $\mathrm{tr}(A,a)(M)\cap \mathrm{tr}(B,b)(M)=0$.

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, "c"] & A \arrow[d, "\beta"] \\ B \arrow[r, "\alpha"'] & C \end{tikzcd}$$

For any non-zero element in the intersection of the traces would induce a non-zero morphism from C to M.

We will say that a finite-dimensional module X is **in** an interval of $\mathbb{R}_0^\infty$ if each of its indecomposable summands has slope in that inverval. Here is a key technical result from [20]. It shows that each pointed finitely presented module has (indeed, the proof produces) a "good" representative in a neighbourhood of any specified irrational.

Theorem 6.3 ([20, Prop. 2, p. 79]). *Let (A, a) be a pointed finitely presented module and let r be a positive irrational. Then there is a pointed finitely presented module (A', a') with $(A, a) \geq (A', a')$ and there is $\epsilon > 0$ such that:*
- *$A' \in \mathrm{add}(\mathcal{P}_{r-\epsilon})$;*
- *$\mathrm{coker}(a') \in \mathrm{add}(\mathcal{Q}_{r+\epsilon})$;*
- *$(A, a) \sim_X (A', a')$, that is, $\mathrm{tr}(A, a)(X) = \mathrm{tr}(A', a')(X)$, for all X in $(r-\epsilon, r+\epsilon)$;*
- *the map $f \mapsto fa'$ induces a bijection $(A', X) \simeq \mathrm{tr}(A', a')(X)$ for all X in $(r - \epsilon, r + \epsilon)$.*

In fact, the conclusion also holds for infinite-dimensional (indecomposable) modules in place of X. The proof given in [20] and [21] already is in terms of pointed modules, so I don't repeat or explicate it here.

Using this, the following result, extending the above to pairs ("pp-pairs" in the model-theoretic terminology) and extracting the information about dimensions is derived.

Corollary 6.4 ([20, Prop. 3, p. 82]). *Let $(A, a) \geq (B, b)$ be a pair of pointed finitely presented modules and let r be a positive irrational. Then there is $\epsilon > 0$ and a vector $v \in \mathbb{Z}^n$ such that $\dim(\mathrm{tr}(A, a)(X)/\mathrm{tr}(B, b)(X)) = v \cdot \underline{\dim}(X)$ for all X in $(r - \epsilon, r + \epsilon)$.*

Say that a pair $(A, a) \geq (B, b)$ of pointed finitely presented modules is **closed** on a module M if $\mathrm{tr}(A, a)(M) = \mathrm{tr}(B, b)(M)$, otherwise the pair is **open** on M. Say that this pair is **closed near the left of** r if there is $\epsilon > 0$ such that $(A, a) \geq (B, b)$ is closed on every indecomposable X in $(r-\epsilon, r)$; otherwise say that $(A, a) \geq (B, b)$ is **open near the left of** r. The latter says only that $(A, a) \geq (B, b)$ is open on "cofinally many" modules near to, and to the left of, r but it is proved eventually (6.5 then 6.7) that, in this case, $(A, a) \geq (B, b)$ is open on *every* module in some interval $(r - \epsilon', r)$. Similarly say that $(A, a) \geq (B, b)$ is **closed near the right of** r if there is $\epsilon > 0$ such that $(A, a) \geq (B, b)$ is closed on every indecomposable X in $(r, r + \epsilon)$; otherwise say that $(A, a) \geq (B, b)$ is **open near the right of** r.

Corollary 6.5 ([20, Cor. 9, p. 81, Thm. 30, p. 83]). *Let $(A, a) \geq (B, b)$ be a pair of pointed finitely presented modules and let r be a positive irrational.*

If $(A, a) \geq (B, b)$ is open near the left of r then there is $\epsilon > 0$ such that $(A, a) \geq (B, b)$ is open on every module in $(r - \epsilon, r)$ which lies in a homogeneous tube.

Similarly, if $(A, a) \geq (B, b)$ is open near the right of r then there is $\epsilon > 0$ such that $(A, a) \geq (B, b)$ is open on every module in $(r, r + \epsilon)$ which lies in a homogeneous tube.

Proof. Apply 6.3 to obtain pointed finitely presented modules $(A', a') \leq (A, a)$ and $(B', b') \leq (B, b)$, and ϵ_1, ϵ_2 satisfying the conclusions of that result. Set $\epsilon' = \min(\epsilon_1, \epsilon_2)$.

Suppose that there is $\gamma \in (r-\epsilon', r) \cap \mathbb{Q}$ and a module $E[k]$ in a homogeneous tube $\mathcal{T}(\rho)$ in $\mathcal{T}_\gamma$ such that $(A,a) \geq (B,b)$ is closed on $E[k]$. Here, if E is a quasisimple module (that is, a module at the mouth of a tube) then we will denote by $E[k]$ the module in the same tube which has quasisimple length k and quasisimple socle E. We shall prove that $(A,a) \geq (B,b)$ must be closed near the left of r.

We have $\mathrm{tr}(A',a')(E[k]) = \mathrm{tr}(A,a)(E[k]) = \mathrm{tr}(B,b)(E[k]) = \mathrm{tr}(B',b')(E[k])$ and so, by 6.3, $\dim(A',E[k]) = \dim(B',E[k])$. By considering almost split sequences in $\mathcal{T}(\rho)$ and induction, it is easy to check that for all positive integers m, $\dim(A',E[m]) = \dim(B',E[m])$. That is $(A',a') \geq (B',b')$ is closed on every module in $\mathcal{T}(\rho)$ and hence on every module in $\mathrm{add}(\mathcal{T}(\rho))$.

Now, given X in (γ, r) and any $f \in (A,X)$ there is, by 6.3, $f' \in (A',X)$ with $f'a' = fa$. By 6.1, f' factors through a module $Y \in \mathrm{add}(\mathcal{T}(\rho))$ (this being part of the definition of a separating tubular family).

$$\begin{tikzcd}
A' \arrow[rr, "f'"] \arrow[dr, dotted, "\exists g"'] & & X \\
& Y \arrow[ur, dotted, "\exists h"'] &
\end{tikzcd}$$

Since $(A',a') \geq (B',b')$ is closed on Y it follows that $g(a') \in \mathrm{tr}(B',b')(Y)$ and so, composing with h, $fa = f'a' \in \mathrm{tr}(B',b')(X) = \mathrm{tr}(B,b)(X)$. Thus $(A,a) \geq (B,b)$ is closed on every module in (γ, r) - as required (take $\epsilon = r - \gamma$).

The proof of the second statement is similar. □

The next result shows how the lattice of pp formulas/pointed finitely presented modules *at* an irrational r is completely determined by and reflected in what happens *near* r. By saying that a pair $(A,a) \geq (B,b)$ is **open at** r we mean that it is open on some module of slope r[16].

Corollary 6.6. *Let $(A,a) \geq (B,b)$ be a pair of pointed finitely presented modules and let r be a positive irrational. Then the following are equivalent:*
(i) $(A,a) \geq (B,b)$ is open near the left of r;
(ii) $(A,a) \geq (B,b)$ is open near the right of r;
(iii) $(A,a) \geq (B,b)$ is open at r.

Proof. If we have (i) then we can use a compactness argument, very similar to that used earlier, to get (iii). Namely, given finitely many of the (Hom and Ext) conditions cutting out the subcategory $\mathcal{D}_r$, there is $\epsilon > 0$ such that every indecomposable finite-dimensional module with slope in $(r-\epsilon, r)$ satisfies them. By assumption there is such a module on which $(A,a) \geq (B,b)$ is open and that can be expressed by a suitable sentence of the formal language for R-modules. Thus the conditions cutting out $\mathcal{D}_r$ are finitely consistent with the condition that $(A,a) \geq (B,b)$ is open so, by the Compactness Theorem, there is a module which satisfies all these conditions, in particular is of slope r, as required. This argument

[16]Here we follow [21] in saying that a pair is closed at r if it is not open at r; this is not the relation $\sim_r$ in [20] which is defined in terms of neighbourhoods of r.

also shows (ii)⇒(iii)[17].

For the converse, suppose we have (iii), say M is a module of slope r on which $(A,a) \geq (B,b)$ is open, say $m \in \mathrm{tr}(A,a)(M) \setminus \mathrm{tr}(B,b)(M)$. Choose $f : A \to M$ such that $m = fa$. Given $\epsilon > 0$, M is, by [54, Lemma 11], the directed union of its finite-dimensional submodules in $(r-\epsilon, r)$, so there is a finite-dimensional submodule X of M which contains fA and is in $(r-\epsilon, r)$. Then certainly $m \in \mathrm{tr}(A,a)(X)$ and, since $m \notin \mathrm{tr}(B,b)(M)$ it must be that $m \notin \mathrm{tr}(B,b)(X)$. Therefore $(A,a) \geq (B,b)$ is open on X and hence is open on some indecomposable summand of X, and we have proved (i).

Finally, assume (iii) and, in order to prove (ii), continue with notations and assumptions as in the previous paragraph. Again using representation of M as a directed union of submodules, there is a submodule Z of M into which X embeds, such that every indecomposable summand of Z has slope greater than the maximum slope of any indecomposable summand of X. By 6.1 the embedding $X \to Z$ factors through a module X' whose indecomposable summands all lie in some homogeneous tube of slope, γ say, between that of any direct factor of X and r. Note that if m' denotes the image in X' of m regarded as an element of X, then $m' \in \mathrm{tr}(A,a)(X') \setminus \mathrm{tr}(B,b)(X')$. If the vector v is chosen for $(A,a) \geq (B,b)$ as in 6.4 then we therefore have $v.\underline{\mathrm{dim}}(X') > 0$. From the definition of slope, it follows that the dimension vector of X' has the form $c(h_0 + \gamma h_\infty)$ where γ is the slope of X' and c is a positive integer. Therefore $v.(h_0 + \gamma h_\infty) > 0$. Repeating this whole argument (but noting that v can be taken to be fixed), we produce an increasing sequence of rationals (the various γ) with limit r, each satisfying $v.h_0 + \gamma v.h_\infty > 0$. Since r is irrational it cannot be that $v.h_0 + rv.h_\infty = 0$, so $v.h_0 + rv.h_\infty > 0$ and hence there is a rational $\gamma' > r$, which we may take in the interval $(r, r+\epsilon'')$ for any given $\epsilon'' > 0$, with $v.h_0 + \gamma' v.h_\infty > 0$. By 6.4, $(A,a) \geq (B,b)$ is open on the homogeneous modules of slope γ'. Thus, $(A,a) \geq (B,b)$ is open near the right of r. □

We need to remove the restriction in 6.5 to modules in homogeneous tubes. This is done for certain tubular algebras; the consequences, in particular undefined width for the relativised pp/pointed-module lattice (Section 2.2) at irrational r, are derived over these algebras and then extended by tilting functors to arbitrary tubular algebras. Here are some, but by no means most, of the details.

The particular algebras which are dealt with are those appearing in [57, §5.6] and the result proved is the following.

[17] Appeals to the Compactness Theorem can be replaced by using the (algebraic) ultraproduct construction but this is usually more work. To indicate how this might go in this case: we would take a sequence, $(q_i)_i$, of rationals approaching r from below and for each of these choose a module X_i with slope in (q_i, r) such that there is $f_i : A \to X_i$ such that there is no $g : B \to X_i$ with $gb = f_i a$. By assumption we can do this. Then we would choose an ultrafilter $\mathcal{U}$ on the index set $\{i\}_i$ and form the ultraproduct $M = \prod_i X_i/\mathcal{U}$. The morphims f_i together induce a morphism $f : A \to M$ which is such that there is no $g : B \to M$ with $gb = fa$ - because this equality would imply equality on some components in the product - indeed on a set of indices in $\mathcal{U}$. So $(A,a) \geq (B,b)$ is open on M. But we have yet to show that M is, or can be taken to be, of slope r. One way would be to replace M by an elementarily equivalent direct sum of indecomposable pure-injectives and show that any of slope smaller than r are irrelevant, but again the obvious route appeals to model theory!

Corollary 6.7 ([20, Lemma 63, p. 102]). *Let R be one of the algebras $C(4,\lambda)$, $C(6)$, $C(7)$, $C(8)$ from [57, §5.6]. Let $(A,a) \geq (B,b)$ be a pair of pointed finitely presented R-modules and let r be a positive irrational. If $(A,a) \geq (B,b)$ is open at r then there exists $\epsilon > 0$ such that $(A,a) \geq (B,b)$ is open on every module in $(r-\epsilon, r+\epsilon)$.*

The proof uses 6.4 to relate what happens on homogeneous tubes to modules in inhomogeneous tubes, computing relevant estimates on slopes and dimensions for these particular algebras. Further computations (none of this part uses pp formulas or model theory) yield the following key result.

Theorem 6.8 ([20, Lemma 66, p. 106]). *Let R be one of the algebras $C(4,\lambda)$, $C(6)$, $C(7)$ or $C(8)$. Given any positive irrational r, any $\epsilon > 0$ and any $d \geq 1$, there exists an inhomogeneous tube $\mathcal{T}(\rho)$ of rank $\langle h_0, h_\infty \rangle$ and with slope γ in $(r-\epsilon, r)$ such that, if E is a quasisimple of $\mathcal{T}(\rho)$ and X is an indecomposable finite-dimensional module in (γ, r), then $\dim(X) \geq \dim(E) + d$.*

In particular, any nonzero morphism from E to a module in (γ, r) is an embedding.

Now we have the ingredients we need for the proof that over these algebras width at an irrational is undefined. Let $M(r)$ be any module which generates the definable category $\mathcal{D}_r$, so we have the lattice $\mathrm{pp}(M(r))$ of pointed finitely presented modules at r. (We remark that, by 6.6, this can be defined in terms of modules with slope near r, hence purely in terms of mod-R.)

Theorem 6.9 ([20, Thm. 31, p. 107]). *Let R be one of the algebras $C(4,\lambda)$, $C(6)$, $C(7)$ or $C(8)$. For each positive irrational r the lattice $\mathrm{pp}/\sim_r$ of pointed finitely presented modules relativised to $\mathcal{D}_r$ has width ∞. Indeed, every non-trivial interval in this lattice contains incomparable elements.*

Proof. Take any pp-pair $(A,a) \geq (B,b)$ which is open at r, hence open in a neighbourhood of r. By 6.3 we may replace each of these pointed finitely presented modules by one to which it is $\sim_r$-equivalent and with properties as in 6.3. We assume that we have made these replacements from the outset.

Let $d = \dim(B)$ and apply 6.8 to obtain $\gamma \in (r-\epsilon, r) \cap \mathbb{Q}$ and a tube $\mathcal{T}(\rho)$ of index γ as in the statement of that result. Pick any quasisimple module E in $\mathcal{T}(\rho)$ and let E' be any other quasisimple module in that tube. Fix $f : A \to E$ such that there is no $g : B \to E$ with $gb = fa$ and similarly fix $f' : A \to E$ witnessing that $\mathrm{tr}(A,a)(E') > \mathrm{tr}(B,b)(E')$. We shall show that the images of $(B,b)+(E,x)$ and $(B,b)+(E',x')$ in $\mathrm{pp}/\sim_r$ are incomparable.

So suppose, for a contradiction, that, say, $(B,b)+(E,x) \leq_r (B,b)+(E',x')$, that is, $(E,x) \sim_r (E,x) \wedge ((B,b)+(E',x'))$. Therefore, by 6.7, there is $\delta > 0$ such that for all X in $(r-\delta, r)$ we have $\mathrm{tr}(E,x)(X) = \mathrm{tr}\big((E,x) \wedge ((B,b)+(E',x'))\big)(X)$; we may take $\delta < \epsilon$. Recalling what the operations in the lattice of pointed finitely presented modules are, the right-hand side of this equation is $\mathrm{tr}(L,l)(X)$ where L is the pushout module shown and $l = g(x) = g'(b,x')$.

$$\begin{tikzcd} R \arrow[r, "x"] \arrow[d, "{(b,x')}"'] & E \arrow[d, "g"] \\ B \oplus E' \arrow[r, "g'"] & L \end{tikzcd}$$

Note that $\dim(L) < \dim(E) + \dim(E') + \dim(N)$.

Write L as $L' \oplus L''$ where each summand of L' has slope $< r$ and each summand of L'' has slope $> r$. So, by 6.1, there are nonzero morphisms, embeddings by 6.8, from E to modules X in $(r-\delta, r)$, $(L, X) \neq 0$. Therefore $L' \neq 0$ and so, if we set $f' = \pi' g$ where $\pi' : L \to L'$ is the induced projection, then $f'(x) \neq 0$. We will show that L' has no summand of slope $> \gamma$; we do this by showing that this is also true of $\mathrm{coker}(f')$, noting that by choice of γ and since L' is in (γ, r), f' is, by 6.8, an embedding, so we have the exact sequence

$$0 \to E \xrightarrow{f'} L' \to \mathrm{coker}(f') \to 0.$$

Pick $\gamma' \in (\gamma, r)$ such that L' is in $(0, \gamma')$ (indeed, in $[\gamma, \gamma')$). First we show that $\mathrm{coker}(f')$ is in $[0, \gamma')$.

Suppose that $h : E \to Z$ where Z is in $(r-\delta, r)$; then h factors through g. Indeed, since $h(x) \in \mathrm{tr}(E, x)(Z) \leq \mathrm{tr}((B, b) + (E', x'))(Z)$, there must exist a map $h' : B \oplus E' \to Z$ with $h'(b, x') = h(x)$. The pushout property then gives a factorisation of h through g, hence through f', as claimed.

Pick Z of slope γ'; then we have the long exact sequence

$$0 \to (\mathrm{coker}(f'), Z) \to (L', Z) \xrightarrow{\mathrm{coker}(f', Z)} (E, Z) \to \mathrm{Ext}(\mathrm{coker}(f'), Z) \to \mathrm{Ext}(L', Z) = 0$$

(the last term is 0 since, by the Auslander-Reiten formula, it has the same dimension as $(Z, L') = 0$). We have just seen that $\mathrm{coker}(f', Z)$ is surjective so it follows that $\mathrm{Ext}(\mathrm{coker}(f'), Z) = 0$ and then, by the Auslander-Reiten formula and 6.1, that $(Z, \mathrm{coker}(f')) = 0$. This is so for every Z of slope γ' so, by 6.1, $\mathrm{coker}(f')$ must be in $[0, \gamma')$.

Now, notice that $\dim(\mathrm{coker}(f')) = \dim(L') - \dim(E) \leq \dim(L) - \dim(E) < \dim(B) + \dim(E') + \dim(E) - \dim(E) = \dim(N) + \dim(E')$. So, by choice of γ, $\mathcal{T}(\rho)$ to satisfy 6.8, and the fact that E' is in $\mathcal{T}(\rho)$, it must be that every summand of $\mathrm{coker}(f')$ with slope $> \gamma$ has slope $> r$. We saw above that every summand of $\mathrm{coker}(f')$ has slope $< \gamma' < r$, so we deduce that $\mathrm{coker}(f')$ has slope γ.

It follows that L' also has slope γ and hence that the exact sequence

$$0 \to E \xrightarrow{f'} L' \to \mathrm{coker}(f') \to 0$$

lies in $\mathrm{add}(\mathcal{T}(\rho))$. If L' is not indecomposable then we can replace it by a direct summand to which the induced map from E is nonzero hence still an embedding, and that sequence will, up to isomorphism, have the form

$$0 \to E \xrightarrow{f''} E[k] \xrightarrow{p} \tau^{-1} E[k-1] \to 0.$$

Since $E[k]$ has quasisimple socle E, it follows that $\pi_1 g' E' = 0$, where $\pi_1 : L \to E[k]$ is the projection, and hence $\pi_1 g'(b) = f''(x)$, so $p \pi_1 g'(b) = 0$. Therefore

$p\pi_1 g'$ factors through coker(b) which, by choice of B, b and ϵ to satisfy 6.3, must be in $(r,\infty]$. Therefore $p\pi_1 g' = 0$ and $\pi_1 g$ takes B to $f''E \simeq E$. We deduce that $f''(x) \in \mathrm{tr}(B,b)(f''E)$ and hence that $x \in \mathrm{tr}(B,b)(E)$ - a contradiction to the choice of x.

Thus $(B,b)+(E,x)$ and $(B,b)+(E',x')$ have incomparable images in the lattice $\mathrm{pp}/\sim_r$ as claimed. This is true for arbitrary pairs $(A,a) \geq (B,b)$ as at the start of the proof, so we have shown that this lattice is wide. □

The final stage is to use tilting functors to move the conclusion of the result above to other tubular algebras. This is done in [20] and [21] using the language of pp formulas but the proofs actually use pointed modules (in fact n-pointed modules since the generator R_R will in general be tilted to a module which is not cyclic). We don't repeat the arguments here. An alternative would be to use general results about interpretation functors [37, §18.2.1] or [39, §25] and the fact that tilting functors are such ([33, 4.8], see [37, 18.2.22]). In any case the conclusion is as follows.

Theorem 6.10 ([20, Thm. 34, p. 119]). *Let R be a tubular algebra and let r be a positive irrational. Then the width of the lattice of pointed finitely presented modules for the definable category $\mathcal{D}_r$ is undefined. If R is countable then there is a superdecomposable pure-injective module of slope r.*

Corollary 6.11. *Let R be a tubular algebra and let r be a positive irrational. Then the Krull-Gabriel dimension of the definable category $\mathcal{D}_r$ is undefined. If R is countable then there are continuum many indecomposable pure-injective modules of slope r.*

For the second assertion see the comment at the end of Section 4.1.

7. Final remarks

Here we have stopped at the point of proving existence of a superdecomposable pure-injective and we haven't considered the structure of such a module. For example if N is superdecomposable it may or may not be the case that $N \simeq N^2$. There is a structure theory for injective modules over von Neumann regular rings and that can be reflected into the structure of pure-injective modules, as in [9]. Puninski, see e.g. [47], also has developed some ways of understanding the structure of these modules and has formulated conjectures about their structure. There seems, however, to have been rather limited development in this direction. A specific question we can ask, in the context of modules over tubular algebras, is whether every (superdecomposable) pure-injective is the pure-injective hull of a direct sum of modules with slope.

It would be good to know whether the countability hypothesis currently needed to deduce from having width undefined that there is a superdecomposable pure-injective can be removed. An obvious approach, if R is an uncountable ring with pp_R having width ∞, is to take a countable elementary subring R_0 (which, by the downwards Löwenheim-Skolem Theorem of model theory, may be chosen to contain any specified countably many elements of R) and which is such that pp_{R_0} also has width ∞ (this can be done: see [34, Prop. 10]). Then there will be a superdecomposable pure-injective R_0-module N_0. Can one, for example, obtain a superdecomposable pure-injective R-module from $N_0 \otimes_{R_0} R$?

References

[1] I. Assem, D. Simson and A. Skowróński, Elements of the Representation Theory of Associative Algebras, Vol. 1: Techniques of Representation Theory, Cambridge University Press, London Math. Soc. Student Texts, Vol. 65, 2006.

[2] M. Auslander, Functors and objects determined by objects, pp. 1-244 *in* R. Gordon (Ed.), Representation Theory of Algebras, Philadelphia 1976, Marcel Dekker, New York, 1978.

[3] V. Bavula, Generalised Weyl algebras and their representations (*in Russian*), Algebra and Analiz, 4(1) (1992), 75-97; *English transl. in* St. Petersburg Math. J., 4(1), (1993), 71-92.

[4] K. Burke, Some Model-Theoretic Properties of Functor Categories for Modules, Doctoral Thesis, University of Manchester, 1994.

[5] K. Burke, Co-existence of Krull filtrations, J. Pure Appl. Algebra, 155(1) (2001), 1-16.

[6] K. Burke and M. Prest, The Ziegler and Zariski spectra of some domestic string algebras, Algebras and Representation Theory, 5(3) (2002), 211-234.

[7] W. Crawley-Boevey, Infinite-dimensional modules in the representation theory of finite-dimensional algebras, pp. 29-54 *in* I. Reiten, S. Smaløand Ø. Solberg (Eds.), Algebras and Modules I, Canadian Math. Soc. Conf. Proc., Vol. 23, Amer. Math. Soc, 1998.

[8] S. Ebrahimi Atani, On pure-injective modules over pullback rings, Comm. Algebra, 28(9) (2000), 4037-4069.

[9] A. Facchini, Decompositions of algebraically compact modules, Pacific J. Math., 116(1) (1985), 25-37.

[10] A. Facchini, Module Theory: Endomorphism rings and direct sum decompositions in some classes of modules, Progress in Math., Vol. 167, Birkhäuser, Basel, 1998.

[11] E. Fisher, Abelian Structures, Yale University, 1974/5, partly published as Abelian Structures I, pp. 270-322 *in* Abelian Group Theory, Lecture Notes in Mathematics, Vol. 616, Springer-Verlag, 1977.

[12] S. Garavaglia, Dimension and rank in the model theory of modules, *preprint*, University of Michigan, 1979, *revised* 1980.

[13] W. Geigle, The Krull-Gabriel dimension of the representation theory of a tame hereditary artin algebra and applications to the structure of exact sequences, Manus. Math., 54(1-2) (1985), 83-106.

[14] R. Göbel and J. Trlifaj, Approximations and Endomorphism Algebras of Modules, W. de Gruyter, 2006.

[15] R. Gordon and J. C. Robson, Krull Dimension, Mem. Amer. Math. Soc., 133, American Mathematical Society, Providence, R.I., 1973.

[16] R. Gordon and J. C. Robson, The Gabriel dimension of a module, J. Algebra, 29(3) (1974), 459-473.

[17] G. Grätzer, General Lattice Theory, Birkhäuser, 1978.

[18] L. Gruson and C. U. Jensen, Modules algébriquement compact et foncteurs $\varprojlim^{(i)}$, C. R. Acad. Sci. Paris, 276 (1973), 1651-1653.

[19] L. Gruson and C. U. Jensen, Dimensions cohomologiques reliées aux foncteurs $\varprojlim^{(i)}$, pp. 243-294 *in* Lecture Notes in Mathematics, Vol. 867, Springer-Verlag, 1981.

[20] R. Harland, Pure-injective Modules over Tubular Algebras and String Algebras, Doctoral Thesis, University of Manchester, 2011, *available at* www.maths.manchester.ac.uk/∼mprest/publications.html

[21] R. Harland and M. Prest, Modules with irrational slope over tubular algebras, *preprint*, University of Manchester, 2012.

[22] B. Huisgen-Zimmermann, Purity, algebraic compactness, direct sum decompositions and representation type, pp. 331-367 *in* [28].

[23] C. U. Jensen and H. Lenzing, Model Theoretic Algebra; with particular emphasis on Fields, Rings and Modules, Gordon and Breach, 1989.

[24] I. Kaplansky, Infinite Abelian Groups, Univ. of Michigan Press, Ann Arbor, 1954. *Also revised edition*, Ann Arbor, 1969.

[25] S. Kasjan and G. Pastuszak, On two tame algebras with super-decomposable pure-injective modules. Colloq. Math., 123(2) (2011), 249-276

[26] L. Klingler and L. Levy, Wild torsion modules over Weyl algebras and general torsion modules over HNPs, J. Algebra, 172(2) (1995), 273-300.

[27] H. Krause, Generic modules over artin algebras, Proc. London Math. Soc. (3), 76(2) (1998), 276-306.

[28] H. Krause and C. M. Ringel (Eds.), Infinite Length Modules, Birkhäuser, 2000.

[29] H. Lenzing, Generic modules over tubular algebras, pp. 375-385 *in* M. DrR. Göbel (Ed.) Advances in Algebra and Model Theory, Gordon and Breach, 1997.

[30] L. S. Levy, Modules over pullbacks and subdirect sums, J. Algebra, 71(1) (1981), 50-61.

[31] M. Prest, Model Theory and Modules, London Math. Soc. Lecture Notes Ser. Vol. 130, Cambridge University Press, 1988.

[32] M. Prest, Duality and pure-semisimple rings, J. London Math. Soc. (2), 38 (1988), 403-409.

[33] M. Prest, Interpreting modules in modules, Ann. Pure Applied Logic, 88(2-3) (1997), 193-215.

[34] M. Prest, The representation theories of elementarily equivalent rings, J. Symbolic Logic, 63(2) (1998), 439-450.

[35] M. Prest, Morphisms between finitely presented modules and infinite-dimensional representations, pp. 447-455 *in* Canad. Math. Soc. Conf. Proc., Vol. 24 (1998).

[36] M. Prest, Ziegler spectra of tame hereditary algebras, J. Algebra, 207(1) (1998), 146-164.

[37] M. Prest, Purity, Spectra and Localisation, Encyclopedia of Mathematics and its Applications, Vol. 121, Cambridge University Press, 2009.

[38] M. Prest, Pure-injective modules, AJSE-Mathematics, 1(1) (2009), 175-191.

[39] M. Prest, Definable additive categories: purity and model theory, Mem. Amer. Math. Soc., Vol. 210/No. 987, 2011.

[40] M. Prest and G. Puninski, Some model theory over hereditary noetherian domains, J. Algebra, 211(1) (1999), 268-297.

[41] G. E. Puninski, Superdecomposable pure-injective modules over commutative valuation rings (*in Russian*), Algebra i Logika, 31 (1992), 655-671, *English transl. in* Algebra and Logic, 31(6) (1992), 377-386.

[42] G. Puninski, Finite length and pure-injective modules over a ring of differential operators, J. Algebra, 231(2) (2000), 546-560.

[43] G. Puninski, Pure-injective and finite length modules over certain rings of differential polynomials, pp. 171-185 in Y. Fong *et al.* (Eds.), Proc. Moscow-Taiwan Workshop, Springer-Verlag, 2000.

[44] G. Puninski, Serial rings, Kluwer, Dordrecht, 2001.

[45] G. Puninski, Some model theory over an exceptional uniserial ring and decompositions of serial modules, J. London Math. Soc. (2), 64(2) (2001), 311-326.

[46] G. Puninski, Some model theory over a nearly simple uniserial domain and decompositions of serial modules, J. Pure Applied Algebra, 163(3) (2001), 319-337.

[47] G. Puninski, Pure-injective modules over a commutative valuation domain, Algebras and Representation Theory, 6(3) (2003), 239- 250.

[48] G. Puninski, The Krull–Gabriel dimension of a serial ring, Comm. Algebra, 31(12) (2003), 5977-5993.

[49] G. Puninski, Superdecomposable pure-injective modules exist over some string algebras, Proc. Amer. Math. Soc., 132(7) (2004), 1891-1898.

[50] G. Puninski, When a superdecomposable pure injective module over a serial ring exists, pp. 449-463 *in* A. Facchini, E. Houston and L. Salce (eds.), Rings, Modules and Abelian Groups, Marcel Dekker, 2004.

[51] G. Puninski, How to construct a 'concrete' superdecomposable pure-injective module over a string algebra, J. Algebra, 212(4) (2008), 704-717.

[52] G. Puninski, V. Puninskaya and C. Toffalori, Superdecomposable pure-injective modules and integral group rings, J. London Math. Soc. (2), 73(1) (2006), 48-64.

[53] G. Puninski, V. Puninskaya, and C. Toffalori, Krull-Gabriel dimension and the model-theoretic complexity of the category of modules over group rings of finite groups, J. London Math. Soc., 78(1) (2008), 125-142.

[54] I. Reiten and C. M. Ringel, Infinite dimensional representations of canonical algebras, Canad. J. Math., 58(1) (2006), 180-224.

[55] R. Rentschler and P. Gabriel, Sur la dimension des anneaux et ensembles ordonnés, C. R. Acad. Sci. Paris, Ser. A5 (1967), 712-715.

[56] C. M. Ringel, Infinite-dimensional representations of finite-dimensional hereditary algebras, Symp. Math. inst. Naz. Alta. Mat., 23 (1979), 321-412.

[57] C. M. Ringel, Tame Algebras and Integral Quadratic Forms, Lecture Notes in Mathematics, Vol. 1099, Springer-Verlag, 1984.

[58] C. M. Ringel, Some algebraically compact modules I, pp. 419-439 *in* Abelian Groups and Modules, Eds. A. Facchini and C. Menini, Kluwer, 1995.

[59] C. M. Ringel, The Ziegler spectrum of a tame hereditary algebra, Colloq. Math., 76 (1998), 105-115.

[60] C. M. Ringel, Tame algebras are Wild, Algebra Coll., 6 (1999), 473-480.

[61] J. Schröer, Hammocks for String Algebras, Doctoral thesis, Universität Bielefeld, 1997.

[62] J. Schröer, On the Krull-Gabriel dimension of an algebra, Math. Z. 233(2) (2000), 287-303.

[63] J. Schröer, The Krull-Gabriel dimension of an algebra - open problems and conjectures, pp. 419-424 *in* [28].

[64] D. Simson and A. Skowrónski, Elements of the Representation Theory of Associative Algebras, Vol. 3: Representation-Infinite Tilted Algebras, Cambridge University Press, London Math. Soc. Student Texts, Vol. 72, 2007.

[65] J. Trlifaj, Two problems of Ziegler and uniform modules over regular rings, pp. 373-383 *in* Abelian Groups and Modules, M. Dekker, New York, 1996.

[66] R. B. Warfield, Purity and algebraic compactness for modules, Pacific J. Math., 28(3) (1969), 699-719.

[67] M. Ziegler, Model theory of modules, Ann. Pure Appl. Logic, 26(2) (1984), 149-213.

[68] B. Zimmermann-Huisgen and W. Zimmermann, Algebraically compact rings and modules, Math. Z., 161(1) (1978), 81-93.

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